INDUCTIVE METHODS FOR BALANCED INCOMPLETE BLOCK DESIGNS

By R. G. Stanton and R. C. Mullin

University of Waterloo

1. Introduction. When working with balanced incomplete block designs, one notes with disappointment that inductive methods are usually unavailable, and in those instances where induction is available it is complicated by the fact that the parameter of induction usually increases by some integral constant greater than unity. It is our purpose here to embed such designs in larger systems where inductive techniques are possible. These larger systems permit block sizes to vary but retain the other axioms of block designs. In this paper we examine two such general systems which are closely related; one is more suitable as a generalization of finite geometries and the other more convenient for use with block designs. The idea of using variable block size is not new. For example it has been used in references [1] and [2]. However it does not seem to have been exploited fully as an inductive technique. We shall introduce a notational system which is favourable to inductive constructions.

2. Balanced incomplete block designs. A balanced incomplete block design is a system consisting of a set V of v objects called varieties, and a collection of b subsets of V called blocks satisfying the following conditions:

B_1 : every block contains precisely $k < v$ (distinct) varieties,

B_2 : every variety occurs in precisely r blocks,

B_3 : every pair of varieties occurs in precisely $\lambda > 0$ blocks.

We shall see directly that B_2 is a redundant axiom. To do so let us introduce the notion of a λ-system.

3. λ-systems. Let us define a λ-system as consisting of a set V of v varieties and a collection of b subsets (called blocks) of V which satisfies the following axioms:

L_1 : every pair of varieties occurs in precisely λ blocks.

L_2 : every block contains at least 2 varieties.

Note that, for $\lambda = 1$, we have the fact that every pair of varieties (points) determines a unique block (line); thus the connection with geometries. We have adopted the terminology of block designs to make the extension of λ-systems to (r, λ)-systems (to be discussed later) more natural.

Associated with every λ-system L there is a sequence of non-negative integers $B = (b_1, b_2, b_3, \cdots)$ where b_i is the number of blocks containing exactly i varieties. Although B is formally an infinite vector, all but a finite number of entries are zero. We shall call such a vector the B-vector of L. Let us note that $b_1 = 0$, and that counting occurrences of pairs of varieties gives

Received 10 January 1966.

$$\sum_{i=2}^{\infty} b_i\binom{i}{2} = \lambda\binom{v}{2}, \quad (1)$$

where v is the number of varieties.

Let x be a fixed variety of V. Using x we can find another vector, $R_x = R = (r_0, r_1, r_2, \cdots)$, associated with L; in this vector r_i is the number of blocks of size $i + 1$ which contain x. Thus $r_i \leqq b_{i+1}$ ($i = 0, 1, \cdots$); since every variety appears with x exactly λ times, we obtain

$$\sum_{i=1}^{\infty} ir_i = \lambda(v - 1). \quad (2)$$

Let us delete x from L and omit any blocks of single elements that this operation creates. This yields a λ-system with $v - 1$ varieties and B-vector (b_i^*), in which $b_1^* = 0, b_i^* = b_i + r_i - r_{i-1}$ ($i = 2, 3, \cdots$). The inverse operation may be called the induction principle for λ-systems and is contained in the following:

THEOREM 1. *A λ-system with B-vector $(b_1, b_2, \cdots)$ exists if and only if there exists a sequence of integers $(0, r_1, r_2, \cdots)$ and a λ-system L' with B-vector $(0, b_2 + r_2 - r_1, b_3 + r_3 - r_2, \cdots)$ and a collection of blocks C of L', containing r_i blocks of size i ($i = 2, 3, \cdots$) such that every variety v of L' occurs $\lambda_v < \lambda$ times in C.*

PROOF. The necessity of the conditions was established above. Let us assume that we have a system L' with collection C as defined above. Let us adjoin a new variety x to every block in C. Further, for every variety v of L', adjoin $\lambda - \lambda_v$ blocks (x, v). Thus we have the required system. It is easily verified that r_1 new pairs (x, v) are adjoined. As a corollary we have immediately

THEOREM 2. *Let L be a λ-system in which every block has constant size $k < v$. Then L is a balanced incomplete block design.*

PROOF. The B-vector of L has one non-zero component b_k only. Let $R = (r_i)$ be an R-vector relative to any variety x of L. Since $0 \leqq r_i \leqq b_{i+1}$, there is only one non-zero component $r_{k-1} = r$ of R. Thus, by (2), $(k - 1)r = \lambda(v - 1)$; r is independent of the choice of x; and the theorem follows.

