A STOCHASTIC CHARACTERIZATION OF WEAR-OUT FOR COMPONENTS AND SYSTEMS

By Z. W. Birnbaum[1], J. D. Esary and A. W. Marshall

University of Washington; Boeing Scientific Research Laboratories; Boeing Scientific Research Laboratories

0. Summary. It is well known that the future life distribution of a device remains the same regardless of the time it was previously in use, if and only if the life distribution of that device is exponential. For this reason exponential life distributions are accepted as characterizing the phenomenon of no wear. The problem of finding a class of life distributions which would similarly reflect the phenomenon of wear-out has been under investigation for some time. In answer to this problem we introduce in this paper the class of IHRA (Increasing Hazard Rate Average) distributions and show that it has, among others, the following optimal properties: (i) it contains the limiting case of no wear, i.e., all exponential distributions, (ii) whenever components which have IHRA life distributions are put together into a coherent system, this system again has an IHRA life distribution, i.e., a system wears out when its components wear out, and (iii) the IHRA class is the smallest class with properties (i) and (ii).

1. Introduction. The nature of many physical devices is to wear out in time. In this paper we endeavor to determine how the wear-out process is reflected as a property of the corresponding life distributions.

Wear-out is an intuitively suggestive concept, but its precise stochastic meaning is not immediately evident. However, as a beginning, we review the long accepted stochastic characterization of the phenomenon of no wear. Let $T \geqq 0$ be the failure time of a device. Let $\bar{F}(t) = P\{T > t\}$ be the complement of the usual distribution function, which we will call the "survival probability." The conditional survival probability for remaining life, given that the device has survived to age x, is $\bar{F}_x(t) = \bar{F}(x+t)/\bar{F}(x)$ if $\bar{F}(x) > 0$, $\bar{F}_x(t) = 0$ if $\bar{F}(x) = 0$. A device does not wear if regardless of age an unfailed device is like new, i.e. $\bar{F}_x(t) = \bar{F}_0(t)$ for all $x, t \geqq 0$. It follows that the class of no wear survival probabilities is the class of solutions of the functional equation

$$\bar{F}(x+t) = \bar{F}(x)\bar{F}(t), \qquad x, t \geqq 0, \tag{1.1}$$

i.e. the class of exponential survival probabilities $\bar{F}(t) = \exp(-\lambda t)$, $0 \leqq \lambda \leqq +\infty$. *Our first requirement of the class of survival probabilities describing wear-out is that it should admit the boundary case of no wear*, i.e. *contain the exponential survival probabilities.*

It is tempting to characterize wear-out in a way similar to no wear, by supposing

Received 1 February 1966.

[1] The research of Z. W. Birnbaum was supported in part by the Office of Naval Research.

that increasing age has a deleterious effect on the conditional survival probability of a device, i.e. $\bar{F}_x(t) \geqq \bar{F}_y(t)$, all $t \geqq 0$, whenever $x \leqq y$. This condition defines the increasing hazard rate (IHR) class of survival probabilities. The name derives from their property that when there is a density f, the hazard rate (failure rate) $r(t) = f(t)/\bar{F}(t)$ is non-decreasing. The IHR class meets our first requirement, since it does contain the exponential survival probabilities. It has been extensively studied and a number of interesting results about it are now known (see [1] for a survey and bibliography).

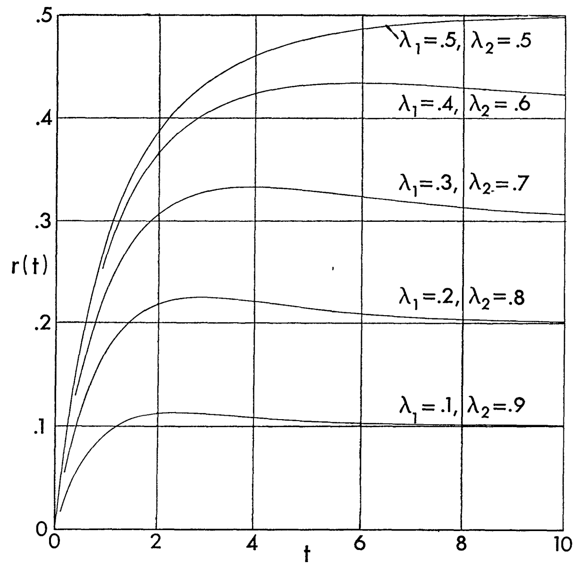

FIG. 1. Representative shapes of the system hazard rate for two exponential lives in parallel.

