ABSTRACTS OF PAPERS

(Abstracts of papers not connected with any meeting of the Institute.)

1. The number of linearly inducible orderings of points in Euclidean n-space. THOMAS M. COVER, Stanford University.

A collection of k points in E^n is ordered by orthogonal projection onto a freely chosen reference vector $w \varepsilon E^n$. If σ is a permutation of the integers $\{1, 2, \cdots, k\}$, $w \varepsilon E^d$ induces the ordering σ if $w \cdot x_{\sigma(1)} > w \cdot x_{\sigma(2)} > \cdots > w \cdot x_{\sigma(k)}$; and σ is said to be linearly inducible if there exists such a w. In this paper it is demonstrated that there are precisely $Q(k, n)$ linearly inducible orderings of k points in general position in E^n, where $Q(k, n) = 2\sum_{j=0}^{n-1} {}_kS_j$ and ${}_kS_j$ is the sum of products of numbers taken j at a time without repetition from the set $\{2, 3, \cdots, k-1\}$. Thus $Q(k, n)$ is the number of ways an art judge may rank k paintings, each having n numerical attributes, by forming weighted averages of the attributes. (Received 20 June 1966.)

2. Statistical inference. V. P. GODAMBE, Johns Hopkins University.

Fisherian statistics and for that reason most of the current mathematical statistics deals with *hypothetical* populations having no real existance at all. This hypothetical character of the populations has been emphasized above all by Fisher himself, on several occasions. On the other hand statisticans often have to deal with *real* populations, i.e. the populations consisting of a large number of *identified* individuals each having some variate value. Clearly, only for such real populations can one use just a random number table to draw a random sample. Sample-survey populations serve a good illustration. The first indication, that for such real populations of identified individuals, the results of current mathematical statistics are inadequate, was provided by the author's [*J. Roy. Statist. Soc.* (1955)] demonstration of the non-existance of the UMV estimation for such populations, *regardless* of the distribution of the variate values. This point was further emphasized by the author subsequently [*Sankhyā* (1960), *Rev. Inter. Statist. Inst.* (1965), *Ann. Math. Statist.* (1965), *J. Roy. Statist. Soc.* (1966)] several times. Now it is shown that the very basic concepts of mathematical statistics such as "significance level" or "power", need complete revision before they can be applied to real populations of identified individuals. For instance a test is constructed for the mean of a population, which has neither "significance level" or "power" and yet looks reasonable enough in terms of some prior knowledge. In other words just the frequency function of the variate in a real population of identified individuals does *not* determine the test criteria for the unknown parameter of the frequency function. (Received 13 June 1966.)

3. Bayesian sufficiency in survey-sampling. V. P. GODAMBE, Johns Hopkins University.

The main constituents of Bayesian inference, in order of their importance are as follows: (1) Bayes theorem of inverse probability. (2) A specific prior distribution on the parameter space. (3) A loss function. Now for several problems of inference loss function does not at all exist or is only very vaguely defined. Next often the prior knowledge about the parameter in (2) can at best be characterized by a *class* of prior distributions and not by any specific prior distribution. Hence the following relaxation of (2) and (3) above is suggested: Principle of Baysian Sufficiency. "*If* Ω is the class of prior distributions of the parameter characterising our prior knowledge and $t(x)$ is a statistic (defined for all possible samples x), such that the posterior distribution of the parameter, given the sample x, for every prior in Ω, depends on x only through t, *then* any inference about the parameter should depend on the sample only through the statistic t." In survey-sampling, under the conditions when simple random sampling is appropriate, Ω can be assumed to be the class of all distributions ξ, such that the variate values associated with different units of the survey-population, when jointly distributed as ξ are probabilistically independent. It is then shown that the Principle of Baysian Sufficiency implies that all inference about the population mean, must depend exclusively on *sample mean* and the *units in the sample.* That is the variate values in the sample should be used only through their mean, for any inference about the population mean. This provides some justification for the use of ratio and regression estimates which are functions of sample mean. (Received 17 June 1966.)

4. A table of sums of discrete right triangular random variables (or, alternatively, of a measure of rank differences between two particular objects). HANS K. URY, California State Department of Public Health and University of California, Berkeley.

The table lists the coefficients of x^D in the expansion of $S = (nx + [n-1]x^2 + \cdots + 2x^{n-1} + x^n)^m$, for $D = m, m+1, \cdots, mn$. Treating these coefficients as frequencies, it also gives cumulative probabilities over D for fixed m, n to 3 decimals. Range of the table: $[n=]2$, $[m \leqq]24$; 3, 16; 4, 12; 5, 11; 6–7, 10; 8–9, 9; 10–11, 8; 12–13, 7; 14–15, 6; 16–19, 5; 20–24, 4; 25–29, 3. Since $(mn)^{-1}S$ is the pgf of a sum $S_{m,n}$ of m iid variates X_i with $\Pr(X_i = j) = p_j = (n+1-j)/[(n+1)n/2]$, $j = 1, 2, \cdots, n$, the coefficients of x^D thus give the frequencies with which a sum of m independent discrete right triangular random variables of this type takes on the values D. The table is applicable to the following m-rankings situation: $n+1$ objects are ranked m times (mutually independent rankings). One is interested in the sum, over the m rankings, of the absolute rank difference between two particular objects. (The case of *signed* rank differences was treated by Whitfield [*British J. Statist. Psych.* **7** (1954) 45–49] and Stuart [ibid., pp. 50–51].) Then for a single ranking, the absolute rank difference between the two objects is distributed according to p_j under the null hypothesis of equality of all $n+1$ objects. (Received 9 June 1966.)

(Abstracts of papers presented at the European Regional meeting, London, England, September 5–10, 1966. Additional abstracts appeared earlier issues.)

6. On partial sufficiency and partial ancillarity. ERLING BERNHARD ANDERSEN, Copenhagen School of Economics.

The paper contains a discussion of some implications in probability theory of the use of conditional procedures in test theory. Given a family of probability measures Π indexed by a product parameterspace $\Theta \times E$. A σ-field $\mathcal{B}$ is called partially sufficient for ϵ if $\mathcal{B}$ is sufficient for the subfamily $\Pi_\theta = \{P_{\theta,\epsilon} \mid \epsilon \varepsilon E\}$ for all $\theta \varepsilon \Theta$. It is proved that if Π is dominated and the P's have constant carrier then the compound σ-field $\mathcal{B}_1\mathcal{B}_2$ is sufficient for (θ, ϵ) if $\mathcal{B}_1$ is part suff for ϵ and $\mathcal{B}_2$ is part suff for θ. The theorem remains true if formulated in terms of minimal sufficiency. In general there exists no minimal part suff σ-field for ϵ. In fact a construction of a min part suff σ-field leads to a class $\{\mathcal{B}_\theta\}$, each $\mathcal{B}_\theta$ min suff for the corresponding Π_θ. As a consequence a definition of partial sufficiency in terms of classes of σ-fields is proposed. A definition of partial ancillary σ-fields similar to the definition of part suff is given and some generalizations of a theorem due to Basu [*Sankhyā* **15** 377–380] concerning the relationship between part suff, part anc and independency of

σ-fields is proved. As an application of the developed theory the two-dimensional analysis of variance and related models is considered. (Received 5 July 1966.)

7. On the similarity of factor matrices. FRIEDRICH GEBHARDT, Deutsches Rechenzentrum.

Consider two $n \times k$ matrices A and B, $n > k$. The similarity of A and B is defined by $R = \max \operatorname{tr}(A'BL)/(\operatorname{tr}(A'A) \operatorname{tr}(B'B))^{\frac{1}{2}}$, the maximum being taken with respect to all orthogonal $k \times k$ matrices L. Let all elements of A and B be normally distributed with expectations 0 and equal variances. This assumption shall reflect a notion of "no similarity between factor matrices" and may not be too realistic. By using a special orthogonal matrix, one finds $E(R) \geqq k/2n$; it is also shown that $E(R) \leqq (k/n)^{\frac{1}{2}}$. Monte Carlo computations suggest that the upper limit is closer to the real value than the lower one yielding $E(R) \approx 0.82(k/n)^{\frac{1}{2}}$ and var $(R) \approx 0.35/n \cdot k$ for $4 \leqq k \leqq 15$, $5 \leqq n/k \leqq 10$. (Received 5 July 1966.)

8. Some efficiency comparisons for normal location problems. J. L. HODGES, JR., and E. L. LEHMANN, University of California, Berkeley.

Asymptotic efficiency comparisons are obtained (i) between one-sided, symmetric two-sided and asymmetric two-sided t-tests; (ii) between Student's t-test and the corresponding normal test suitable when the variance is known. In the latter case corrections of order $1/n$ to the asymptotic power functions are used to obtain a closer approximation than the Pitman efficiency. It is shown that the difference of the sample sizes required to match the corrected power functions tends to a constant, the *asymptotic difference efficiency* (ADE), and that for the usual significance levels the efficiency loss is on the order of 1 to 3 observations. The ADE is also obtained for Scheffé's Behrens-Fisher test relative to the t-test for equal variances. Efficiency in terms of the number of observations required to minimize the maximum difference between the power functions being compared is used to obtain an alternative approach to Pitman efficiency. (Received 12 July 1966.)

9. Randomized rules for the two-armed bandit with finite memory. S. M. SAMUELS, Purdue University.

We are given two coins with unknown probabilities, p_1 and p_2, of heads. At each stage, based only on the results of the previous r tosses we must decide which coin to toss next. Our goal is to find the rule which maximizes the limiting proportion of heads. Successively better non-randomized rules have been proposed by Robbins [*Proc. Nat. Acad. Sci* **42** 920–933], Isbell [*Ann. Math. Statist.* **30** 606–610], and Smith and Pyke [*Ann. Math. Statist.* **36** 1375–1386]. There is, however, a randomized rule which is much better than these. In particular, unlike the earlier rules it has the property that as the smaller of the p's goes to zero, the proportion of tosses with the better coin diverges. (Received 12 July 1966.)