Thus axiom B_2 for balanced incomplete block designs is redundant.

4. (r, λ)-systems. It is of interest to investigate systems similar to λ-systems in which every variety occurs in precisely r blocks. Let us define an (r, λ)-system as a set V of v varieties and a collection of b subsets of V called blocks, satisfying the following axioms:

R_1 : every pair of varieties occurs in precisely λ blocks,

R_2 : every variety occurs in precisely r blocks,

$R_3 : r > \lambda \geqq 0$.

In a BIBD, $k < v$ is equivalent to $r > \lambda$; thus block designs are special cases of (r, λ)-systems. However, λ-systems do not include (r, λ)-systems, since blocks consisting of a single element are permitted in the latter. (r, λ)-systems can be used to investigate embedding, isomorphism, and other properties of BIBD's. B-vectors and R-vectors relative to some variety are defined as for λ-systems, but, in this case, b_1 and r_0 are not necessarily zero. The conditions (1) and (2) still hold; however, we now have

(3) $$\sum_{i=0}^{\infty} r_i = r.$$

The total number of blocks b is given by

(4) $$b = \sum_{i=1}^{\infty} b_i.$$

Also

(5) $$vr = \sum_{i=1}^{\infty} ib_i,$$

since each side represents the total number of occurrences of all varieties in the blocks. The following inductive principle for BIBD's is analogous to Theorem 1.

THEOREM 3. *An (r, λ)-system S with B-vector (b_i), where $i = 1, 2, \cdots$, exists if and only if there exists a sequence of non-negative integers $r_0, r_1, \cdots$, satisfying $\sum_{i=0}^{\infty} r_i = r$ and an (r, λ)-system S' with B-vector (b_i^*), $b_i^* = b_i + r_i - r_{i-1}$ $(i = 1, 2, \cdots)$, which contains a set C of blocks, with r_i blocks of size i, such that every variety of S' occurs in precisely λ blocks of C.*

PROOF. Clearly the condition is necessary. The converse is obtained by adjoining a new variety x to every block of C, then adding r_0 blocks consisting of x itself to S' to get the required system.

5. Applications. So far we have discussed B-vectors and R-vectors only as they arise from λ-systems and (r, λ)-systems. For given positive integers r and λ, let us refer to any vector of non-negative integers as a B-vector if it satisfies (1) and (5) for some integer v, and to any such vector which satisfies (2) and (3) for the same value of v as an associated R-vector.

Let us also say that a B-vector β^* is derived from a B-vector β if there is an associated R-vector (r_i) of $\beta = (b_i)$ such that

$$\beta^* = (b_i + r_i - r_{i-1}), \quad \text{where} \quad i = 1, 2, 3, \cdots.$$

Define a partial ordering on B-vectors β_i (for given r and λ) as follows: $\beta_m \leqq \beta_n$ if $\beta_m = \beta_n$, or if there exists a sequence of B-vectors $\beta_m, \beta_a, \beta_b, \cdots, \beta_n$ such that each B-vector is derived from its successor. Now we determine when there is an (r, λ)-system corresponding to a given B-vector (λ-systems can be dealt with similarly).

