ON COMBINABILITY OF INFORMATION FROM UNCORRELATED LINEAR MODELS BY SIMPLE WEIGHTING[1]

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0. Summary. Within the set of all linear parametric functions, $\lambda'\beta$, estimable from either or both of two uncorrelated sets of data, $y_1 = X_1\beta + e_1$ and $y_2 = X_2\beta + e_2$ with known non-singular variances, a general characterization is presented of those $\lambda'\beta$'s for which the best linear unbiased estimator (b.l.u.e.) is obtainable from one source of information alone or by simple weighting of respective b.l.u.e.'s from each of the two sources. It is shown that if the intersection of the row spaces of X_1 and X_2 has rank r then in the intersection space there are exactly r independent λ' vectors for which the b.l.u.e. of $\lambda'\beta$ is obtainable by simple weighting. Some related statements are made for $k > 2$ uncorrelated sources of information.

In the case of incomplete block designs the reduced intrablock normal equations and the interblock normal equations may be regarded as originating from two uncorrelated sources of information on the treatment parameter vector τ. It is shown that an estimable treatment contrast, $\gamma'\tau$, is best estimated from one source alone or by simple weighting of b.l.u.e.'s from the respective sources if and only if γ is an eigenvector of $\Lambda = (\lambda_{ij})$, where λ_{ij} is the number of times treatments i and j occur together in a block. For symmetric factorial or quasifactorial designs, it is shown that any effect or interaction degree of freedom contrast is an eigenvector of Λ, and hence is best estimated by simple weighting of its interblock and intrablock estimates.

1. Introduction. Consider two uncorrelated sets of data, $y_i = X_i\beta + e_i$, $i = 1, 2$, where y_i is an $n_i \times 1$ column of observations, X_i is an $n_i \times p$ design matrix, β is a $p \times 1$ column of parameters and e_i is a $n_i \times 1$ column of errors with means zero and known non-singular $n_i \times n_i$ variance matrices V_i. A linear combination, $\sum_{j=1}^{p} \lambda_j\beta_j = \lambda'\beta$, of the parameters is said to be estimable in set i if and only if there is a linear combination $\sum_{j=1}^{n_i} a_j y_{ij} = a'y_i$ whose expectation is $\lambda'\beta$, i.e., $a'X_i\beta = \lambda'\beta$ identically in β. Thus $\lambda'\beta$ is estimable in set i if and only if λ' is a linear combination of the rows of X_i, i.e., λ' is in the row space of X_i denoted by χ_i.

Let y represent the combined data with variance matrix V where

$$(1.1) \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \beta + \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = X\beta + e \quad \text{and} \quad V = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}.$$

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It is well-known (see for example Chapter 5 of [9]) that the b.l.u.e. of a $\lambda'\beta$ estimable from the combined data $y = X\beta + e$ is given by $\lambda'\beta^* = \rho'X'V^{-1}y$ where ρ is any solution to the conjugate normal equation $X'V^{-1}X\rho = \lambda$. Similarly, if they exist, we may obtain the separate b.l.u.e.'s of $\lambda'\beta$ from the first and second sources and denote them by $\lambda'\hat{\beta}$ and $\lambda'\tilde{\beta}$ respectively. It is also well-known that here β^*, $\hat{\beta}$, and $\tilde{\beta}$ are any solutions to their corresponding normal equations, i.e., any vectors such that $X'V^{-1}X\beta^* = X'V^{-1}y$, $X_1'V_1^{-1}X_1\hat{\beta} = X_1'V_1^{-1}y_1$ and $X_2'V_2^{-1}X_2\tilde{\beta} = X_2'V_2^{-1}y_2$.

DEFINITION 1.1. An estimable parametric function $\lambda'\beta$ is said to be best combinable by simple weighting (b.c.s.w.) if

$$\lambda'\beta^* = w\lambda'\hat{\beta} + (1 - w)\lambda'\tilde{\beta}, \quad 0 < w < 1, \qquad \text{or} \qquad \lambda'\beta^* = \lambda'\hat{\beta} \qquad \text{or} \qquad \lambda'\beta^* = \lambda'\tilde{\beta}.$$

The extension of this definition to $k > 2$ uncorrelated sources of information is obvious.