10. The joint assessment of normality of several independent samples. S. S. SHAPIRO and M. B. WILK, General Electric Co. and Bell Telephone Laboratories.

Statistical methods are presented for the joint assessment of the supposed normality of a collection of independent (small) samples, which may derive from populations having differing means and variances. The procedures are based on the use of the W statistic [S. S. Shapiro and M. B. Wilk (1965). An Analysis of Variance Test for Normality. *Biometrika* **52**

591–611.] as a measure of departure from normality. Two modes of combination of a collection of W statistics are considered, namely the standardized mean of the normal transforms of W and the sum of the $\chi^2(2)$ transforms of the significance levels of W; and these are proposed for use in conjunction with the probability plotting of the collection of these transforms. Tables and formulae for practical implementation are provided. Some summary empirical sampling results are given on the comparative sensitivities of these procedures, along with detailed consideration of several specific examples which illustrate the additional informative value of probability plotting. The proposed techniques appear to have substantial data analysis value as adjuncts to other statistical methodology. (Received 14 July 1966.)

11. Statistical techniques for effective condensation of talker identification data. M. B. WILK and R. GNANADESIKAN, Bell Telephone Laboratories.

The short-time spectral decomposition of the waveform of speech, yielding a two-way (Frequency $\times$ Time) classification of energies, may be used as a basis for identifying talkers. The overall problem involves data collection, condensation, specification of discrimination spaces and associated metrics, and development of practical strategies for classification. The current paper presents statistical analyses and methods for preliminary condensation of the basic energy data which involves approximately 15,000 observations per utterance, classified by 57 frequency and 275 time channels. The condensation procedures should take into account the arbitrariness, due to experimental artefacts, of both the time origin and the overall level of intensity in different utterances. Analysis of variance methods, including various plotting techniques, are applied to the two-way table of energies to study nonadditivities and variance heterogeneity. A transformation yielding a more nearly additive structure and diminished variance heterogeneity is estimated, leading to an effective summarization of the 57×275 table by $(57 + 275)$ marginal effects. The latter are completely determined by the two "cumulative functions" obtained from exponential transforms of their values. These cumulated functions are themselves effectively summarized by a set of interpolated "quantiles", far fewer in number than the $(57 + 275)$ effects. Moreover, these quantiles are invariant under multiplicative utterance effects and contrasts amongst them eliminate the arbitrary time origin. (Received 12 July 1966.)

(Abstracts of papers presented at the Western Regional meeting, Los Angeles, California, August 15–17, 1966. Additional abstracts appeared in earlier issues.)

18. On the solution of Bechhofer's general goal in the indifference zone formulation of the ranking and selection problem (preliminary report). DAVID R. BARR, Aerospace Research Laboratories, Wright-Patterson AFB.

In 1954, Bechhofer proposed the following general goal for the indifference zone formulation of the ranking and selection problem: to partition k distributions into s categories, containing the k_s "best", the k_{s-1} "second best", $\cdots$, and the k_1 "worst" distributions, respectively, "bestness" being in terms of magnitude of a parameter. A solution is found for this general goal for a scale parameter family with an absolutely continuous distribution function by assuming a preference zone of the form $\theta_{[1]} \leqq \cdots \leqq \theta_{[k_1]} \leqq \psi_1(\theta_{[k+1]}) < \theta_{[k+1]}$; $\theta_{[k_j+1]} \leqq \cdots \leqq \theta_{[k_{j+1}]} \leqq \psi_{j+1}(\theta_{[k_{j+1}]}) < \theta_{[k_{j+1}]}$, $j = 1, \cdots, s-2$; $\psi_{s-1}(\theta_{[k_{s-1}+1]}) < \theta_{[k_{s-1}+1]} \leqq \cdots \leqq \theta_{[k_s]}$, and selecting that preference zone of this form in which the infimum of the probability of a correct selection over the preference zone is easiest to compute. This preference zone has $\psi_r(\theta) = \rho_r\theta$, $r = 1, \cdots, s-1$, $\rho_r < 1$. Extension of the results to a location parameter family with an absolutely continuous distribution function is immediate. (Received 20 June 1966.)

19. Some selection and ranking procedures for multivariate normal populations. M. R. GNANADESIKAN, Bell Telephone Laboratories.

In the present paper, some selection procedures following the approach of S. S. Gupta [University of North Carolina, Inst. of Stat. Mimeo No. 150, (1956)] are proposed for multivariate normal populations with respect both to location and to dispersion characteristics. Firstly, given a single multivariate normal population, with known or unknown covariance matrix and unknown mean vector, procedures are discussed for selecting a subset of the components of the population to include (with at least a prespecified probability) the component having the largest population mean. An application of these procedures in multiple regression is described. Secondly, for a set of k multivariate normal populations, a procedure is proposed for selecting a subset of these populations which would include the "uniformly best" population (i.e., each component mean is largest). Thirdly, a selection procedure is defined in terms of generalized variances for selecting a subset of the populations which would include the population with the smallest (largest) generalized variance. The associated problem of the distribution of the generalized variance and approximations to it are also studied. Finally, the problem, including the definition of appropriate criteria, of statistically assessing the performance of the procedures is studied. Some generalizations of the above procedures are also proposed. (Received 13 June 1966.)

20. Further consideration of the distribution of the multiple correlation coefficient. JOHN GURLAND, University of Wisconsin.

The form of the probability density of the multiple correlation coefficient based on a sample from a multivariate normal population was expressed by R. A. Fisher (1928) in terms of a hypergeometric function. For $N - p$ an even integer, where N is the sample size and p the number of components in the random vector, Fisher expressed the probability integral in terms of a finite series, but the number of terms is too large for this to be of great practical value. In the present article, the probability density is expressed in terms of a confluent hypergeometric function which is considerably simpler than Fisher's expression based on the hypergeometric function. Further, when $N - p$ is even the probability integral is expressible as a finite series with only $(N - p)/2 + 1$ terms, and the coefficients are interpretable as binomial probabilities. Finally, an approximation is presented for the general distribution of the multiple correlation coefficient which for the particular case $p = 2$ is evidently more accurate than Fisher's transform over a large set of values of the parameters involved. (Received 22 June 1966.)

21. On bias in variance component estimation. DAVID A. HARVILLE, Aerospace Research Laboratories, Wright-Patterson AFB.

This paper deals with certain aspects of variance component estimation for the unbalanced one-way random classification where the number (N) of observations per class is treated as a random variable not necessarily independent of the class effect (A). It is assumed that in general $P\{N = 0\} > 0$. A general expression is derived for the expected value of that estimator of the between variance component yielded by analysis of variance of class means. The expectation of the estimator is a function of the between variance component (σ_a^2), the number (I) of classes, $P\{N = 0\}$, $P\{N = 1\}$, $E\{A \mid N = 0\}$, $E\{A \mid N = 1\}$, $E\{A^2 \mid N = 0\}$, and $E\{A^2 \mid N = 1\}$. The limit (as $I \to \infty$) of the expected value of the estimator is obtained. Sufficient conditions for this limit to be less than σ_a^2 are that (1) A be symmetrically distributed about zero and either (2i) $P\{N = 0 \mid A = a\}$

$+ P\{N = 0 \mid A = -a\}$ be a strictly increasing function of a for $0 < a < \infty$ or (2ii) $P\{N = 0 \mid A = a\}$ be $\leqq$ or $\geqq$ $P\{N = 0 \mid A = -a\}$ for $0 < a < \infty$ with strict inequality holding for a subinterval of nonzero measure and $P\{N = 0 \mid A = a\}$ $+ P\{N = 0 \mid A = -a\}$ be a monotonic increasing function of a for $0 < a < \infty$. Conditions (2i) and (2ii) are both satisfied when $P\{N = n \mid A = a\}$ is given by a Poisson distribution with parameter ce^{ka} where c and k are constants, $c \geqq 1$. (Received 21 June 1966.)

22. Factorization of probability measures on locally compact groups. HERBERT K. HEYER, University of Washington.

Let $\mathfrak{M}$ be the set of all probability measures (probabilities) on a locally compact group G having a countable base for its topology. $\mathfrak{M}$ can be made a topological semigroup, if convolution is introduced as operation; the topology being the vague topology of measures $(\mathfrak{J}_v)$ (1) $\mu \varepsilon \mathfrak{M}$ is *decomposable*, if there exist two non degenerate (nd) measures ν, $\lambda \varepsilon \mathfrak{M}$ such that $\mu = \nu * \lambda$. Measures in $\mathfrak{M}$ which are not decomposable are called *prime*. Let $\mathcal{P}$ denote the set of all prime probabilities in $\mathfrak{M}$. Along the lines of the work of Parthasarathy, Rao and Varadhan [*Trans. Amer. Math. Soc.* **12** (1961)] as well as Vorobyov [*Mat. Sborn.* **34** (1954)] prime probabilities are constructed and the category of $\mathcal{P}$ is determined (using in part a method suggested by Choquet's representation theorem for compact convex subsets of locally convex topological vector spaces). (2) $\mu \varepsilon \mathfrak{M}$ is called *weakly right decomposable*, if there exists a sequence $(\mu_n)_{n \geqq 1}$ of nd measures in $\mathfrak{M}$ such that $\mu = \lim_{n\to\infty} \mu_1 * \cdots * \mu_n$, where the limit is meant in the sense of the topology $\mathfrak{J}_v$. $\mu \varepsilon \mathfrak{M}$ is said to be *weakly divisible*, if there exists an nd measure $\lambda \varepsilon \mathfrak{M}$ such that $\mu = \lim_{n\to\infty} \lambda^{*n}$. Using a theorem of the author's on the characterization of idempotents in $\mathfrak{M}$ it is possible to extend results of Stromberg [*Trans. Amer. Math. Soc.* **94** (1960)] from compact to arbitrary locally compact groups: (i) necessary conditions for $\mu \varepsilon \mathfrak{M}$ to be weakly right decomposable; (ii) sufficient conditions for $\mu \varepsilon \mathfrak{M}$ to be weakly right decomposable; (iii) necessary and sufficient conditions for $\mu \varepsilon \mathfrak{M}$ to be weakly divisible. (Received 24 June 1966.)