$r(t) = [e^{\lambda_1 t} + e^{\lambda_2 t} - 1]^{-1} \cdot [\lambda_1 e^{\lambda_2 t} + \lambda_2 e^{\lambda_1 t} - 1]$ is the system hazard rate where λ_1 and λ_2 are the component hazard rates, $\lambda_1 + \lambda_2 = 1$. The condition $\lambda_1 + \lambda_2 = 1$ always holds if the system hazard rate is measured in units equal to the sum of the two actual component hazard rates, and time is measured in the reciprocal units.

If two components, with failure times T_1, T_2 form a series system, i.e. the system failure time T equals min (T_1, T_2), and if the components have IHR survival probabilities, then it is immediate that the system has an IHR survival probability. We can say that the IHR class is closed under the formation of series systems. On the other hand if two components form a parallel system, i.e. $T = \max(T_1, T_2)$, the system need not have an IHR survival probability even though the components do. This may be seen by an argument given in [7] or from Figure 1 which exhibits the shape of the hazard rate for parallel systems of two components with exponential survival probabilities. Thus the IHR class is not closed under the formation of parallel systems. The family of coherent (Section 2) systems contains the series and parallel systems, and many that are more complex. Coherent systems are, from one view, those in which the shortening of any component life tends to shorten the life of the system. For such systems it seems reasonable that if all the components in the system wear out, then the system should wear out. *Our second requirement on the class of survival probabilities describing wear-out is that it should be closed under the formation of coherent systems.*

The increasing hazard rate average (IHRA, Section 4) class of survival probabilities $\bar{F}$ is named from its property that the hazard rate $r(t)$ when defined has a non-decreasing average $\int_0^t r(u)\,du/t$. The class could be alternately defined by a condition intuitively related to wear-out, that for each $x \geqq 0$ and λ chosen so that $\bar{F}(x) = \exp(-\lambda x)$,

$$
\begin{aligned}
(1.2) \qquad \bar{F}(t) &\geqq \exp(-\lambda t) \quad \text{if} \quad t \leqq x \\
&\leqq \exp(-\lambda t) \quad \text{if} \quad t \geqq x.
\end{aligned}
$$

This condition says that if an IHRA device and a device that is not wearing have the same chance of surviving some period of time $[0, x]$, then the IHRA device has the better chance of surviving any shorter period and the worse chance of surviving any longer period. The IHRA class contains the exponential survival probabilities. In fact, it contains all IHR survival probabilities. We show that the IHRA class is closed under the formation of coherent systems, and that it is essentially the smallest class containing the exponentials which is so closed.

REMARK. We have said it is reasonable to expect that if the components of a coherent system wear out, then the system will wear out. It might seem equally reasonable to expect that if the components of a coherent system do not wear, then the system should not wear, i.e. to expect the class of exponential survival probabilities to be closed under the formation of coherent systems. This, as we have seen by example, is not the case. An intuitive explanation is that components which do not wear may still fail; their failure is a form of cumulative damage to their system, and when there is cumulative damage there is wear-out.

2. A reliability inequality for coherent systems. We need to introduce the family of coherent systems, and obtain for it an inequality which is necessary to our later results. This is most conveniently done if we ignore, for the present, the role time plays in our problem.

A system has some finite number, n, of components, each capable of just two modes of performance. The performance of the ith component is represented by an indicator variable x_i with $x_i = 1$ if the component is functioning and $x_i = 0$ if the component is failed. The system is capable of the same two modes of performance, and we assume that the indicator of system performance is a function $\phi(x_1, x_2, \cdots, x_n)$ of the component indicators. ϕ is the *structure function* of the system. The system is *coherent* ([9], [2], [1]) if $\phi(1, 1, \cdots, 1) = 1$, $\phi(0, 0, \cdots, 0) = 0$, and $\phi(x_1, x_2, \cdots, x_n) \leqq \phi(y_1, y_2, \cdots, y_n)$ whenever $x_i \leqq y_i$ for $i = 1, \cdots, n$.