The discussion in standard theory texts, [2] and [5], and the literature, [4] and [8], of the simple weighting of interblock and intrablock information on treatment contrasts in b.i.b. and factorial designs prompted a deeper investigation of $\lambda'\beta$'s enjoying the b.c.s.w. property in incomplete block designs. It was natural to subsume this investigation to the case of any two known matrices X_1 and X_2, a problem of general interest. Sprott [8] established conditions under which sets of treatment effects or differences, estimable in both the interblock and intrablock sources of information, are b.c.s.w. However, his methods relied on heavy algebraic manipulations of the solutions to the normal equations and did not bring out the essential role played by eigenvectors of the interblock information matrix. Sprott's results, as well as a rigorous justification of the customary procedure of simple weighting of interblock and intrablock estimates of degree of freedom contrasts or interactions in symmetric factorials, follow as a simple consequence of the theory presented in this paper.

2. A general characterization of b.c.s.w. linear parametric functions.

LEMMA 2.1 (Newcomb [7]). *Given two real $p \times p$ positive semidefinite symmetric matrices, say, $X_1'V_1^{-1}X_1$ and $X_2'V_2^{-1}X_2$, with rank $(X_1'V_1^{-1}X_1) = a \leqq$ rank $(X_2'V_2^{-1}X_2) = b$, then there exists a real non-singular matrix T and real diagonal matrices D_a and D_b such that $D_a = T'X_1'V_1^{-1}X_1T$ and $D_b = T'X_2'V_2^{-1}X_2T$ where*

$$(2.1) \qquad D_a = \begin{pmatrix} I_a & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad D_b \begin{pmatrix} u_1 & & & & & & 0 \\ 0 & \ddots & & & & & \\ & & u_r & & & & \\ & & & O_{a-r} & & & \\ & & & & I_{b-r} & & \\ & & & & & & O_{p-a-b+r} \end{pmatrix}$$

and u_i, $i = 1, \cdots, r$ are positive where r is the dimension of $\chi_1 \cap \chi_2$.

Consider the reduction of the design matrices, X_i, to these "canonical coordinates" by

$$(2.2)\qquad \begin{aligned} y_1 &= (X_1T)(T^{-1}\beta) + e_1, & y_2 &= (X_2T)(T^{-1}\beta) + e_2 \\ &= Z_1\tau + e_1 & &= Z_2\tau + e_2 \end{aligned}$$

where we now write $\lambda'\beta = \lambda'T\tau = (T'\lambda)'\tau = \eta'\tau$. The b.l.u.e., $\eta'\tau^*$, of $\eta'\tau$ estimable from the combined data (1.1) is given by

$$(2.3)\qquad \eta'\tau^* = \rho'Z_1'V_1^{-1}y_1 + \rho'Z_2'V_2^{-1}y_2$$

where ρ is any solution to the conjugate normal equation

$$(2.4)\qquad (D_a + D_b)\rho = \eta.$$

If $\eta'\tau$ is separately estimable in each set of data, then the respective b.l.u.e.'s $\eta'\hat{\tau}$ and $\eta'\tilde{\tau}$, are given by

$$(2.5)\qquad \eta'\hat{\tau} = \alpha'Z_1'V_1^{-1}y_1 \text{ and } \quad \eta'\tilde{\tau} = \gamma'Z_2'V_2^{-1}y_2$$

where α and γ are any solutions to the pair of conjugate equations

$$(2.6)\qquad D_a\alpha = \eta \quad \text{and} \quad D_b\gamma = \eta.$$

If $\lambda'\beta = \eta'\tau$ estimable in both sources, is b.c.s.w. then

$$(2.7)\qquad \begin{aligned} \eta'\tau^* &= w\eta'\hat{\tau} + (1 - w)\eta'\tilde{\tau} \\ &= w\alpha'Z_1'V_1^{-1}y_1 + (1 - w)\gamma'Z_2'V_2^{-1}y_2 \\ &= \rho'Z_1'V_1^{-1}y_1 + \rho'Z_2'V_2^{-1}y_2 \end{aligned}$$

from (2.3). Since (2.7) is an identity in y_1 and y_2, $\rho'Z_1' = w\alpha'Z_1'$ and $\rho'Z_2' = (1 - w)\gamma'Z_2'$, and so it follows that η must be such that

$$(2.8)\qquad D_a\rho = w\eta \quad \text{and} \quad D_b\rho = (1 - w)\eta.$$

From (2.1) the only possible solutions to (2.8) are given by the classes of vectors $\rho_i = c(0, \cdot, (1 + u_i)^{-1}, \cdot, 0_{a+b-r}, x \cdot x_p)', i = 1, \cdots, r$, with $(1 + u_i)^{-1}$ in the i position, the x's arbitrary, and zeros elsewhere and with corresponding class $\eta_i = c(0, \cdots, 1_i, \cdots, 0)'$, where $w_i = (1 + u_i)^{-1}$ and c is any scalar $\neq 0$, or by linear combinations of these ρ_i with corresponding u_i all equal, in which case entire subspaces of λ's in $\chi_1 \cap \chi_2$ are such that $\lambda'\beta$ is b.c.s.w.