Let $P = (\beta_m \leqq \beta_a \leqq \beta_b \leqq \cdots \leqq \beta_n)$ be a maximal chain between β_m and β_n. We say a system L with B-vector β_m can be *integrated* along the chain P to a system with B-vector β_n if to each member of the sequence (except the first) there corresponds a system which can be obtained from the system corresponding to its predecessor by means of Theorem 3 and which can be extended to a correspondent of its successor (except for the last) also by Theorem 3. Clearly a system L_m with B-vector β_m can be embedded in a system L_n with B-vector β_n if and only if L_m can be integrated along some chain from β_m to β_n. As a special case, a design exists with B-vector β if and only if the design $L_1(r)$ consisting of one variety replicated r times can be integrated from (r) to β, where (r) is the B-vector $(r, 0, 0, 0, \cdots)$. (We adopt the usual convention of omitting the zeros to the right of the last non-zero entry). Clearly a necessary condition for the existence of a system corresponding to β is that $(r) \leqq \beta$. It can be shown that this rules out the existence of (20, 6)-systems for the following B-vector with $v = 3$: (27, 15, 1).

For small systems, one can check uniqueness by performing all integrations along all paths (i.e., maximal chains) from (r) to β, and identifying isomorphic members. For the convenience of the reader, the uniqueness of the Fano geometry is proved by way of example in Section 7. For slightly larger systems it is more reasonable to look for existence, since the number of paths may increase greatly as the size vr of the system increases.

When it is known that a certain system must have a given subconfiguration, it is only necessary to check the paths through the B-vector of the subconfiguration. In some instances, such as projective planes, one chain may be necessary and sufficient. However, since the integration time increases rapidly with v, the plane of order 10 (with $v = 11$) is probably beyond reach by this method.

To test the method, a Fortran programme was written which constructed all BIBD's with parameters (8, 14, 7, 4, 3) (the parameters are in standard order $(v\ b\ r\ k\ \lambda)$), since this was the first member of the family $H_2(x)$, discussed in [5], which admitted the possibility of more than one design for the given parameters. The answer was checked by theoretical methods in that case. Four solutions were found, and it was shown in [6] that this is indeed the correct number.

6. Properties of (r, λ)-systems. Let S^* be a subset of the variety set V of an (r, λ)-system L. Restriction of L to S^* is obtained by expunging all elements of $V - S^*$. Clearly a restriction of an (r, λ)-system is an (r, λ)-system. If a system L^* is isomorphic to a restriction of a system L, we say that L^* is embedded in L.

THEOREM 4. *A* BIBD *with given k, r, and λ can be embedded in another (r, λ)-system only if k divides $r - \lambda$.*

PROOF. In order to be embeddable, D must be embeddable in a one-element extension of D. The B-vector of D has only one non-zero element $b_k = b$. By the condition for extensibility, any R-vector for the extension has only two non-negative components $x = r_k$ and $r_0 = r - x$. Also

$$xk = \lambda v,$$

$$r(k - 1) = \lambda(v - 1).$$

Thus $x = k^{-1}(rk - r + \lambda)$ and $k \mid (r - \lambda)$. Let $r - \lambda = nk$. Then $x = r - n$ and $r_0 = n > 0$.

A remarkable theorem by Ryser [3] proves that if in an (r, λ)-system $v = b$, then $\lambda(v - 1) = r(r - 1)$, and the system is a BIBD with $k = r$. We may use this theorem to give an inductive proof of the Fisher inequality.

THEOREM 5. *In any (r, λ)-system, $b \geqq v$.*

PROOF. The theorem is trivial for $v = 1$, and any r and λ. Let us assume there is an (r, λ)-system with $v \geqq 2$, such that $v > b$. Let us choose a system L with the least value of v for which the theorem fails. Then, in L, $v = b + 1$; for otherwise removal of a variety would produce a system with a smaller value for v in which

$v > b$, which is contrary to the choice of L. Similarly removing a variety from L must produce a system in which $v = b$. But such a system is a BIBD with $k = r$. Thus k does not divide $r - \lambda$, and the BIBD cannot be extended to produce L. This contradiction establishes the theorem.

Let us call (r, λ)-systems elliptic, parabolic, or hyperbolic accordingly as the expression $\lambda(v - 1) - r(r - 1)$ is negative, zero, or positive. All BIBD's are either elliptic or parabolic, and are parabolic only if they are symmetric, that is, if $b = v$.