23. On degradation of combination locks and the maximum trial time to open them. MARY D. LUM, Aerospace Research Laboratories, Wright-Patterson AFB.

This paper describes the comparative performance of combination locks (in five states ranging from excellent to poor) using the maximum trial time ("trial and error" with no repeats) to arrive at the correct combination, thus opening the lock. The 3-number combination of a $10p$-point (p is a positive integer) lock is randomly chosen subject only to the possibility of one of four increasingly severe restrictions (including unrestricted randomization). Two kinds of trial times are investigated: "L-time" (each 3-number combination is dialed completely) and "S-time" (all possible third combination numbers are tried before changing the first two combination numbers). The order in which the combinations are tried has no effect on the uniform distribution of L-time or of S-time with balanced restrictions. However, for unbalanced restrictions, combination ordering affects S-time; the optimal (minimum expected S-time) ordering(s) occurs when the rate-of-change of the first two combination numbers is made monotone nondecreasing. Since the maximum S-time does not depend on combination ordering, it yields simple calculations of the maximum L-time, expected L-time, expected S-time (balanced case), and an upper bound for the (minimum) expected S-time under optimal ordering (unbalanced case). A numerical example is given for the 100-point ($p = 10$) lock. (Received 24 June 1966.)

24. One-order statistic conditional estimators of the shape parameters of the limited and Pareto distributions and the scale parameters of the Type II asymptotic distribution of smallest values. Albert H. Moore and H. Leon Harter, Air Force Institute of Technology and Aerospace Research Laboratories, Wright-Patterson AFB.

One-order-statistic estimators are derived for the shape parameter K of the limited distribution function $F_1(x, \omega, K) = 1 - (\omega - x)^K$ and the Pareto distribution function $F_2(y, \epsilon, K) = 1 - (y - \epsilon)^{-K}$, given the location parameters ω and ϵ, respectively. A similar estimator is derived for the scale parameter v of the Type II asymptotic distribution of smallest values, $F_3(z, v, K) = 1 - \exp[-(z/v)^{-K}]$, given the shape parameter K. The one-order-statistic estimators are $\tilde{K} \mid \omega = -1/c_{mn} \ln(\omega - x_{mn})$ for the limited distribution, $\tilde{K} \mid \epsilon = 1/c_{mn} \ln(y_{mn} - \epsilon)$ for the Pareto distribution, and $\tilde{v} \mid K = c_{mn}^{-1/K} z_{mn}$ for the Type II asymptotic distribution of smallest values, where x_{mn}, y_{mn}, and z_{mn} are the mth order statistics of samples of size n from the respective distributions and c_{mn} is the coefficient for a one-order-statistic estimator of the scale parameter of an exponential distribution, which has been tabled in an earlier paper. It is shown that exact confidence bounds can be easily derived for the above parameters, using exact confidence bounds for the scale parameter of the exponential distribution. Use of the above estimators is illustrated by numerical examples. (Received 24 June 1966.)

25. Conditional maximum-likelihood estimation, from singly censored samples, of the shape parameters of the limited and Pareto distributions and the scale parameter of the Type II asymptotic distribution of smallest values. Albert H. Moore and H. Leon Harter, Air Force Institute of Technology and Aerospace Research Laboratories, Wright-Patterson AFB.

Use of the functional relationships between the exponential and the limited, Pareto, and Type II extreme-value distributions enables one to obtain conditional maximum-likelihood estimators, from singly censored samples, of the shape parameters of the limited distribution, $F_1(x, \omega, K) = 1 - (\omega - x)^K$, and the Pareto distribution, $F_2(y, \epsilon, K) = 1 - (y - \epsilon)^{-K}$, and the scale parameter of the Type II asymptotic distribution of smallest values, $F_3(z, v, K) = 1 - \exp[-(z/v)^{-K}]$, by a simple transformation of the corresponding estimator, based on the first m order statistics of a sample of size n, of the scale parameter θ of the exponential distribution. Use is made of the fact that $\hat{K}_{mn} \mid \omega = 1/\hat{\theta}_{mn}$, $\hat{K}_{mn} \mid \epsilon = 1/\hat{\theta}_{mn}$, and $\hat{v}_{mn} \mid K = -\hat{\theta}_{mn}^{-1/K}$, where $2m\hat{\theta}_{mn}/\theta$ has the chi-square distribution with $2m$ degrees of freedom, to set confidence bounds on the shape parameter K of the limited and Pareto distributions and the scale parameter v of the Type II asymptotic distribution of smallest values. The probability densities of $\hat{K}_{mn} \mid \omega$, $\hat{K}_{mn} \mid \epsilon$, and $\hat{v}_{mn} \mid K$, which for given m are the same for any $n \geqq m$, are obtained by a simple transformation of $\hat{\theta}_{mn}$. (Received 24 June 1966.)

26. Bayesian confidence limits for the reliability of cascaded exponential subsystems. Melvin D. Springer and William E. Thompson, General Motors Corporation Defense Research Laboratories.

The problem treated here is that of deriving exact Bayesian confidence intervals for the reliability of a cascade system consisting of independent subsystems whose failure probabilities are estimated from life test data. The posterior probability density function of the reliability of N independent cascaded exponential subsystems is derived in closed form, using the method of the Mellin integral transform. The posterior distribution function is

obtained, which yields Bayesian confidence limits on the total system reliability. These results, which are believed to be new for $N > 3$, have an immediate application to problems of reliability evaluation and test planning. (Received 27 June 1966.)

27. Some rules for a combinatorial method for multiple products of generalized k-statistics. DERRICK S. TRACY, University of Windsor.

Dwyer and Tracy [*Ann. Math. Statist.* **35** (1964) 1174–1185] gave some rules for pattern functions useful in obtaining formulae for products of two generalized k-statistics in terms of linear combinations of such statistics. All these can be generalized to give rules for pattern functions when taking products of more than two generalized k-statistics. This paper indicates the generalization of the four rules of Dwyer and Tracy and gives four further rules, some of which involve interesting algebraic identities. (Received 27 June 1966.)

(Abstracts of papers presented at the Annual meeting, New Brunswick, New Jersey, August 30–September 2, 1966. Additional abstracts appeared in earlier issues.)

10. The distribution of the smallest sample spacing (preliminary report). LEE R. ABRAMSON, Columbia University Electronics Research Laboratories.

Let $X_1, \cdots, X_n$ be independent random variables with a common distribution function $F(x)$ and let $Y_1 \leqq \cdots \leqq Y_n$ be their ordered values. Then $U_n = \min\{Y_{i+1} - Y_i, i = 1, \cdots, n-1\}$ is the smallest sample spacing, with distribution function $G_n(u)$. (If $X_1, \cdots, X_n$ are the arrival times of n customers at a single-server counter with a fixed service time of τ, then $1 - G_n(\tau)$ is the probability that no customer waits for service.) Suppose that $F(x)$ is absolutely continuous with density $f(x)$ and $\phi = \int f^2(x)\,dx$ is finite. Let $\theta = n(n-1)u$ be fixed. Then $G_n(u) = 1 - e^{-\theta\phi} + O(n^{-1})$. (The error term vanishes if F is the exponential distribution.) This asymptotic result is a generalization of one obtained by Lionel Weiss (*Ann. Math. Statist.* **30** 590–593). (Received 5 July 1966.)

11. Likelihood ratio test for equal correlation. MURRAY A. AITKIN and W. C. NELSON, University of North Carolina and Virginia Polytechnic Institute.

Tests for the equality of the correlations in a p-variate normal distribution have been proposed by Anderson (*Ann. Math. Statist.* **34** 122–148) and Lawley (*Ann. Math. Statist* **34** 149–151). These tests depend on the equality of $p - 1$ characteristic roots of the correlation matrix. The likelihood ratio test for equality is somewhat complicated and difficult to evaluate; however, a simple approximation to it is easily evaluated, has the same asymptotic χ^2 distribution, and is slightly more powerful than the above tests. (Received 5 July 1966.)

12. Minimax unbiased estimator of mixing distribution for finite mixtures. DUANE C. BOES, Colorado State University.

Let $\mathcal{H} = \{H_\theta(x):H_\theta(x) = \sum_{i=1}^{k+1} \theta_i F_i(x), \theta_i > 0, i = 1, \cdots, k+1, \sum_{i=1}^{k+1} \theta_i = 1\}$ be the family of finite mixtures of any fixed set of $k + 1$ (distinct) distribution functions $F_1, \cdots, F_{k+1}$. The minimax unbiased estimator of the parameter $\theta = (\theta_1, \cdots, \theta_k)$ for identifiable families $\mathcal{H}$ is derived. An estimator is called minimax unbiased if it is unbiased and if it is minimax (minimizes the maximum risk) within the family of all unbiased estimators. $\sum_{i=1}^{k} a_i \text{ Var } (\hat{\theta}_i)$, $a_i > 0$, was used as the risk function. Here, the weighted sum (weighted with respect to the a_i's) of the diagonal elements of the inverse of the information matrix is a concave function that provides a greatest lower bound for the risk function of unbiased estimators; and, the unbiased estimator that is best (smallest risk) at the point where this greatest lower bound for the risk is maximal is minimax unbiased. (Received 8 July 1966.)