Conversely, if η is such that the equations (2.8) have a solution then ρ is also a solution of (2.3) and $\eta'\hat{\tau} = w^{-1}\rho'Z_1'V_1^{-1}y_1$ and $\eta'\tilde{\tau} = (1 - w)^{-1}\rho'Z_2'V_2^{-1}y_2$. Hence

$$(2.9)\qquad \begin{aligned} \eta'\tau^* &= \rho'Z_1'V_1^{-1}y_1' + \rho'Z_2'V_2^{-1}y_2 \\ &= w\eta'\hat{\tau} + (1 - w)\eta'\tilde{\tau}. \end{aligned}$$

This discussion leads to the following sequence of theorems.

THEOREM 2.2. *If $\chi_1 \cap \chi_2$ has dimension r, then there are exactly r linearly independent vectors λ' in $\chi_1 \cap \chi_2$ such that $\lambda'\beta$ is* b.c.s.w.

Such vectors λ are given by $\lambda_i = (T^{-1})'\eta_i$, $i = 1, \cdots, r$, where $\eta_i =$

$c(0, \cdots, 1_i, \cdots, 0)$, i.e., the set of linearly independent b.c.s.w. $\lambda'\beta$'s, λ' in $\chi_1 \cap \chi_2$, is given by non-zero scalar multiples of the first r rows of T^{-1}.

THEOREM 2.3. *A linear combination,* $\sum_{i=1}^{k} a_i\lambda_i'\beta$, *of* b.c.s.w. $\lambda'\beta$'s *with* λ_i' *in* $\chi_1 \cap \chi_2$, *and corresponding weight* w_i, *is itself* b.c.s.w. *if and only if* $w_1 = w_2 = \cdots = w_k$.

Consider now the set η_i, $i = r + 1, \cdots, a$, defining a set of parametric functions $\eta'\tau$ estimable in y_1 but not y_2. For such η_i the set of all possible solutions to the combined conjugate equation (2.4) is given by the vectors $\rho_i = (0, \cdots, 0_r, 0, \cdots, 1_i, \cdots, 0_{a+b-r}, x, \cdots, x_p)'$, but it is immediate that this set is also the set of all solutions to the pair of equations (2.8) with $w = 1$. Thus for $\eta_i'\tau$, $i = r + 1, \cdots, a$, it follows from the previous discussion that $\eta_i'\tau^* = \eta_i'\hat{\tau}$. Since $w_i = 1$, $i = r + 1, \cdots, a$, it also follows that the b.l.u.e. of any linear combination, $\sum_{i=r+1}^{a} c_i\eta_i'\tau$, is given by

$$\sum_{i=r+1}^{a} c_i\eta_i'\tau^* = \sum_{i=r+1}^{a} c_i\eta_i'\hat{\tau}. \tag{2.10}$$

Equation (2.10) may be formalized in the following theorem.

THEOREM 2.4. *If* $\chi_1 \cap \chi_2$ *has dimension* r *then there exists an* $a - r$ *dimensional subspace in* $\chi_1 - \chi_2$ *such that for any* λ *in this subspace* $\lambda'\beta^* = \lambda'\hat{\beta}$ *and there exists a* $b - r$ *dimensional subspace in* $\chi_2 - \chi_1$ *such that for any* λ *in this subspace* $\lambda'\beta^* = \lambda'\tilde{\beta}$.

COROLLARY 2.5. *If* $\chi_1 \cap \chi_2 = 0$, *then for every* $\lambda'\beta$ *estimable in source* $i = 1, 2$ *its* b.l.u.e. *is obtainable from that source alone.*

It should be noted that there is no essential reason for D_a to be an upper left identity except that an algorithm exists for this resolution. For any non-singular matrix Q which simultaneously diagonalizes both information matrices, one need only look to the rows of Q^{-1} to determine the b.c.s.w. $\lambda'\beta$'s. The case for Q orthogonal is exploited in Sections 4 and 5.

To simplify notation and facilitate the discussion we shall hereafter, with no real loss of generality, restrict V_i to be of the form $\sigma_i^2 I$. Returning to the original coordinate system, the discussion of Theorems 2.2 and 2.4 establishes the following theorem useful for further development.