We assume that the component performance indicators are independent binary random variables $X_1, X_2, \cdots, X_n$. Then $p_i = P\{X_i = 1\}$ is the reliability of the ith component. The *reliability function* of the system is

$$h(\mathbf{p}) = P\{\phi(\mathbf{X}) = 1\},$$

where $\mathbf{p} = (p_1, p_2, \cdots, p_n)$, $\mathbf{X} = (X_1, X_2, \cdots, X_n)$. A standard decomposition of a system structure function

$$\phi(\mathbf{X}) = X_i\phi(1_i, \mathbf{X}) + (1 - X_i)\phi(0_i, \mathbf{X})$$

where $(1_i, \mathbf{X}) = (X_1, \cdots, X_{i-1}, 1, X_{i+1}, \cdots, X_n)$ and $(0_i, \mathbf{X})$ is similarly defined, leads immediately to a corresponding decomposition of the reliability function

$$h(\mathbf{p}) = p_i h(1_i, \mathbf{p}) + (1 - p_i)h(0_i, \mathbf{p}). \tag{2.1}$$

From (2.1),

$$\partial h(\mathbf{p})/\partial p_i = h(1_i, \mathbf{p}) - h(0_i, \mathbf{p}). \tag{2.2}$$

For coherent systems, $h(1, 1, \cdots, 1) = 1$, $h(0, 0, \cdots, 0) = 0$, and $h(1_i, \mathbf{p}) \geqq h(0_i, \mathbf{p})$ for all $\mathbf{p}$, $i = 1, \cdots, n$. Thus $\partial h(\mathbf{p})/\partial p_i$ is always non-negative.

THEOREM 2.1. *If h is the reliability function of a coherent system and ψ is defined on* [0, 1] *by either $\psi(u) = -u \log u$ or $\psi(u) = -(1 - u) \log (1 - u)$, the inequality*

$$\textstyle\sum_{i=1}^{n} \partial h(\mathbf{p})/\partial p_i \cdot \psi(p_i) \geqq \psi[h(\mathbf{p})] \tag{2.3}$$

holds for all $\mathbf{p}$.

PROOF. The proof is by induction on n. For $n = 1$ there is just one coherent system, $\phi(x) = x$ with $h(p) = p$, and (2.3) is true.

Let h be the reliability function of a n-component coherent system with structure function ϕ. $h(1_n, \mathbf{p})$ is the reliability function of the $(n - 1)$-component system with structure function $\phi(1_n, \mathbf{x})$. This system is either coherent or $\phi(1_n, \mathbf{x}) \equiv 1$, e.g. when the n-component system is a parallel system. If it is coherent, (2.3) holds for $h(1_n, \mathbf{p})$ by inductive hypothesis. If it is not coherent, then $h(1_n, \mathbf{p}) \equiv 1$ and (2.3) holds since $\psi(1) = 0$. $h(0_n, \mathbf{p})$ is the reliability function of the $(n - 1)$-component system with structure function $\phi(0_n, \mathbf{x})$, which is coherent unless $\phi(0_n, \mathbf{x}) \equiv 0$, $h(0_n, \mathbf{p}) \equiv 0$, e.g. if the n-component system is a series system. Since $\psi(0) = 0$, (2.3) holds for $h(0_n, \mathbf{p})$. Then, from (2.1) and (2.2),

$$\begin{aligned}
\textstyle\sum_{i=1}^{n} \partial h(\mathbf{p})/\partial p_i \cdot \psi(p_i) &= p_n \textstyle\sum_{i=1}^{n-1} \partial h(1_n, \mathbf{p})/\partial p_i \cdot \psi(p_i) \\
&\quad + (1 - p_n) \textstyle\sum_{i=1}^{n-1} \partial h(0_n, \mathbf{p})/\partial p_i \cdot \psi(p_i) \\
&\quad + [h(1_n, \mathbf{p}) - h(0_n, \mathbf{p})]\psi(p_n) \\
&\geqq p_n \psi[h(1_n, \mathbf{p})] + (1 - p_n)\psi[h(0_n, \mathbf{p})] \\
&\quad + [h(1_n, \mathbf{p}) - h(0_n, \mathbf{p})]\psi(p_n).
\end{aligned}$$