13. Strongly consistent estimator for mixtures of distribution functions.
KEEWHAN CHOI, Cornell University.

Let $\{F(x, \theta); \theta \in \Theta\}$ be a known family of distribution functions and G an (unknown) probability measure on Θ. Assume $F(x, \theta)$ is continuous and strictly increasing in x for each θ and continuous in θ for each x. The correspondence between G and $P_G(\cdot) = \int F(\cdot, \theta)\, dG(\theta)$ is assumed to be one-to-one. The problem is to estimate G from n independent observations on P_G. Denoting the empirical distribution by F_n, we prove that $G^*_{(n)}$ which minimizes $\int [P_{G_{(n)}}(x) - P_G(x)]^2\, dF_n(x)$ is a strongly consistent estimator of G, i.e. $G^*_{(n)}$ converges weakly to G with probability one. We show that when Θ consists of a finite number of points, $G^*_{(n)}$ is asymptotically normal (multivariate) and give its covariance matrix. It turns out that the minimization for $G^*_{(n)}$ has to be carried out only among all those discrete probability measures which put all their mass at $(n + 1)$ or less points in Θ. (Received 13 June 1966.)

14. On a class of matrices arising in the study of Markov renewal processes.
ERHAN ÇINLAR, Northwestern University.

A square matrix $A(s)$ of elements $A_{ij}(s)$ is called a semi-Markov matrix if $A_{ij}(s)$ is a completely monotonic function for $0 \leqq s < \infty$, $(i, j = 1, \cdots, n)$ and if $\sum_j A_{ij}(0) = 1$ for all i. To every semi-Markov matrix there corresponds a semi-Markov process (and a Markov renewal process). Let $A(s)$ be a semi-Markov matrix. Every eigenvalue of $A(s)$ is an analytic function of s $(s \geqq 0)$. $A(s)$ is irreducible if and only if $A(0)$ is irreducible. If $A(s)$ is irreducible, then there exists a simple eigenvalue $\lambda(s)$ which is positive and is such that no other eigenvalue exceeds it in absolute value. Further, $\lambda(s)$ is a non-increasing function of s, $\lambda(0) = 1$, and $\lambda'(0) = aA'(0)e$, where $e^T = [1 \cdots 1]$, $aA(0) = a$, $ae = 1$. $I - A(s)$ is non-singular for $s > 0$; and if $A(s)$ is irreducible, then $\lim_{s \to 0} s(I - A(s))^{-1} = (-\lambda'(0))^{-1}ea$. These results are generalized to reducible semi-Markov matrices. The theorems of Smith (1955) and Pyke (1961) on the classification of states and on the limiting probabilities for semi-Markov processes with finitely many states can be obtained by using these results. (Received 27 June 1966.)

15. Some characterizations of normality. T. CACOULLOS, New York University.

Normality is characterized by the property of constant regression of the square of a linear statistic V on another linear statistic U. MAIN THEOREM. *Let $X_1, \cdots, X_n$ be independent and identically distributed scalar random variables with* df $F(x)$ *and suppose that $F(x)$ has moments of every order. Consider the linear forms $U = a_1X_1 + \cdots + a_nX_n$, $V = b_1X_1 + \cdots + b_nX_n$, where the constants a_j, b_j satisfy the conditions: $a_1b_1 + \cdots + a_nb_n = 0$ and $a_ja_k > 0$ for all $j, k = 1, \cdots, n$. Then V^2 has constant regression on U,* i.e., $E(V^2/U) = E(V^2)$ *if, and only if, F is the normal* df. *The same conclusion holds if the condition $a_ja_k > 0$ is replaced by the assumption that F is symmetric.* Multivariate analogues of these theorems are also given. Two other characterizations of multinormality are based on corresponding univariate results due to Laha [Lukacs-Laha, *Applications of Characteristic Functions*, p. 117, Griffin (1964) and C. R. Rao (*Le Calcul des probabilities et ses applications*, Paris (1959)]. (Received 11 July 1966.)

16. Some results related to the Darmois-Koopman problem. J. L. DENNY, University of California, Riverside.

Let $\{P_t\}$ be a family of probabilities with the same support Ω. For fixed t_0 the Darmois-Koopman problem is concerned with sufficient conditions that a minimal real linear space containing versions of $f(t, \cdot) \equiv \ln dP_t/dP_{t_0}$ has finite dimension. We fix $x \varepsilon \Omega$ and consider local versions of $f(t, \cdot)$ and distinguish two cases: for some open set containing x the essential linear dimension (smallest dimension of the minimal linear spaces containing versions) of the $f(t, \cdot)$ on the open set equals the essential linear dimension of the $f(t, \cdot)$ on each positive (P_t) subset of the open set; inequality is the other case. Equality of linear dimensions is related to sufficient statistics T which are *not reducible*: T is sufficient for $\{P_t\}$ and moreover on each positive (P_t) set $A \subset \Omega$ no proper subcollection of the real-valued components of T is sufficient for the family $Q_{t,A}(\cdot) = P_t(A \cap \cdot)/P_t(A)$. When Ω is locally Euclidean we obtain conditions that $\{P_t\}$ be an exponential family, conditions that there are versions of the $f(t, \cdot)$ which are continuous almost everywhere, conditions that there are versions of the $f(t, \cdot)$ which are continuous, etc. Some of these results extend and use the methods of L. Brown, *Ann. Math. Statist.* **35** (1964) 1456–1474, and employ the existence of a sufficient statistic which is not reducible. (Received 20 June 1966.)

17. A property of positive random variables (preliminary report). S. W. DHARMADHIKARI, University of Arizona.

Let X be a positive random variable. For $k \geqq 0$, let $g(k) = E(e^{-kX})$. THEOREM *For any* $\delta > 0$, $[g(\delta) - g(0)]/g'(\delta)$ *is not less than* $[g(2\delta) - g(\delta)]/g'(2\delta)$. Let $Z = e^{-\delta X}$, $A = E(Z)$, $B = E(XZ)$, $U = (1 - Z)/(1 - A)$ and $V = (XZ)/B$. Then the assertion of the theorem is equivalent to: the coefficient of regression of V on Z is not less than the coefficient of regression of U on Z. This observation, followed by a geometrical argument, leads to the desired result. Generalizations of this result are under investigation. (Received 8 July 1966.)

18. Moments of some rank order measures of correlation (preliminary report). WALTER J. DICK, and BRYANT CHOW, Rutgers–The State University.

Pitman [*J. Roy. Statist. Soc. Suppl.* **4** (1937), 225–232] presents the lower moments and the corresponding approximate distribution for rank correlation coefficients when the sample pairings are uncorrelated. Using the generalized correlation coefficient of Daniels [*Biometrika* **33** (1944), 129–135] some numerical results are given for the lower moments of the Fisher-Yates correlation coefficient (a measure of association based on the expected value of normal order statistics) in the case of correlated pairings from a bivariate normal parent population. From these results the corresponding approximate distributions are found using Pearson's method of moments. A comparison is made of the numerical moments to the empirical findings of Fieller and Pearson [*Biometrika* **48** (1961), 29–40]. A natural extension of the method of Spearman's ρ and Kendall's τ is also included. (Received 11 July 1966.)

19. Matric-variate generalizations of the multivarite t distributions and the inverted multivariate t distribution. JAMES M. DICKEY, Yale University.

Consider the $p \times q$ random matrix $T_0 = (U^{\frac{1}{2}})^{-1}X$, $U^{\frac{1}{2}}(U^{\frac{1}{2}})' = U \sim W(\Sigma, m - q)$, U distributed independently of the column p-vectors of X, which are identically and independently $N(0, \Sigma)$ distributed. $T = (P^{\frac{1}{2}\prime})^{-1}T_0Q^{\frac{1}{2}\prime} \sim (V^{\frac{1}{2}\prime})^{-1}Y$, $V \sim W(P, m - q)$, the row vectors of Y being $N(0, Q)$. The density of T is $k\ |P|^{q/2}\ |Q|^{(m-p)/2}\ |Q + T'PT|^{-m/2}$,

$k^{-1} = \pi^{pq/2}\Gamma_p[\frac{1}{2}(m - q)]/\Gamma_p(\frac{1}{2}m)$. $T_0'T_0$ is multivariate beta distributed (the generalization of F). Consider, also, $R_0 = [(I + T_0T_0')^{\frac{1}{2}}]^{-1}T_0 = [(XX' + U)^{\frac{1}{2}}]^{-1}X$. $R = (P^{\frac{1}{2}'})^{-1}R_0Q^{\frac{1}{2}'} \sim [(P^{\frac{1}{2}}Y'Q^{-1}YP^{\frac{1}{2}'} + V)^{\frac{1}{2}'}]^{-1}Y$ and has density $k\,|P|^{q/2}\,|Q|^{-(m-q-1)/2}\,|Q - R'PR|^{(m-p-q-1)/2}$, $I - R_0'R_0 > 0$, k as above. $R_0'R_0$ is multivariate beta distributed (the generalization of beta). The density of T' (of R') can be written in the T form (R form), implying further representations for T (for R). The marginal and conditional distributions of submatrices of T (of R) are of the $T + C$ form ($R + C$ form), C constant. A posterior distribution for several unknown normal-mean vectors, corresponding to scale-proportional unknown covariance matrices, is of the form $T + C$. The usual estimate of a matrix β of regression coefficients has a distribution of the form $T + \beta$ (Kshirsagar, *Proc. Camb. Phil. Soc.*, **57** (1961) 80–85).

20. On the asymptotic representation of sample quantiles. FRIEDHELM EICKER, Columbia University.

R. R. Bahadur recently has proved an asymptotically almost surely valid linear representation for sample quantiles in terms of the sample distribution function (df) at a fixed point, and has provided an asymptotic bound for the error term involved. The bound was shown to be $O(n^{-\frac{3}{4}}(\log n)^{\frac{1}{2}}(\log \log n)^{\frac{1}{4}})$ where n is the size of the random sample and the constant involved in 0 depends on the sample. Bahadur stated, however, that his proof did not provide a satisfactory explanation of the bound. Based on a different, in principal very simple method of proof such an explanation is given in the paper abstracted here. The proof utilizes the well known conditional probability distribution of the sample df given an order statistic and standard normal approximation methods. The explanation given consists in showing that Bahadur's bound cannot be improved upon as long as the Borel-Cantelli lemma is used to establish a.s. convergence. It is shown, moreover, that the sequence of (random) error terms tends to zero in probability like $N_n n^{-\frac{3}{4}}$ with an arbitrary sequence of constants $N_n \to \infty$. This result is optimal, since the assertion is false if N_n does not tend to infinity. (Received 15 July 1966.)