THEOREM 2.6. *A necessary and sufficient condition for* $\lambda'\beta$ *to be* b.c.s.w. *is that the set of solutions to the conjugate equation*

$$(X_1'X_2')\begin{pmatrix}(\sigma_1^2)^{-1}I & 0 \\ 0 & (\sigma_2^2)^{-1}I\end{pmatrix}\begin{pmatrix}X_1 \\ X_2\end{pmatrix}\rho = \lambda \tag{2.11}$$

is identical to the set of solutions to exactly one of the 3 *pairs of conjugate equations,*

$$\begin{matrix} X_1'X_1\rho = w\sigma_1^2\lambda & & X_1'X_1\rho = \sigma_1^2\lambda & & X_1'X_1\rho = 0 \\ & & \text{or} & & \text{or} \\ X_2'X_2\rho = (1 - w)\sigma_2^2\lambda & & X_2'X_2\rho = 0 & & X_2'X_2\rho = \sigma_2^2\lambda. \end{matrix} \tag{2.12}$$

We point out that for the purpose of determining the b.c.s.w. $\lambda'\beta$'s, one may arbitrarily assign values to σ_1^2 and σ_2^2, say $= 1$, since in effect only common directions of the image of ρ are the ones of interest. In order to determine the actual weight w it is necessary to know at least the ratio of σ_1^2 to σ_2^2.

Restricting attention to only those λ' in $\chi_1 \cap \chi_2$ one may also state the following formulation.

THEOREM 2.7. *For λ' in $\chi_1 \cap \chi_2$, $\lambda'\beta$ is* b.c.s.w. *if and only if λ is the image under either $X_1'X_1$* or *$X_2'X_2$ of a vector ρ, such that ρ is a generalized eigenvector of the pencil*

$$(X_1'X_1 - kX_2'X_2)\rho = 0 \tag{2.13}$$

for some generalized eigenvalue $k \neq 0$.

We remark that in the original development we derived Theorem 2.6 first, directly from standard linear model considerations, and that in the search for further characterization we have availed ourselves of Lemma 2.1. We have developed the present section beginning with Lemma 2.1 as a result of a simplifying suggestion from Dr. A. P. Dempster.

For the case of $k > 2$ sources of information, it is obvious, by an extension of Theorem 2.6 that if a $\lambda'\beta$ is b.c.s.w. for the entire set of k sources then $\lambda'\beta$ is pairwise b.c.s.w. for any possible pair of sources. The converse is not true, as can be shown by easily constructed counter examples for $k = 3$ (for an example see [6]).

3. The full rank case. If X_1 and X_2 are both of full rank, the theorems and corollaries in Section 2 apply as they stand. The proofs may be simplified.

In [1], Fraser showed that a necessary and sufficient condition for each component β_i of β to be b.c.s.w. is that the information matrix from source one be a diagonal matrix multiple of the information matrix from source two. By application of Theorem 2.3, there will be an s dimensional subspace of b.c.s.w. linear parametric functions if and only if s elements of this diagonal matrix are equal. For any full rank X_1 and X_2 there will always be p independent b.c.s.w. linear parametric functions by Theorem 2.2.

For the case of $k > 2$ full rank sources of information, the condition that $\lambda'\beta$ be b.c.s.w. for all possible pairs is sufficient for $\lambda'\beta$ to be b.c.s.w. throughout the whole set, since for any i, j the solution ρ, to the set of equations

$$\begin{aligned} X_i'X_i\rho &= w_{ij}\lambda \\ X_j'X_j\rho &= (1 - w_{ij})\lambda \end{aligned} \tag{3.1}$$

is unique.

4. Common eigenvectors. If ρ is a common eigenvector of $X_1'X_1$ and $X_2'X_2$, it is an immediate consequence of Theorem 2.6 that $\rho'\beta$ is b.c.s.w. If $\rho_1, \cdots, \rho_r$ is a set of common eigenvectors, it follows from Theorem 2.3 that a linear combination $\sum_{i=1}^{r} a_i\rho_i'\beta$ is b.c.s.w. if and only if the weights of $\rho_i'\hat{\beta}$ and $\rho_i'\tilde{\beta}$ are the same for all $\rho_i'\beta$. We therefore state the following theorems.

THEOREM 4.1. *A sufficient condition that $\lambda'\beta$ be* b.c.s.w. *is that λ be a common eigenvector of $X_1'X_1$ and $X_2'X_2$.*

THEOREM 4.2. *If* $\lambda_1, \cdots, \lambda_r$ *is a set of common eigenvectors of* $X_1'X_1$ *and* $X_2'X_2$ *then* $(\sum_{i=1}^r a_i\lambda_i)'\beta$ *is* b.c.s.w. *if and only if* $k = c_{11}c_{12}^{-1} = \cdots = c_{r1}c_{r2}^{-1}$, *where* c_{i1} *and* c_{i2} *are the eigenvalues of* λ_i *under* $X_1'X_1$ *and* $X_2'X_2$ *respectively.*

Theorems 4.1 and 4.2 apply to the case of $k > 2$ uncorrelated sources of information.