We must now show that

$$r\psi(h_1) + (1 - r)\psi(h_0) + (h_1 - h_0)\psi(r) \geqq \psi(rh_1 + (1 - r)h_0),$$

where $r = p_n$, $h_1 = h(1_n, \mathbf{p})$, $h_0 = h(0_n, \mathbf{p})$, and $h(\mathbf{p}) = rh_1 + (1 - r)h_0$. When $\psi(u) = -u \log u$, the inequality above is equivalent to $\psi(rh_1 + (1 - r)h_0) - \psi(h_0) \leqq \psi(rh_1) - \psi(rh_0)$. Since, for a coherent system $h_1 \geqq h_0$, and $\psi(u)$ is concave on [0, 1], this last inequality is true. When $\psi(u) = -(1 - u) \log (1 - u)$ the argument is similar. □

The inequality (2.3) holds also when $\psi(u) = u(1 - u)$ ([6], [1]). For this choice of ψ it was first obtained by E. F. Moore and C. E. Shannon [10] for the case $p_1 = p_2 = \cdots = p_n$. The functions ψ defined on [0, 1] for which (2.3) is valid form a convex cone. It is fairly easy to show that these functions must be concave with $\psi(0) = 0$ and $\psi(1) = 0$, and that if a function ψ is in the cone, then its dual $\psi^D(u) = \psi(1 - u)$ is in the cone. It may also be shown that (2.3) becomes an identity in $\mathbf{p}$ only for $h = p_1 p_2 \cdots p_n$ (a series system, $\phi(\mathbf{x}) = x_1 x_2 \cdots x_n$) and $\psi(u) = -u \log u$, or for $1 - h(\mathbf{p}) = (1 - p_1)(1 - p_2) \cdots (1 - p_n)$ (a parallel system, $1 - \phi(\mathbf{x}) = (1 - x_1)(1 - x_2) \cdots (1 - x_n)$) and $\psi(u) = -(1 - u) \log (1 - u)$.

3. Closed classes of life distributions. Now we do consider time and assume that each component in a system functions until some time at which it fails, and that subsequently it remains failed. We assume that the failure times of the components are given by independent, non-negative, and possibly infinite random variables $T_1, T_2, \cdots, T_n$. The reliability at time t of the ith component is $\bar{F}_i(t) = P\{T_i > t\}$. We let $X_i(t) = 1$ if $T_i > t$, $X_i(t) = 0$ if $T_i \leqq t$. If the system is coherent, it functions until some time of failure and then remains failed. We let T be the failure time of a coherent system with structure function ϕ. Then $T > t$ if, and only if, $\phi[\mathbf{X}(t)] = 1$. The reliability at time t of the system is

$$\bar{F}(t) = P\{T > t\} = P\{\phi[\mathbf{X}(t)] = 1\} = h[\bar{\mathbf{F}}(t)]. \tag{3.1}$$

Systems, as considered here, must be coherent to have the "once failed, stays failed" property for all possible independent distributions for $T_1, T_2, \cdots, T_n$ [5].

Suppose $\mathcal{A}$ is some class of survival probabilities. We define the *closure* $\mathcal{A}^{CS}$ *of* $\mathcal{A}$ *under the formation of coherent systems* to be the class of all survival probabilities $\bar{F}(t) = h[\bar{\mathbf{F}}(t)]$ where h is the reliability function of some coherent system and $\bar{F}_i \in \mathcal{A}$, $i = 1, \cdots, n$. Closure under coherent systems is a legitimate closure operation, i.e. $\mathcal{A} \subset \mathcal{A}^{CS}$ (since the system consisting of just one component is coherent), $\mathcal{A} \subset \mathcal{B}$ implies $\mathcal{A}^{CS} \subset \mathcal{B}^{CS}$, $(\mathcal{A}^{CS})^{CS} = \mathcal{A}^{CS}$ (since a coherent system whose components are in turn coherent systems is, after composition, a coherent system), and the closure of the empty set is empty. We define the *closure* $\mathcal{A}^{LD}$ *of* $\mathcal{A}$ *under limits in distribution* to be the class of survival probabilities which are, at their continuity points, the limit of a sequence of survival probabilities chosen from $\mathcal{A}$. Closure under limits in distribution is, of course, also a closure operation.