21. Sequential evaluation of graded preferences for two treatments. W. J. HALL, University of North Carolina.

Suppose a sequence of subjects each in turn indicates a graded preference for one of two treatments, A and B; for example, a strong or weak preference may be registered for either of the treatments, or no preference registered. A sequential test of the hypothesis that, in each preference grade, A is just as likely to be preferred as B, is presented. OC and ASN functions are derived. Allowance is made for certain subject-to-subject differences. The test is a *conditional sequential probability ratio test* and may be considered as a simple extension of Wald's sequential test for double dichotomies, applicable in this context if only one preference grade (and a no preference grade) were considered. Two-sided alternative versions and analogous non-sequential tests are also described. Possible applications include various kinds of consumer testing, psychological testing, and subjective aspects of clinical trials. (Received 11 July 1966.)

22. Testing the homogeneity of a set of correlated variances. CHIEN-PAI HAN, Harvard University.

To test $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_p^2$ from a p-variate normal distribution with common (known or unknown) pairwise correlation coefficient, four test criteria are constructed and their asymptotic distributions are compared. (1) Approximations to the maximum likelihood estimators of $\log \sigma_i^2$, $i = 1, 2, \cdots, p$, are obtained and a corresponding test criterion is found which is asymptotically equivalent to the likelihood ratio criterion. (2) Bartlett test is well known when the variates are uncorrelated. A modification of it is the statistic $M_1/(1 - \rho^2)$ which has an asymptotic $\chi^2_{(p-1)}$ distribution, where ρ is known and M_1 is the logarithm of the ratio of the arithmetic mean and the geometric mean of the sample variances. When ρ is unknown, it can be replaced by the average of all sample correlation coefficients. (3) When $p = 2$, Pitman (*Biometrika* **31** (1939) 9–12) has found that the null hypothesis is equivalent to the simple correlation coefficient of two transformed variates being zero. An extension is to test the multiple correlation coefficient of $u_1 = \bar{x} = (x_1 + x_2 + \cdots + x_p)/p$ on $u_i = x_i - \bar{x}$, $i = 2, 3, \cdots, p$. (4) A short-cut test statistic when $\rho = 0$ is $S^2_{\max}/S^2_{\min}$ and may be used in large samples for general ρ. (Received 20 June 1966.)

23. Infinite product Markov processes. T. E. Harris, University of Southern California.

$X^i = (x_t^i, M_t^i, P_x^i{}^i)$, $i = 1, 2, \cdots$, are standard nonterminating Markov-Feller processes (E. B. Dynkin terminology) on compact separable metric spaces E^i with Borel fields B^i in E^i. Let $X = (x_t, \bar{M}_{t+0}, P_x)$, $x_t = (x_t^1, x_t^2, \cdots)$, be a product process corresponding to independence of $x_t^1, x_t^2, \cdots$, with $M_t = M_t^1 \times M_t^2 \times \cdots$; $B = B^1 \times B^2 \times \cdots$; and $B_\infty = \bigcap_n (B^n \times B^{n+1} \times \cdots)$. Then X is standard. Let $H \varepsilon B$ be SC (stochastically closed; i.e., $P_x(x_t \varepsilon H, t \geqq 0) = 1$, $x \varepsilon H$). Suppose $x', x'' \varepsilon H$, $\Gamma \varepsilon B_\infty$, $P(t, x', \Gamma) = 1$ for a.e. t (Lebesgue), and $P(t, x'', \Gamma) = 0$ for a.e. t. Then $H = H' \cup H''$, H' and H'' disjoint and SC, $x' \varepsilon H'$, $x'' \varepsilon H''$. If each E^i is R_1 compactified by addition of an absorbing point, if the part of X^i on R_1 is an infinitely divisible process (same law for each X^i), if $H \varepsilon B$ is a SC subset of $R_1 \times R_1 \times \cdots$, and if x' and x'' are points of H such that $\Sigma(x_i' - x_i'')^2 = \infty$, then H can be decomposed as above. (Received 11 July 1966.)

24. A Tchebycheff type of inequality. David B. Hill, University of Vermont.

In this paper we establish an inequality similar in form to the Tchebycheff inequality but involving statistics rather than the population mean and variance. The inequality is particularly applicable in biological experimentation where often times many "pre-treatment" measurements can be made but due to the nature of the procedure only a single "post-treatment" reading can be obtained. Given a density function $f(t)$, for $i = 1, \cdots, n$, let X_i be a random variable with mean μ_i, variance σ^2, and with density function $f(t - \mu_i)$. Further let $P[X_i < \mu_i] = q$, $P[S^2 < \sigma^2] = \Pi$ where S^2 is the sample variance for random samples of size m. Suppose further that a random sample of size m is chosen from each of the n populations and let $\hat{\sigma} = [\max_j S_j^2]^{\frac{1}{2}}$ and $\hat{\mu}_p = \max_j x_{pj}$ where x_{pj} is the jth observation from the pth population. Let an additional observation y be chosen from one of the populations, which without loss of generality can be taken to be the first. Then the following theorems hold: Theorem 1. $\Pr[y > \hat{\mu}_1 + k\hat{\sigma}] \leqq 1/k^2 + q^m + \Pi^{n-1} - q^m\Pi^{n-1}$. Theorem 2. *If* $f(x) = f(-x)$ (*and hence* $q = \frac{1}{2}$), $\Pr[y > \hat{\mu}_1 + k\hat{\sigma}] \leqq 1/2k^2 + 1/2^m + \Pi^n - \Pi^n/2^m$. (Received 8 July 1966.)

25. Strict efficiency excludes superefficiency. Peter J. Huber, Swiss Federal Institute of Technology.

Let P_θ be a real-parameter family of probability measures having densities p_θ with respect to some fixed σ-finite measure. Assume that $(p_{\theta+\delta} - p_\theta)/(p_\theta \delta)$ has a limit p_θ'/p_θ in the $L_2(P_\theta)$-sense for $\delta \to 0$, and that the information $I(\theta) = E_\theta(p_\theta'/p_\theta)^2$ is continuous and $\neq 0$ at θ_0. Let $X_1, X_2, \cdots$ be a sequence of independent random variables with common distribution P_θ. THEOREM. *If a sequence of estimates* $T_n = T_n(X_1, \cdots, X_n)$ *of* θ *is "strictly efficient" at* θ_0, i.e. *if for some* $c > 0$ $\lim_{k\to\infty} \limsup_{n\to\infty} \sup_{|\theta-\theta_0| \leqq ckn^{-\frac{1}{2}}} I(\theta)E_\theta \cdot \min(n(T_n - \theta)^2, k^2) \leqq 1$, *then* $n^{\frac{1}{2}}(T_n - \theta_0)$ *is asymptotically normal with mean* 0 *and variance* $I(\theta_0)^{-1}$ *at* θ_0. *In particular,* T_n *cannot be superefficient at* θ_0. This result supplements a theorem of Stein and Rubin (Theorem 1 on p. 12 of Chernoff, *Ann. Math. Statist.* **27** (1956) 1–22). (Received 5 July 1966.)

26. A note on the asymptotic form of *a posteriori* distributions (preliminary report). RICHARD A. JOHNSON, University of Minnesota.

Let $X_1, X_2, \cdots$ be iid each having density $p_\phi(x) = C(\phi) \exp[\phi R(x)]$ with respect to a σ-finite measure μ. Suppose that $\rho(\phi)$ is an *a priori* probability density defined on the parameter space. Let ϕ_0 be the unobserved value of the parameter ϕ, let $\hat{\phi}$ be the maximum likelihood estimate of ϕ (assumed to be interior to the parameter space), and let $b^2(\hat{\phi})$ be the Fisher information evaluated at $\hat{\phi}$. Denote the *a posteriori* distribution of $n^{\frac{1}{2}}[\phi - \hat{\phi}]b(\hat{\phi})$ given $(x_1, \cdots, x_n)$ by F_n. If $\rho(\phi_0) > 0$ and ρ is continuous in some neighborhood of ϕ_0, then F_n converges in law to the standard normal distribution $\Phi(\cdot)$ for almost all sequences $x = (x_1, x_2, \cdots)$ where the measure is generated by $\prod_k p_{\phi_0}(x_k)$. Our results show that, subject to further restrictions on ρ, there exists an asymptotic expansion for $F_n(\cdot)$ in powers of $n^{-\frac{1}{2}}$ for almost all sequences. In particular, using only one correction term, we have almost surely $F_n(\xi) \sim \Phi(\xi) + \varphi(\xi)[a_1(\hat{\phi})(\xi^2 + 2) + \rho'(\hat{\phi})/\rho(\hat{\phi})]/b(\hat{\phi})n^{\frac{1}{2}} + O(n^{-1})$ $(n > N_x)$ where $\varphi(\cdot)$ is the standard normal density, N_x depends on x, and $a_1(\hat{\phi})$ is a continuous function of $\hat{\phi}$ determined by $C(\cdot)$. The result is uniform in ξ. This shows clearly how the *a priori* distribution influences the *a posteriori* distribution through the term of order $n^{-\frac{1}{2}}$. Related results of LeCam [*Publ. Inst. Statist. Univ. Paris* **7** (1958) 17 and *Univ. Calif. Publ. Statist.* **1** (1953) 277] assume a more general likelihood, but give only the first (i.e. $O(1)$) term. (Received 11 July 1966.)

