ON THE BLOCK STRUCTURE OF SINGULAR GROUP DIVISIBLE DESIGNS

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1. Introduction and summary. Shah [5] obtained an upper bound for the number of disjoint blocks in certain PBIB designs. The results obtained there [5] were based on theorems given in References [1], [3], and [6]. It should be pointed out here that theorems on Semi-Regular Group Divisible designs [5] are also implicitly given by Saraf [4]. In this note, the similar results are given for Singular Group Divisible (SGD) designs and the main tool used to establish them is Theorem 3.1 given below.

2. Definition and preliminaries. An incomplete block design with v treatments each replicated r times in b blocks of size k is said to be group divisible [2], if the treatments $v = mn$ can be divided into m groups, each with n treatments, so that the treatments (conventionally called first associates) belonging to the same group occur together in λ_1 blocks and treatments (called second associates) belonging to different groups occur together in λ_2 blocks ($\lambda_1 \neq \lambda_2$). The parameters of such designs are $v = mn, b, r, k, \lambda_1, \lambda_2, n_1 = n - 1$ and $n_2 = n(m - 1)$ where n_i's ($i = 1, 2$) are exactly the number of ith associates of each treatment and further, the following relationships can be established:

$$bk = vr = mnr \tag{2.1}$$

$$(n-1)\lambda_1 + n(m-1)\lambda_2 = r(k-1) \quad \text{or} \quad rk - \lambda_2 v = r - \lambda_1 + n(\lambda_1 - \lambda_2) \tag{2.2}$$

$$r \geqq \lambda_1 \quad \text{and} \quad rk - \lambda_2 v \geqq 0. \tag{2.3}$$

3. On the block structure of the SGD design. The GD designs are said to be singular [2] if $r = \lambda_1$ and $rk - \lambda_2 v > 0$ and for such designs the following Theorem 3.1 holds, the proof of which can be derived from the Theorem 2 [1].

THEOREM 3.1. *For a* SGD *design, k is divisible by n. If $k = cn$, then the number of groups in each block must be c.*

THEOREM 3.2. *A given block of the* SGD *design cannot have more than*

$$b - 1 - [k(\lambda_1 - 1)^2/((\lambda_1 - 1)n + (k - n)(\lambda_2 - 1))]$$

disjoint blocks with it and if some block has that many disjoint blocks, then $n + [(k - n)(\lambda_2 - 1)]/(\lambda_1 - 1)$ is an integer and each non-disjoint block has $n + [(k - n)(\lambda_2 - 1)]/(\lambda_1 - 1)$ treatments common with that given block.

PROOF. Following the procedure given in Shah [5] and Theorem 3.1 above, we obtain

$$\textstyle\sum_{i=1}^{b-d-1} x_i = k(\lambda_1 - 1), \tag{3.1}$$

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$$(3.2)\quad \sum_{i=1}^{b-d-1} x_i(x_i - 1) = k[(n-1)(\lambda_1 - 1) + (k-n)(\lambda_2 - 1)], \quad \text{and}$$

$$(3.3)\quad \sum_{i=1}^{b-d-1} (x_i - \bar{x})^2 = k[(\lambda_1 - 1)n + (k-n)(\lambda_2 - 1)] - [k^2(\lambda_1 - 1)^2/(b-d-1)]$$

where $\bar{x} = \sum_{i=1}^{b-d-1} x_i/(b-d-1)$. Hence the theorem follows immediately from (3.3).

The following theorems and corollaries can be easily derived from Theorem 3.2.

THEOREM 3.3. *The necessary and sufficient condition that a block of a* SGD *design has the same number of treatments common with each of the remaining blocks is that* (i) $b = m$.

PROOF. In (3.3), substituting $d = 0$, it follows that

$$(3.4)\quad \sum_{i=1}^{b-1} (x_i - \bar{x})^2 = rk(v-k)^2(b-m)/m(b-1)(v-n).$$

From (3.4) the result of Theorem 3.3 follows immediately.

THEOREM 3.4. *If a block of the* SGD *design with parameters* $v = mn = tk$, $b = tr$ (t *an integer greater than* 1) *has* $(t-1)$ *blocks disjoint with it, then the necessary and sufficient conditions that it has the same number of treatments common with each of the non-disjoint blocks is that* (i) $b = r + m - 1$. *Another necessary condition is that* (ii) k/t *is an integer.*

PROOF. In (3.3) letting $d = t - 1$, it follows that

$$(3.5)\quad \sum_{i=1}^{b-t} (x_i - \bar{x})^2 = k^2(t-1)[bn + n - kt - nr]/t(v-n)$$

where $\bar{x} = k/t$. From (3.5), the results follow immediately.

COROLLARY 3.4A. *For a resolvable* SGD *design* $b \geqq r + m - 1$.

COROLLARY 3.4B. *The necessary and sufficient condition that a resolvable* SGD *design be affine resolvable is that it has a block which has the same number of treatments common with each block not belonging to its own replication.*

THEOREM 3.5. *If* x *be the number of treatments common between any two blocks of a* SGD *design, then* $\max(0, T_1) \leqq x \leqq \min(k, T_2)$ *where* $T_1 = [k(r-1)/(b-1)] - (b-2)^{\frac{1}{2}}A$, $T_2 = [k(r-1)/(b-1)] + (b-2)^{\frac{1}{2}}A$, *and* $A^2 = kr(b - r - m + (k/n))(v-k)/(b-1)^2(m-1)$.

PROOF. Following the method given in Shah [6] and Theorem 3.1 above, we have

$$(3.6)\quad \sum_{i=3}^{b} x_i = k(r-1) - x$$

$$(3.7)\quad \sum_{i=3}^{b} x_i(x_i - 1) = k[(n-1)(\lambda_1 - 1) + (k-n)(\lambda_2 - 1)] - x(x-1)$$

$$(3.8)\quad \sum_{i=3}^{b} (x_i - \bar{x})^2 = k[(n-1)(\lambda_1 - 1) + (k-n)(\lambda_2 - 1) + (r-1)] - x^2 - \{[k(r-1) - x]^2/(b-2)\} \geqq 0$$

where $\bar{x} = \sum_{i=3}^{b} x_i/(b-2)$. Hence Theorem 3.5 follows immediately from (3.8).

COROLLARY 3.5. *If* $b = r + m - (k/n)$, *then any two blocks have* $k - rk + \lambda_2 v$ *treatments common.*

REMARKS. It is shown in [1] that any SGD design can be obtained from a BIB design with parameters v^*, k^*, r^*, b^*, λ^* given by

$$v^* = m, \qquad k/n = c, \qquad r^* = r = \lambda_1, \qquad b^* = b, \qquad \lambda^* = \lambda_2$$

by putting treatments of the BIB design in one-to-one correspondence with the groups of treatments in the group divisible design, and replacing each block of the BIB design with the corresponding k^* groups. Then if x is the number of treatments common to two blocks of the SGD design and y is the number of treatments common to the two corresponding blocks of the BIB design, we have $x = ny$. This could be used to give different proofs of several results of this paper, based on known theorems for BIB designs. In the case of the SGD designs of Theorem 3.3, the BIB design is symmetric, with exactly λ^* treatments common to any two blocks. It follows that the number of treatments common to any two blocks of the SGD designs of Theorem 3.3 is $n\lambda^* = n\lambda_2$. In the case of the SGD designs of Theorem 3.4, a stronger necessary condition than (ii) is that k/nt is an integer, representing the number of groups of n treatments common to the given block and a non-disjoint block.

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