We write $\mathcal{A}^{LD,CS}$ for the closure of $\mathcal{A}^{LD}$ under coherent systems, and $\mathcal{A}^{CS,LD}$ for the closure of $\mathcal{A}^{CS}$ under limits in distribution.

THEOREM 3.1. *For any class of survival probabilities* $\mathcal{A}$,

$$\mathcal{A}^{LD,CS} \subset \mathcal{A}^{CS,LD}.$$

PROOF. If $\bar{F} \in \mathcal{A}^{LD,CS}$, $\bar{F}(t) = h[\bar{\mathbf{F}}(t)]$ for all $t \geqq 0$, where $\bar{F}_1, \bar{F}_2, \cdots, \bar{F}_n \in \mathcal{A}^{LD}$. For each $\bar{F}_i$ there is a sequence of survival probabilities $\bar{F}_i^{(k)} \in \mathcal{A}$ such that, at continuity points of $\bar{F}_i$, $\bar{F}_i^{(k)}(t) \to \bar{F}_i(t)$ as $k \to \infty$. The collection of discontinuity points of $\bar{F}_1$ or $\bar{F}_2$ or $\cdots$ or $\bar{F}_n$ being countable, the points of simultaneous continuity of $\bar{F}_1, \bar{F}_2, \cdots, \bar{F}_n$ are a dense set C in $[0, \infty)$. From the continuity of h, $\bar{F}^{(k)}(t) \to \bar{F}(t)$ on C where $\bar{F}^{(k)}(t) = h[\bar{\mathbf{F}}^{(k)}(t)] \in \mathcal{A}^{CS}$. Thus $\bar{F} \in \mathcal{A}^{CS,LD}$. ☐

If, for example, $\mathcal{A} = \{deg\}$, i.e. the class of all survival probabilities which are degenerate at some $t \in [0, +\infty]$, equality holds in Theorem 3.1, since $\{deg\}^{LD} = \{deg\}$ and $\{deg\}^{CS} = \{deg\}$. The inclusion can also be proper, e.g. for $\mathcal{A} = \{exp\}$, the class of all exponential survival probabilities. Using (1.1) it is easy to show that $\{exp\}^{LD} = \{exp\}$. Since every survival probability in $\{exp\}$ is continuous on $(0, +\infty)$, and coherent reliability functions are continuous, every survival probability in $\{exp\}^{LD,CS} = \{exp\}^{CS}$ is continuous on $(0, +\infty)$. With the exception of the survival probabilities degenerate at 0 and $+\infty$, no degenerate survival probability can be in $\{exp\}^{LD,CS}$. On the other hand

$$\{exp\}^{CS,LD} \supset \{deg\}. \tag{3.2}$$

One can show (3.2) for any class such as $\{exp\}$ which contains a continuous survival probability $\bar{F}_0$ with support $[0, +\infty)$ as follows. For a sequence of "k out of n" systems ($\phi_{k,n}(\mathbf{x}) = 1$ if $S_n(\mathbf{x}) = \sum_{i=1}^{n} x_i \geqq k$, $\phi_{k,n}(\mathbf{x}) = 0$ if $S_n(\mathbf{x}) < k$, $1 \leqq k \leqq n$) chosen so that $n \to \infty$ and $k/n \to \theta$, $0 \leqq \theta \leqq 1$; and for $\bar{F}_1 = \bar{F}_2 = \cdots = \bar{F}_n = \bar{F}_0$,

$$\begin{aligned} h_{k,n}[\bar{\mathbf{F}}(t)] = P\{S_n[\mathbf{X}(t)]/n \geqq k/n\} &\to 1 \quad \text{if} \quad t < \bar{F}_0^{-1}(\theta) \\ &\to 0 \quad \text{if} \quad t > \bar{F}_0^{-1}(\theta). \end{aligned}$$

4. Closure properties of the IHRA distributions. *A life distribution with survival probability* $\bar{F}$ *is* IHRA *if* $-\log \bar{F}(t)/t$ *is non-decreasing as t increases.* It is easily seen that this condition is equivalent to (1.2). It is also equivalent to say that $-\log \bar{F}$ is starshaped [4], i.e., $-\log \bar{F}(\alpha t) \leqq \alpha[-\log \bar{F}(t)]$, $0 \leqq \alpha \leqq 1$.