Let us note that from relations (1), (4) and (5), we obtain

$$(6) \qquad 0 \leqq \sum_{i=1}^{\infty} (i - r)^2 b_i = v[\lambda(v - 1) - r(2r - 1)] + r^2 b.$$

Thus for elliptic designs we have the stronger inequality

$$b \geqq \{[r(2r - 1) - \lambda(v - 1)]/r^2\}v.$$

If we ask whether the condition $\lambda(v - 1) = r(r - 1)$ is enough to guarantee that $v = b$; we find that such is not the case, as shown by the following example: 1234567, 124, 235, 346, 457, 561, 672, 713, in which $\lambda = 2$, $r = 4$, and $v = 7$.

However this system contains a complete block, which, when removed, produces the Fano geometry.

Henceforth, for convenience, the term design refers to an (r, λ)-system unless otherwise stated. Let us call a design irreducible if it contains neither a complete block nor a set of v single element blocks whose union is V. Otherwise it is reducible. By removing complete blocks and/or units of single blocks from a reducible design, one obtains either a proper irreducible design or the null design; if the latter occurs, the original design is called trivial. We make the following conjectures:

CONJECTURE 1. *For $\lambda \leqq 2$ (and perhaps all λ), $\lambda(v - 1) = r(r - 1)$ implies $v = b$ if the corresponding design is irreducible.*

CONJECTURE 2. *For $\lambda \leqq 2$ (and perhaps all λ), all hyperbolic systems are reducible.*

We prove the conjectures for $\lambda = 1$.

LEMMA 1. *Let l be the longest block in a design with $\lambda = 1$ which contains no complete block. Then $r \geqq l$.*

PROOF. Let B_1 be a block of length l, and x be a variety not in B_1. Then every variety of B_1 must occur with x, and since no pair of varieties from B_1 can appear together outside B_1, there must be an occurrence of x for each member of B_1. Thus $r \geqq l$.

THEOREM 6. *If in a $(r, 1)$-system, $\lambda(v - 1) = r(r - 1)$, the design is trivial or symmetric.*

PROOF. If the design contains a complete block and $\lambda = 1$, it is trivial. Let us assume that there is no complete block. Let $R_w = (r_0, r_1, \cdots)$ be the r-vector associated with the variety w. By the above lemma, $r_i = 0$ for $i > r - 1$. Thus

$$\sum_{i=1}^{r-1} i r_i = \lambda(v - 1) = r(r - 1),$$

which together with $\sum_{i=0}^{r-1} r_i = r$, implies $r_i = r$, and all blocks containing any variety w contain r varieties. But every non-empty block contains at least one variety. Thus all blocks have r varieties, and the system is a BIBD with $k = r$. Thus $v = b$.

COROLLARY. *Any hyperbolic $(r, 1)$-system is reducible. Select any $(r, 1)$-system L with $v > r^2 - r + 1$ varieties.*

PROOF. If $r = 1$, the result is trivial. Let us assume $r > 1$. Select a set S of $r^2 - r + 1$ varieties. The restriction S^* of L to S must be trivial since, by Theorem 4, it cannot be symmetric. Thus S^* contains a complete block B^*. Let w be a variety of $V - S$. If w is not adjoined to B^* in the restriction of L to $S \cup w$, it must occur in $r^2 - r + 1$ single blocks of S^*, and since w only occurs r times, $r \geqq r^2 - r + 1$. But for $r \neq 1$, $r^2 - r + 1 > r$. Therefore we have a contradiction. Hence every element w of $V - S$ belongs to the extension of B^* in L, that is, all hyperbolic designs with $\lambda = 1$ are trivial, and thus reducible.

This establishes the validity of the conjectures for $\lambda = 1$.

The following example shows that Lemma 1 is not valid for $\lambda = 2$. Consider the set V of $v = 2z + 1$ varieties $(\infty, 1, 2, \cdots z, 1^*, 2^*, \cdots, z^*)$. Form the following blocks:

$$V - \infty,$$

$$(\infty, 1, 2, \cdots, z),$$

$$(\infty, 1^*, 2^*, \cdots, z^*),$$

$$(\infty, i, i^*), \qquad i = 1, 2, \cdots, z;$$

$$(i, j^*), \qquad i \neq j.$$

In such a system $\lambda = 2$ and $r = z + 2$, whereas l, the length of the longest block is $2z$. Thus, if $z > 2$, $l > r$. None of the above systems is embeddable in a BIBD, although all are elliptic. Indeed the above example serves as counterexample for various conjectures.