27. Asymptotic expansions associated with *n*th power of a density. RICHARD A. JOHNSON, University of Minnesota. (By title)

Let $k(\cdot)f(\cdot)$ be the density function of an absolutely continuous variate X. Then kf^2, $kf^3, \cdots$ can be normalized to define a sequence of random variables $X_2, X_3, \cdots$. Suppose that $f(\cdot)$ has a unique mode at m. Buehler [*Ann. Math. Statist.* **36** (1965) 1878] has shown that, under mild regularity conditions, $Z_n = n^{\frac{1}{2}}(X_n - m)$ converges in law to a normal distribution. In the present work it is shown that, subject to differentiability conditions, the distribution function $F_n(\cdot)$ of Z_n possesses an asymptotic expansion in powers of $n^{-\frac{1}{2}}$. More specifically, $F_n(\xi) \sim \Phi(\xi) + \varphi(\xi)\sum_{j=0}^{\infty} \gamma_j(\xi)n^{-\frac{j}{2}}$ where $\Phi(\cdot)$ is the limiting normal distribution, $\varphi(\cdot)$ the corresponding density, and the $\gamma_j(\cdot)$ are polynomials. Denote the α-percentiles of F_n and Φ by ξ and ξ_α respectively. The expansion above has also been inverted to furnish an asymptotic expansion for ξ in powers of $n^{-\frac{1}{2}}$ with coefficients which are polynomials in ξ_α. Examples of specific distributions are given. (Received 11 July 1966.)

28. An example in which the preliminary test of significance leads to a uniformly better estimator. B. K. KALE, Iowa State University.

In case of estimation subsequent to a preliminary test of significance in the theory of incompletely specified models, we have several examples in which the resulting estimator has smaller mean square error than that of the usual estimators in the certain ranges of the parameter space. We give here an example in which the estimator subsequent to the preliminary test has smaller mean square error than that of the maximum likelihood estimator (or the minimum variance unbiased estimator) through out the natural range of the parameters involved. (Received 5 July 1966.)

29. Some aspects of the statistical analysis of the "mixed model" (preliminary report). GARY G. KOCH and P. K. SEN, University of North Carolina.

In this paper, the authors discuss the statistical analysis (both parametric and nonparametric) of "mixed model" experiments. The general structure of such experiments involves n randomly chosen subjects who respond to each of p distinct treatments. Let the joint distribution of the responses of the ith subject be

$$F_i(x_1, x_2, \cdots, x_p) = G_i(x_1 + m_{i1}, x_2 + m_{i2}, \cdots, x_p + m_{ip}) \tag{1}$$

where $m_{ij} = b_i + t_j$, $t_1 + t_2 + \cdots + t_p = 0$. Together with (1), the basic assumption is (A.1) the joint distribution of any linearly independent set of contrasts among the observations on any particular subject is diagonally symmetric. Two additional conditions which may or may not be assumed are: (A.2) the subject effects are purely "additive"; (A.3) the compound symmetry of the error vectors. Thus, the four cases of interest are described as follows:

	Not A.2.	A.2.
Not A.3.	Case I	Case II
A.3.	Case III	Case IV

For each of the above cases, the hypothesis

$$H_0: t_1 = t_2 = \cdots = t_p \tag{2}$$

is considered, and appropriate test procedures are given. For Case II and Case IV, these results are compared with those obtained for the corresponding parametric situations in which the error vectors is assumed to be normally distributed. (Received 29 June 1966.)

30. A unified treatement of various representations of the distribution of positive definite quadratic forms in normal variables. S. KOTZ, N. L. JOHNSON and D. W. BOYD, University of Toronto, University of North Carolina, and University of Alberta.

The probability density function (pdf) of a positive definite quadratic form in (central or non-central) normal variables can be represented as a series expansion in a number of different ways. Among these, one of the most important is that of a series of pdf's of non-central x^2's or of central x^2's with increasing degrees of freedom. These expansions have been discussed by Ruben [*Ann. Math. Statist.* **33** (1962) 542–570] [*Ann. Math. Statist.* **34** (1963) 1582–1584] who has given convenient recurrence formulae for determining the coefficients. Expansion in terms of Laguerre series and Maclaurin series (powers of the argument) have been discussed for central variables by Gurland [*Ann. Math. Statist.* **24** (1953) 416–427] and Pachares [*Ann. Math. Statist.* **26** (1955) 128–131] respectively, and in the general (non-central) case by Shah [*Ann. Math. Statist.* **34** (1963) 186–190] and Shah and Khatri [*Ann. Math. Statist.* **32** (1961) 883–887], but the coefficients in their series are not presented in a very convenient form for calculations. It is the purpose of this paper to show how all three kinds of expansion can be derived in a similar way, and incidentally, to obtain convenient recurrence formulae for determining the coefficients in the Laguerre and Maclaurin expansions. (Received 13 June 1966.)

31. A note on confidence bounds for certain ratios of characteristic roots of covariance matrices. P. R. KRISHNAIAH and P. K. PATHAK, Aerospace Research Laboratories, and Indian Statistical Institute. (By title)

Let $\Sigma_1, \Sigma_2, \cdots, \Sigma_k$ be the covariance matrices of k p-variate normal populations. Let λ_{ij} be the jth largest characteristic root of Σ_i $(j = 1, \cdots, p;\ i = 1, \cdots, k)$. We obtain herein simultaneous confidence bounds on (i) $\lambda_{i+1,p}/\lambda_{i1}$ and $\lambda_{i+1,1}/\lambda_{ip}$ $(i = 1, \cdots, k-1)$ and (ii) $\lambda_{i1}/\lambda_{jp}$ and $\lambda_{ip}/\lambda_{j1}$ $(i \neq j = 1, \cdots, k)$ by using methods similar to those of Khatri [*Ann. Inst. Statist. Math.* **17** (1965) 175–184]; in case (ii) the confidence bounds are obtained under the assumption of equal sample sizes. (Received 1 July 1966.)

32. Simultaneous test procedures under intraclass and other correlation models. P. R. KRISHNAIAH and P. K. PATHAK, Aerospace Research Laboratories and Indian Statistical Institute. (By title)

Consider k multivariate normal populations with covariance matrices $\Sigma_1, \Sigma_2, \cdots, \Sigma_k$ where

$$\Sigma_i = \sigma_i^2[(1 - \rho_i)I + \rho_i \mathbf{e}\mathbf{e}'],\quad \mathbf{e}' = (1, \cdots, 1)$$

and I is an identity matrix. Recently Srivastava [*Ann. Math. Statist.* **36** (1965) 1802–1806] has proposed a test procedure to test for the equality of covariance matrices against the alternative that $\Sigma_i \neq \Sigma_j$ for $i \neq j = 1, \cdots, k$ when the sample sizes are equal. In this paper we propose an alternative test procedure for the above problem. Also we consider the problem of testing for the equality of covariance matrices against different alternatives when the sample sizes are unequal. We further extend our results by indicating how the hypothesis of the equality of covariance matrices against different alternatives can be tested under circular and successive correlation models. Finally we point out a procedure for testing the significance of the mean vector of a multivariate normal population under successive and circular correlation models. (Received 1 July 1966.)

33. Inverse least squares estimators used in calibration. RICHARD G. KRUTCHKOFF, Virginia Polytechnic Institute.

Assume a linear response of the form $y = \alpha + \beta x + e$ where e is a normal error. In a calibration problem it is often possible to design an experiment, using known values of x, to obtain estimates of α and β, which are used to calibrate the system. One then observes a y and, using the calibration, estimates the x which gave rise to it. The usual estimates for α and β are simply the least squares estimates. The principal conclusion of the study is that the least squares estimates of parameters in the inverse model $x = \gamma + \delta y + e$ give estimators for x which have uniformly smaller average squared error. (Received 11 July 1966.)

34. Extension of the SPRT to 3 decisions (preliminary report). JAMES A. LECHNER, Westinghouse Defense & Space Center.

One way to characterize the Wald SPRT is as follows: Sample until the posterior probability of one of the two hypotheses reaches its preassigned (high) value; then stop and accept that hypothesis. Therefore, one possible generalization to more than 2 alternative decisions is: Sample until one of the n hypotheses has posterior probability exceeding its preassigned (high) value; then accept that hypothesis. This procedure is being investigated, for $n = 3$. Some results have been obtained, and will be presented. Calculation of the stopping boundaries is straightforward. Certain bounds on the various error rates. analogous to Wald's bounds, are obvious, but analytic determination of the OC and ASN functions has not yet been achieved. (Received 11 July 1966.)

35. Two properties of a subset selection procedure (preliminary report). D. M. MAHAMUNULU, State University of New York, Buffalo.

We have at our disposal $k(\geqq 2)$ populations $\Pi_1, \cdots, \Pi_k$. The distribution function of a single observation x_i from Π_i is $F(\cdot \mid \theta_i)$ where θ_i is an unknown scalar parameter, $i = 1, \cdots, k$. From the available set of populations we wish to select a fixed size subset which contains as far as possible the best ones (those with largest θ-values). We assume that a random sample $u_i = (x_{i1}, \cdots, x_{in})$ is available from Π_i, $i = 1, \cdots, k$. A commonly used procedure, designated here as R_s, based on $u_1, \cdots, u_k$ for the selection of such a subset of size $s(< k)$ is the following: consider suitable statistics $T_1, \cdots, T_k$, where $T_i = T(u_i)$. Select the populations corresponding to the s largest T-values. We prove two properties of the procedure R_s: (i) When T_i is absolutely continuous and its distribution function $G_n(\cdot \mid \theta_i)$ is stochastically increasing in θ_i, the procedure R_s possesses a property of multivariate unbiasedness; (ii) it is the uniformly best decision rule in the class of impartial decision rules based on T_i for a class of loss functions when T_i is continuous and has a density with a monotone likelihood ratio. (Received 11 July 1966.)