We write {IHRA} for the class of IHRA survival probabilities. It is immediate that

$$\{\text{IHRA}\}^{LD} = \{\text{IHRA}\} \tag{4.1}$$

from the preservation of the defining property under limits in distribution.

We write {*a.c.* IHRA} for the class of IHRA survival probabilities which are also absolutely continuous.

THEOREM 4.1. $\{a.c.\ \text{IHRA}\}^{CS} = \{a.c.\ \text{IHRA}\}$.

PROOF. We will show that $\{a.c.\ \text{IHRA}\}^{CS} \subset \{a.c.\ \text{IHRA}\}$. The reverse inclusion is automatic.

An absolutely continuous survival probability $\bar{F}$ is IHRA if, and only if, $-d\bar{F}(t)/dt \geqq -[\bar{F}(t) \log \bar{F}(t)]/t \geqq 0$, all $t \geqq 0$.

Let $\bar{F}_1, \bar{F}_2, \cdots, \bar{F}_n$ be a.c. IHRA and $\bar{F}(t) = h[\bar{\mathbf{F}}(t)]$, where h is the reliability function of a coherent system. Then, using Theorem 2.1,

$$\begin{aligned} -d\bar{F}(t)/dt &= \textstyle\sum_{i=1}^{n} \partial h[\bar{\mathbf{F}}(t)]/\partial \bar{F}_i \cdot [-d\bar{F}_i(t)/dt] \\ &\geqq \textstyle\sum_{i=1}^{n} \partial h[\bar{\mathbf{F}}(t)]/\partial \bar{F}_i \cdot [-\bar{F}_i(t) \log \bar{F}_i(t)]/t \\ &\geqq -[\bar{F}(t) \log \bar{F}(t)]/t. \end{aligned}$$

It follows that $\bar{F}$ is both absolutely continuous and IHRA. □

The following theorem is our principal result, that the IHRA class of survival probabilities is closed under the formation of coherent systems, and that the closure under coherent systems of the exponential class of survival probabilities is dense in the IHRA class with respect to limits in distribution.

THEOREM 4.2. $\{\text{IHRA}\}^{CS} = \{\text{IHRA}\} = \{exp\}^{CS,LD}$.

PROOF. We leave until last a proof of

$$\{\text{IHRA}\} \subset \{exp, deg\}^{CS,LD}, \tag{4.2}$$

where $\{exp, deg\} = \{exp\} \cup \{deg\}$. Given (4.2), the proof is accomplished by verifying

$$\begin{aligned} \{\text{IHRA}\}^{CS} &\subset \{exp, deg\}^{CS,LD,CS} \subset \{exp\}^{CS,LD} \\ &\subset \{a.c.\ \text{IHRA}\}^{CS,LD} = \{a.c.\ \text{IHRA}\}^{LD} \\ &\subset \{\text{IHRA}\}^{LD} = \{\text{IHRA}\}. \end{aligned}$$

The first inclusion follows from (4.2). The second inclusion from (3.2) and the application of Theorem 3.1. The third inclusion from $\{exp\} \subset \{a.c.\ \text{IHRA}\}$. The fourth equality from Theorem 4.1. The fifth inclusion from $\{a.c.\ \text{IHRA}\} \subset \{\text{IHRA}\}$. The last equality from (4.1).

It remains to prove (4.2). Excepting the survival probabilities degenerate at 0 and at $+\infty$, IHRA life distributions cannot concentrate any mass at either 0 or $+\infty$. The survival probabilities degenerate at 0 and at $+\infty$ are in both $\{exp\}$ and $\{deg\}$, so it suffices to show (4.2) for the IHRA survival probabilities $\bar{F}$ for which (a) $\bar{F}(0) = 1$ and (b) whatever $\epsilon > 0$, $\bar{F}(t_\epsilon) \leqq \epsilon$ for some finite t_ϵ. We complete the proof by showing that (i) *an* IHRA *survival probability* $\bar{F}$ *subject to* (a) *and* (b) *can be approximated uniformly from below by a survival probability* $\bar{H}$ *of the form*

$$\begin{aligned} \bar{H}(t) &= \exp[-(\lambda_1 + \lambda_2 + \cdots + \lambda_i)t] \quad \text{if} \quad t_{i-1} \leqq t < t_i, \quad i = 1, \cdots, n, \\ &= 0 \quad \text{if} \quad t_n \leqq t, \end{aligned} \tag{4.3}$$

where $0 = t_0 < t_1 < \cdots < t_n < +\infty$ *and* $0 < \lambda_i < +\infty$, and (ii) piecewise exponential survival probabilities of the form (4.3) are in the class $\{exp, deg\}^{CS}$.