7. Uniqueness of the Fano geometry. This section is included merely to illustrate the methods of this paper by proving the well-known and somewhat trivial result that the Fano geometry or symmetric BIBD (7, 3, 1) is unique up to isomorphism. Since the design contains 7 blocks of 3 varieties, it has B-vector $b_1, b_2, b_3) = (0, 0, 7)$, which seems appropriate in the year 1966. Any associated R-vector (r_0, r_1, r_2) must satisfy

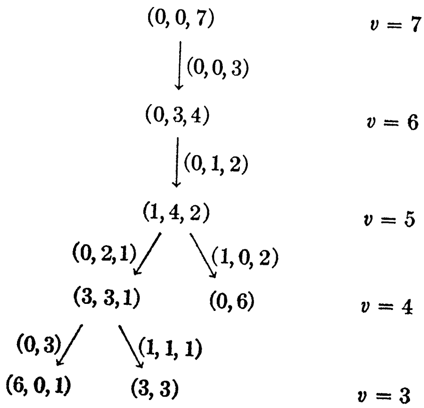
(a) $r_0 + r_1 + r_2 = 3$,

(b) $r_1 + 2r_2 = 6$,

(c) $0 \leqq r_0 \leqq 0, 0 \leqq r_1 \leqq 0, 0 \leqq r_2 \leqq 7$.

Hence the only such R-vector is $(0, 0, 3)$. The derived B-vector is $(0, 3, 4)$, and corresponds to $v = 6$.

Continuing in this way, one obtains the following graph.



Instead of continuing the graph down to (3), we stop with B-vectors (0, 6), (6, 0, 1), and (3, 3), for which the corresponding designs exist and are clearly unique.

Indeed, (0, 6) must correspond to all possible pairs of four elements, whereas (6, 0, 1) corresponds to 123, 1, 2, 3, 1, 2, 3, and (3, 3) corresponds to 12, 23, 31, 1, 2, 3.

Extending these designs in accordance with Theorem 3, we obtain 123, 14, 24, 34, 1, 2, 3 and 124, 13, 23, 43, 1, 2, 4 respectively; these are isomorphic under the interchange (3, 4).

The above design extends in essentially only one way to 123, 145, 24, 34 1, 25, 35, with B-vector (1, 4, 2). Removal of the element 1 gives a representation of the design corresponding to (0, 6). However, this design is completely symmetric and can be extended in only one way; thus the design corresponding to (1, 4, 2) is essentially unique.

The R-vector for extension of (1, 4, 2) to (0, 3, 4) demands that variety 6 be paired with variety 1 to yield a block of length 2. Because of the symmetry between varieties 2 and 3, and varieties 4 and 5, we may, without loss of uniqueness, also assume that we form blocks 246 and 356. The final stage of extension is now completely determined and produces the final result 123, 145, 246, 347, 167, 257, 356. This example, although trivial, serves to illustrate the general method.

REFERENCES

[1] Atiqullah, M. (1961). On a property of balanced designs. *Biometrika* **48** 215–218.
[2] Rao, V. R. (1958). A note on balanced designs. *Ann. Math. Statist.* **29** 290–294.
[3] Ryser, H. J. (1950). A note on a combinatorial problem. *Proc. Amer. Math. Soc.* **1** 422–424.
[4] Sprott, D. A. (1962). Listing of BIB designs from $r = 16$ to $r = 20$. *Sankhyā Ser. A* **24** 203–204.
[5] Stanton, R. G. and Sprott, D. A. (1964). Block intersections in BIBD's *Canada Math. Bull.* **7** 539–548.
[6] Stanton, R. G. and Mullin, R. C. (1966). Uniqueness theorems in BIBD's. Submitted for publication.