36. Some sharp Tchebycheff inequalities. GOVIND S. MUDHOLKAR and PODURI S. R. S. RAO, University of Rochester.

Let $Y = (Y_1, Y_2, \cdots, Y_n)$ be a random vector with $EY = \mu$ and $\varphi \geqq 0$ be a homogeneous concave function on the nonnegative orthant of R^n. It is shown that the inequality $P[\varphi(Y) \geqq \epsilon] \leqq \varphi(\mu)/\epsilon$ is sharp. Using this result a number of multivariate generalizations of one and two sided Tchebycheff inequalities, when only variances are known, are obtained. These contain the corresponding results due to Marshall and Olkin (*Ann. Math. Statist.* **31,** 1001–1014) as particular cases. More specifically, if $X_1, X_2, \cdots, X_n$ be n jointly distributed random variables with $EX_i = 0$ and var $X_i = \sigma_i^2$, $i = 1, 2, \cdots, n$, and $\alpha_i \geqq 0$ are n reals $\sum \alpha_i = 1$, then it is shown that the inequalities $P[(\sum \alpha_i X_i^{2r})^{1/r} \geqq \epsilon] \leqq (\sum \alpha_i X_i^{wr})^{1/r}/\epsilon$, $r \leqq 1$, and $P[(\sum \alpha_i X_i^{2r})^{1/r} \geqq \epsilon, X_i \geqq 0] \leqq (\sum \alpha_i \sigma_i^{2r})^{1/r}/[\epsilon + (\sum \alpha_i \sigma_i^{2r})^{1/r}]$, $r \leqq \frac{1}{2}$, are sharp. It is observed that as r tends, respectively, to $-\infty$, -1, 0, 1, $(\sum \alpha_i t_i^r)^{1/r}$ tends respectively to min (t_i), the weighted harmonic mean, geometric mean, and arithmetic mean of t_i. (Received 14 July 1966.)

37. An extension of the birthday problem. JOSEPH I. NAUS, Rutgers–The State University.

Let N individuals have birthdays distributed at random over D days. The classical birthday problem seeks the probability that at least two of the N birthdays coincide. The probability that at least n of the N birthdays coincide is also well known. We seek the probability that at least n of the N birthdays fall within d adjacent days. For the case where the D days form a circle (the Dth day is adjacent to the first), we find the probability that at least two of the N birthdays fall within d adjacent days to be $1 - (D - Nd + N - 1)!/(D - Nd)!\, D^{N-1}$, for $D \geqq Nd$. For the case where the D days form a line (the Dth day is not adjacent to the first), we find the probability that at least two of the N birthdays fall within d adjacent days to be $1 - [D - (N-1)(d-1)]!/[D - (N-1)(d-1) - N]!\, D^N$, for $D \geqq (N-1)(d-1)$. An approximate solution is given for the more general probability of n birthdays within d adjacent days. (Received 11 July 1966.)

38. Inadmissibility of the best invariant estimate for loss function $W(t) = |t|^k$ (preliminary report). S. K. PERNG, Michigan State University.

By modifying an example due to Charles Stein (private communication), we have an example which shows that, in the fixed sample size case, if the loss function is $W(t) = |t|^k$ for $k \geqq 1$, there exists a probability density such that the best invariant estimator is uniquely determined, but is inadmissible. Furthermore for this example, $E(|X|^{1-\eta}W(X)) < \infty$ where η is an arbitrary positive number. This closes the gap between Brown's theorem (unpublished Ph.D. thesis) giving conditions under which $E(|X|W(X)) < \infty$ implies admissibility and his example showing that $E(|X|^\alpha W(X)) < \infty$ for $0 \leqq \alpha < k/(2^k - 1)$ does not imply admissibility. Let ξ be an unknown real parameter $-\infty < \xi < +\infty$, σ be a random variable distributed according to the known density π with respect to Lebesgue measure, where $\pi(\sigma) = C\sigma^{-(k+2-\eta)}$ for $\sigma > 1$, $= 0$ otherwise, where $\eta > 0$, C is a constant. Assume $X - \xi$ given σ is distributed according to density q_σ with respect to Lebesgue measure where $q_\sigma = \sigma^{-1}q(y/\sigma)$ and $q(z) = \frac{1}{2}\cdot 3^{\frac{1}{2}}$ if $|z| \leqq 3^{\frac{1}{2}}$, and 0 otherwise. Then the following estimator dominates the best invariant estimate X. Let $\varphi(X, \sigma) = X + \sigma f(X/\sigma)$ where $f(z) = -\epsilon\delta z$ if $|z| \leqq 1/\epsilon$; $= -\epsilon\delta(2/\epsilon - |z|) \operatorname{sgn} z$ if $1/\epsilon \leqq |z| \leqq 2/\epsilon$, $= 0$ otherwise. I wish to thank Professor Stein for communicating his unpublished example. (Received 17 June 1966.)

39. On some optimum nonparametric procedures in two-way layout. MADAN L. PURI and PRANAB K. SEN, Courant Institute of Mathematical Sciences, New York University, and University of North Carolina.

For the estimation and testing of contrasts in two-way layout, some optimum nonparametric procedures based on Chernoff-Savage type of rank order statistics (*Ann. Math. Statist.* (1958) 972–994) are considered here. The asymptotic properties of the proposed methods are studied and compared with those of the least square method. (Received 8 July 1966.)

40. Some observations on estimates of the distributions of two stochastically ordered random variables. TIM ROBERTSON and ROBERT V. HOGG, University of Iowa.

Let X and Y have the respective distribution functions F and G such that $F(z) \geqq G(z)$ for all z. With m x-observations and n y-observations, maximum likelihood estimates of F and G, which are of the discrete type, have been found in an earlier paper of Brunk, Franck Hanson, and Hogg; an illustration of this solution is given in this paper. If, instead, it is assumed that the distributions are of the continuous type and unimodal, it is proved that the maximum likelihood estimates of the corresponding densities can be taken to be uniform within intervals which join consecutive observations. Consequently, only estimates, which are uniform within these intervals, are considered. Under this additional restriction, two types of estimates are then found: the first estimates are not required to be unimodal but the second estimates are. In essence, these estimates spread out the probabilities, which are found in the discrete case, over certain intervals until all of the restrictions are fulfilled Examples are given and these estimates are proved to be consistent. (Received 27 June 1966.)

41. The empirical Bayes approach: estimation of posterior quantiles. J. R. RUTHERFORD, Royal Military College of Canada.

In Rutherford and Krutchkoff (submitted to *Biometrika* (1966)) an estimator of the prior distribution is constructed in the following situation: we observe a sequence of random variables $X_1, X_2, \cdots, X_n$ independently distributed according to a known conditional density function $f(x_i \mid \theta_i)$ where $\theta_1, \theta_2, \cdots, \theta_n$ is a sequence of independent realizations of an unobservable random variable Θ distributed according to a prior distribution function $G(\theta)$ which is an unspecified member of the Pearson family of curves. The conditional density function is such that there exist known functions $h_k(x)$, $k = 1, 2, 3, 4$, such that $E[h_k(X) \mid \theta] = \theta^k$. Let $G_n(\theta)$ be the estimate of the prior distribution then an estimate, $q_n(\alpha)$, of the α quantile, $q(\alpha)$, of the posterior distribution of θ given x is constructed. We prove that $q_n(\alpha) \rightarrow q(\alpha)$ for all α. This result finds various applications such as the point estimation problem with the absolute difference as loss function: the estimate of the posterior median is then shown to be asymptotically optimal. (Received 27 June 1966.)

42. The Halmos-Savage theorem in the non-σ-dominated case. Herman Rubin, Michigan State University.

The well-known result of Halmos and Savage (*Ann. Math. Statist.* **20** (1949) 225–241) that in the σ-dominated case if $\mathfrak{J}$ is pairwise sufficient for a $\{P_\theta : \theta \varepsilon \Omega\}$ then there is a factorization $dP_\theta = f(\theta)\, d\mu$ where f is $\mathfrak{J}$-measurable are extended to the case where the P_θ are dominated by a locally finite measure ν or, equivalently, if given any family $\mathcal{S}$ of measurable sets there is a set E such that for all $\theta \varepsilon \Omega$, E is a P_θ-least upper bound for $\mathcal{S}$. The arguments follow closely those of I. E. Segal, (*Amer. J. Math.* **73** (1951) 275–313) and the Halmos-Savage paper. (Received 8 July 1966.)

43. Large-sample approximations to the variance and bias of bispectral estimates. Paul Shaman, New York University and Columbia University.

Let $\{X(t)\}$ be a real-valued, continuous parameter random process that is sixth-order stationary and has mean 0. Given observation of the process for $0 \leqq t \leqq T$, construct an estimate $b_T^*(\omega_1, \omega_2) = c_T^*(\omega_1, \omega_2) + i\, d_T^*(\omega_1, \omega_2)$ of the bispectrum, which is the Fourier transform of the third-order moment function, by forming the Fourier transform of an estimate of the third-order moment function and applying a weighting function. Under appropriate regularity conditions on the process and on the weighting function in the estimate, large-sample approximations are given for the bias of $b_T^*(\omega_1, \omega_2)$ and for var $c_T^*(\omega_1, \omega_2)$, Var $d_T^*(\omega_1, \omega_2)$, and Cov $(c_T^*(\omega_1, \omega_2), d_T^*(\omega_1, \omega_2))$. Also given are bounds on the errors of the approximations for large T. The method of analysis is to work with the spectral functions in the frequency domain rather than with the cumulant functions in the time domain. The regularity conditions on the process include boundedness and boundedness of the first derivative for the spectrum, bispectrum, and trispectrum. For $0 < \alpha < \frac{1}{2}$, $T^\alpha(Eb_T^*(\omega_1, \omega_2) - b(\omega_1, \omega_2)) = O(1)$, $T^{1-2\alpha}$ Var $c_T^*(\omega_1, \omega_2) = O(1)$, $T^{1-2\alpha}$ Var $d_T^*(\omega_1, \omega_2) = O(1)$, and $T^{1-\alpha}$ Cov $(c_T^*(\omega_1, \omega_2), d_T^*(\omega_1, \omega_2)) = O(1)$.