The following construction indicates how (i) may be verified. Its essential

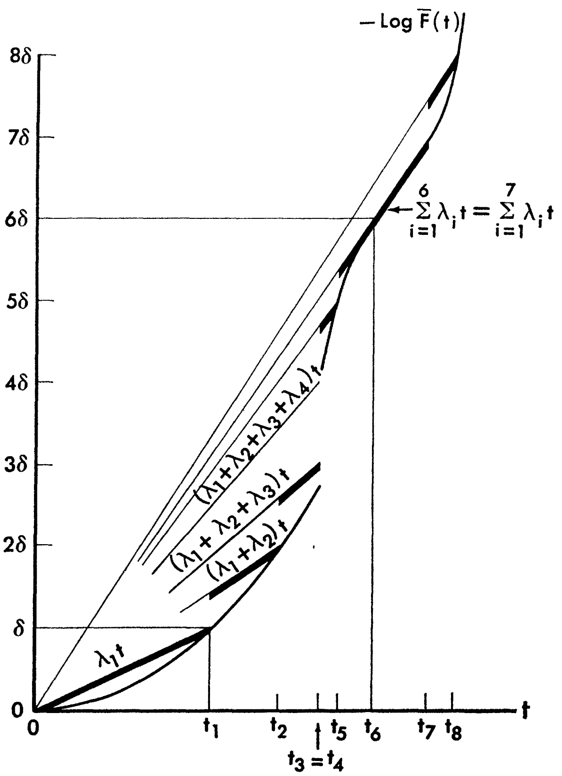

FIG. 2. Piecewise exponential approximation of an IHRA survival probability $\bar{F}$.

details are illustrated in Figure 2. It is convenient, and harmless, to substitute the restrictions $0 < t_1 \leqq t_2 \leqq \cdots \leqq t_n < +\infty$ and $0 < \sum_{i=1}^{k} \lambda_i \leqq \sum_{i=1}^{k+1} \lambda_i < +\infty$, $k = 1, \cdots, n-1$, for those stated in (4.3). We choose some $\epsilon > 0$ and let $\delta = \log(1+\epsilon)$. We determine t_i by $-\log \bar{F}(t_i-) \leqq i\delta \leqq -\log \bar{F}(t_i+)$. That $0 < t_1$ follows from (a). That $t_1 \leqq t_2 \leqq \cdots$ from $-\log \bar{F}$ being non-decreasing. From (b) we can find an n such that $t_\epsilon \leqq t_n < +\infty$. Then from (4.3) $0 \leqq \bar{F}(t) - \bar{H}(t) \leqq \epsilon$ for $t \geqq t_n$. We define $\sum_{i=1}^{k} \lambda_i = k\delta/t_k$. That $0 < \lambda_1$ follows from $t_1 \leqq t_n < +\infty$. That $\sum_{i=1}^{k} \lambda_i \leqq \sum_{i=1}^{k+1} \lambda_i$ from $-\log \bar{F}(t)/t$ being non-decreasing. That $\sum_{i=1}^{n} \lambda_i < +\infty$ from $0 < t_1 \leqq t_n$. From (4.3) $0 \leqq [-\log \bar{H}(t)] - [-\log \bar{F}(t)] \leqq \delta$ for $0 \leqq t < t_n$, and consequently $0 \leqq \bar{F}(t) - \bar{H}(t) \leqq \epsilon$ on $[0, t_n)$.