44. Convergence of a sequence of transformations of distribution functions. R. Shantaram and W. L. Harkness, Pennsylvania State University.

Let $F(x)$ be the distribution function of a positive random variable, all of whose moments $\mu_n = E(x^n)$ exist and are finite. Define $G_1(x) = \mu^{-1} \int_0^x [1 - F(y)]\, dy$ for $x \geqq 0$ and zero elsewhere; and recursively, for $n \geqq 2$, define $G_n(x) = [\mu(G_{n-1})]^{-1} \int_0^x [1 - G_{n-1}(y)]\, dy$ for $x \geqq 0$ and zero elsewhere ($\mu = \mu_1 = E(x)$, and $\mu(G_{n-1})$ is the mean of $G_{n-1}(x)$). Then $G_n(x)$ is the distribution function of a positive random variable, for $n = 1, 2, \cdots$, whose kth moment is given by $\binom{n+k}{k}^{-1}\mu_{n+k}/\mu_n$, $k = 0, 1, 2, \cdots$. If $F(x)$ is distributed on a finite interval, then $G_n(x/n)$ converges, as $n \rightarrow \infty$, to an exponential distribution function. If, on the other hand, $F(x) < 1$ for all $x > 0$ and $G_n(c_n x) \rightarrow G(x)$, a proper distribution function, for a sequence $\{c_n\}$ of normalizers such that $\{c_n/c_{n-1}\}$ is a bounded sequence, then it is shown, among other things, that $G(x)$ is everywhere continuous, $c_n \sim \mu_{n+1}/(n+1)\mu_n$, $(n \to \infty)$, and $\lim_{n\to\infty} c_n/c_{n-1}$ exists. This last limit equals 1 if $F(x)$ has an analytic characteristic function, in which case $G(x)$ is exponential. If $\lim\sup_{n\to\infty} c_n/c_{n-1} = +\infty$ and $G(x)$ is continuous at the origin, then the c_n are not of the form $\mu_{n+1}/(n+1)\mu_n$, (as $n \to \infty$). Finally, certain criteria for existence of the $\{c_n\}$ are derived. (Received 1 July 1966.)

45. Some results on the complete convergence of linear combinations of martingale summands (preliminary report). W. F. STOUT, Purdue University.

Let X_k be a sequence of random variables, $\mathfrak{F}_n$ be the σ-field generated by $X_1, X_2, \cdots, X_n$, a_{nk} be a matrix of real numbers, $A_n = \sum_{k=1}^{\infty} a_{nk}^2$, $S_n = \sum_{k=1}^{n} X_k$, and $T_n = \sum_{k=1}^{\infty} a_{nk} X_k$. T_n is said to converge completely to a constant c if T_n is well defined and if $\sum_{n=1}^{\infty} P[|T_n - c| > \epsilon] < \infty$ for all $\epsilon > 0$. THEOREM 1. *Let X_k be iid, $E|X_k|^{2/\alpha} < \infty$ for some $\alpha > 0$, $a_{nk} = 0$ for $k > n$, $|a_{nk}| \leqq Kn^{-\alpha}$ where K is a finite real number independent of k and n. Then* (i) $0 < \alpha \leqq 1$, $EX_k = 0$, *and* $\sum_{n=1}^{\infty} \exp(-\lambda/A_n) < \infty$ *for all* $\lambda > 0$ *implies that T_n converges completely to zero.* (ii) $\alpha > 1$ *implies that T_n converges completely to zero.* THEOREM 2. *Let $(S_n, \mathfrak{F}_n)$ be a martingale with $|X_i| \leqq 1$, $\sum_{n=1}^{\infty} \exp(-\lambda/A_n) < \infty$ for all $\lambda > 0$. Then T_n converges completely to zero.* Further results are also presented. The stated results are generalizations of results due to Y. S. Chow (*Ann. Math. Statist.* **37** 540, plus unpublished work). (Received 11 July 1966.)

46. On maximum likelihood estimation for two-phase linear regression. DAVID L. SYLWESTER, University of Vermont.

Sprent [*Biometrics* **17** (1961) 634–645] has given some statistical hypotheses relevant to two-phase linear regression models. Behaviour of parameter estimates, especially when the "change-over point" τ is unknown, has received little study. Let $X_1, \cdots, X_n$ be independent normal random variables with $EX_i = \alpha + \beta t_i$ for $t_i \leqq \tau$ and $EX_i = \alpha + \beta\tau$ for $t_i > \tau$ where $\beta \neq 0$ and $\sigma^2 = \text{Var}(X_i) > 0$. We show that for the MLE $\hat{\omega}_n$ of the unknown parameter point $\omega = (\alpha, \beta, \tau, \sigma^2)$, $\hat{\omega}_n \to \omega$ a.s. and $n^{\frac{1}{2}}(\hat{\omega}_n - \omega)$ is asymptotically multivariate normal. To obviate the difficulty caused by the fact that the likelihood function $L(\omega \mid X_1, \cdots, X_n)$ is not everywhere differentiable and even when differentiable the equation $\partial L(\omega \mid X_1, \cdots, X_n)/\partial\tau = 0$ has many solutions, we ignore many of the observations, find a "pseudo-MLE" $\hat{\omega}_n'$ and determine its asymptotic distribution. We then show that $n^{\frac{1}{2}}(\hat{\omega}_n' - \hat{\omega}_n) \to 0$ in probability. A key step in the proof is showing that $Z_n \equiv \sup_{\lambda\epsilon\Lambda} n^{-1} \cdot \sum_{i=1}^{n} \lambda(t_i) Y_i \to 0$ a.s. where the Y_i are iidrv's with zero means and Λ is the collection of all polygonal functions composed of three or fewer straight line segments with $|\lambda(t)| \leqq 1$. (Received 8 July 1966.)

47. Prediction filter coefficients for jointly stationary time series. GRACE WAHBA, IBM Corporation and Stanford University.

Consider $P + 1$ jointly stationary, zero mean, Gaussian time series $\{X_0(t), X_1(t), \cdots, X_P(t)\}$ possessing a positive definite spectral density matrix $F(\omega)$. It is well known that $X_0(t)$ admits the representation $X_0(t) = \sum_{s=-\infty}^{\infty} \sum_{k=1}^{P} b_k(s) X_k(t - s) + \epsilon(t)$ where $\epsilon(t)$ is a stationary Gaussian process independent of $X_k(t)$, $k = 1, 2, \cdots, P$. Suppose that a realization of length T of $\{X_0(t), X_1(t), \cdots, X_p(t)\}$ is available. Under some additional conditions, consistent estimates $\hat{b}_k(s)$ of $b_k(s)$, $|s| \leqq (M-1)/2 \ll T$ are given. These estimates are based on the windowed sample spectral density matrix $\hat{F}(\omega)$ for M equally spaced values of ω, where the bandwidth of the window is proportional to $1/M$. The asymptotic joint distribution of these estimates as $M, T \to \infty$, $M/T \to 0$ is given, as well as a bound on the prediction error σ^2, where $\sigma^2 = E(\tilde{X}_0(t) - \hat{X}_0(t))^2$, $\hat{X}_0(t) = \sum_{|s| \leqq (M-1)/2} \cdot$

$\sum_{k=1}^{P} \hat{b}_k(s)\tilde{X}_k(t-s)$, where $\{\tilde{X}_k(t), k = 1, 2, \cdots, P\}$ is an independent realization of the process. The estimates are a generalization of estimates suggested by Hannan (*Proceedings of the Symposium on Time Series Analysis*, Wiley, 1962). (Received 8 July 1966.)

48. On the existence of an optimal stopping rule for X_n/n (preliminary report).

OSCAR A. WEHMANEN, Purdue University.

Let $X_n \geqq 0$ be a sequence of independent, identically distributed, random variables. Assume that $E(\sup X_n/n) < \infty$, and $E(X_1 \mid X_1 \geqq a) = O(a)$. Define

$$\gamma_n = \text{ess}_{t \in C_n} \sup E(X_t/t \mid \mathfrak{F}_n)$$

where C_n is the class of stopping rules t with $P(n \leqq t < \infty) = 1$. Define $\sigma =$ the first $n \geqq 1$ such that $X_n/n \geqq \gamma_n$ or ∞ if there is no such n. Then, using martingale properties, we can prove that $P(\sigma < \infty) = 1$. By a result of Chow and Robbins (to be published in *Fifth Berkeley Symp. Math. Statist. Prob.* 1965), σ is an optimal stopping rule for X_n/n. (Received 11 July 1966.)

49. Randomized fractional weighing designs.

S. ZACKS, Kansas State University.

The problem studied is that of estimating unbiasedly a given linear function, $\lambda'\omega$, of p unknown weights $\omega_1, \cdots, \omega_p$; where $\omega' = (\omega_1, \cdots, \omega_p)$; when the number of possible weighing operations, n, is smaller than p. The weighing operations are of the chemical type. The randomized fractional weighing designs constructed by choosing, according to a specified randomization scheme, n rows from a $p \times p$ Hadamard matrix (assumed to exist) and performing the corresponding weighing operations. The randomization procedures studied consist of choosing the rows independently and with replacement according to a probability vector, ξ, of order p. Non-randomized designs are special cases. It is shown that any linear function $\lambda'\omega$ can be estimated unbiasedly by a proper choice of ξ. Optimal randomization procedures depend on the given function λ. Each λ specifies a subset of r $(1 \leqq r \leqq p)$ admissible rows, in the sense that if other rows are chosen then either the estimation is biased or has a larger variance than the one attained by choosing the specified rows. It is proved that if each of the r admissible rows is chosen with probability $1/r$, then the corresponding unbiased estimator has a uniformly (in ω and σ^2) minimum variance. The procedure is then extended to random choice of rows without replacement. (Received 5 July 1966.)