To show (ii) we consider the coherent system $\phi(e_1, \cdots, e_n, d_1, \cdots, d_n) = e_1(e_2 \vee d_1)(e_3 \vee d_2) \cdots (e_n \vee d_{n-1})\, d_n$, where $e \vee d = e + d - ed$. A network analogue of this system is shown in Figure 3, in which each component is a closed switch when functioning, an open switch when failed, and the system functions when a "signal" can pass through the network. We assume that the e_i are the performance indicators of components with exponential survival probabilities

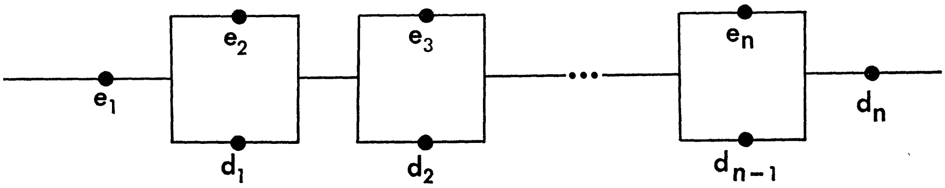

FIG. 3. $\phi(e_1, \cdots, e_n, d_1, \cdots, d_n) = e_1(e_2 \vee d_1)(e_3 \vee d_2) \cdots (e_n \vee d_{n-1})\, d_n$

$\bar{E}_i$ having hazard rates λ_i, and that the d_i are the performance indicators of components with survival probabilities $\bar{D}_i$ degenerate at t_i. The survival probability of the system is

$$\bar{H}(t) = \bar{E}_1(t)[\bar{E}_2(t) \vee \bar{D}_1(t)][\bar{E}_3(t) \vee \bar{D}_2(t)] \cdots [\bar{E}_n(t) \vee \bar{D}_{n-1}(t)]\bar{D}_n(t)$$

which reduces to (4.3). □

5. Remarks. If $T_1, T_2, \cdots, T_n$ are the failure times of a set of n components, the corresponding order statistics $T^{(1)} \leqq T^{(2)} \leqq \cdots \leqq T^{(n)}$ are respectively the failure times of the "k out of n" systems, $k = 1, \cdots, n$, built from the components. It is immediate from Theorem 4.2 that $T_1, T_2, \cdots, T_n$ independent and IHRA implies that $T^{(1)}, T^{(2)}, \cdots, T^{(n)}$ are IHRA. In [7] it is observed that $T_1, T_2, \cdots, T_n$ independent, identically distributed, and IHR implies that $T^{(1)}, T^{(2)}, \cdots, T^{(n)}$ are IHR.

The inequality (2.3) has other applications to reliability theory. One of these is indicated in [3].

We have shown that if a coherent system has components with independent exponentially distributed lives $T_1, \cdots, T_n$, then the system life is IHRA. If instead, $T_1, \cdots, T_n$ are possibly dependent but multivariate exponential as defined in [8], then again the system is IHRA. This follows directly from our result and the fact that if $T_1, \cdots, T_n$ is multivariate exponential, then each T_i is the minimum over some subset of a set of independent exponential random variables.

REFERENCES

[1] BARLOW, R. E. and PROSCHAN, F. (1965). *Mathematical Theory of Reliability.* Wiley, New York.

[2] BIRNBAUM, Z. W., ESARY, J. D. and SAUNDERS, S. C. (1961). Multi-component systems and structures and their reliability. *Technometrics* **3** 55–77.

[3] BIRNBAUM, Z. W. and ESARY, J. D. (1966). Some inequalities for reliability functions. *Proc. Fifth Berkeley Symp. Math. Statist. Prob.* Univ. of California Press.

[4] BRUCKNER, A. M. and OSTROW, E. (1962). Some function classes related to the class of convex functions. *Pacific J. Math.* **12** 1203–1215.

[5] ESARY, J. D. and MARSHALL, A. W. (1964). System structure and the existence of a system life. *Technometrics* **6** 459–462.

[6] ESARY, J. D. and PROSCHAN, F. (1963). Coherent structures of non-identical components. *Technometrics* **5** 191–209.

[7] Esary, J. D. and Proschan, F. (1963). Relationship between system failure rate and component failure rates. *Technometrics* **5** 183–189.

[8] Marshall, A. W. and Olkin, I. 1966. A Multivariate Exponential Distribution. Boeing Scientific Research Laboratories Document, D1-82-0505.

[9] Mine, Hisashi (1959). Reliability of Physical System. *IEEE Trans. Information Theory* IT-S, 138–150.

[10] Moore, E. F. and Shannon, C. E. (1956). Reliable circuits using less reliable relays. *J. Franklin Inst.* **262** 191–208, 281–297.