If there exists an orthogonal matrix O which simultaneously diagonalizes $X_1'X_1$ and $X_2'X_2$, then the columns, $\bar{O}_i$, $i = 1, \cdots, p$, of O with non-zero eigenvalues under at least one of the mappings, determine the b.c.s.w. functions $\bar{O}_i'\beta$. Columns of O with respective eigenvalues in the same ratio determine the subspaces of b.c.s.w. parametric functions. In principle, one need only construct any one such matrix O to determine the complete set of b.c.s.w. $\lambda'\beta$'s.

5. Incomplete block designs $(t, r, b, k, \lambda_{ij})$ **with 1 or 0 incidence in a block.** For the purpose of estimating treatment contrasts we may consider the model $y_{ij} = t_j + b_i + e_{ij}$ where b_i and e_{ij} are random and uncorrelated with mean 0 and variances σ_b^2 and σ^2 respectively. Then the data $Y = X\tau + b + e$ may be written according to blocks as

$$\begin{pmatrix} Y_1 \\ \vdots \\ Y_b \end{pmatrix} = \begin{pmatrix} X_1 \\ \vdots \\ X_b \end{pmatrix} \tau + b + e \tag{5.1}$$

with Var (Y) = block diag $[\sigma^2 I_k + \sigma_b^2 J_k^k, \cdots, \sigma^2 I_k + \sigma_b^2 J_k^k]$ where J_m^n is the $m \times n$ matrix of unit elements. We may demonstrate the way in which the usual intrablock and interblock normal equations are regarded as originating from two uncorrelated sources of data as follows. Transform Y by the matrix O = block diag $[O_1, \cdots, O_b]$ where O_i is a $k \times k$ orthogonal matrix with first row $k^{-\frac{1}{2}}J_1^k$ and O_i^* will denote the $k - 1$ remaining rows. Then

$$OY = Z = \begin{pmatrix} O_1 \ Y_1 \\ \vdots \\ O_b \ Y_b \end{pmatrix} = \begin{pmatrix} O_1 \ X_1 \\ \vdots \\ O_b \ X_b \end{pmatrix} \tau + \eta \tag{5.2}$$

where η has mean zero and Var (Z) = block diag $\left[\sigma^2 I_k + \sigma_b^2 \left(\begin{array}{c|c} k & 0 \\ \hline 0 & 0 \end{array}\right) \cdots, \sigma^2 I_k + \sigma_b^2 \left(\begin{array}{c|c} k & 0 \\ \hline 0 & 0 \end{array}\right)\right]$. Permute Z by P to get

$$PZ = W = \begin{pmatrix} W_1 \\ \hline W_2 \end{pmatrix} = \begin{pmatrix} k^{-\frac{1}{2}} B \\ \hline O_1^* \ Y_1 \\ \vdots \\ O_b^* \ Y_b \end{pmatrix} = \begin{pmatrix} k^{-\frac{1}{2}} N' \\ \hline O_1^* \ X_1 \\ \vdots \\ O_b^* \ X_b \end{pmatrix} \tau + \xi \tag{5.3}$$

where B is the $b \times 1$ vector block totals, N is the $t \times b$ incidence matrix, and Var (W) = block diag $[(\sigma^2 + k\sigma_b^2)I_b, \sigma^2 I_{b(k-1)}]$. W_1 and W_2 are two uncorrelated sources exhausting all the available information on τ. The normal equations for W_1 are given by

$$NN'\tau = NB = R \tag{5.4}$$

where R_j is the total yield of blocks containing treatment j. The normal equations for W_2 are

$$\begin{aligned} (\textstyle\sum_{i=1}^{b} X_i'O_i^{*\prime}O_i^{*}X_i)\tau &= \textstyle\sum_{i=1}^{b} X_i'O_i^{*\prime}O_i^{*}Y_i, \text{ i. e.} \\ [\textstyle\sum_{i=1}^{b} X_i'(I - k^{-1}J_k^{k})X_i]\tau &= \textstyle\sum_{i=1}^{b} X_i'(I - k^{-1}J_k^{k})Y_i, \text{ i. e.} \\ (X'X - k^{-1}\textstyle\sum_{i=1}^{b} N_iN_i')\tau &= X'Y - k^{-1}\textstyle\sum_{i=1}^{b} X_i'(B_iJ_1^{k}), \text{ i. e.} \\ (rI - k^{-1}NN')\tau &= T - k^{-1}R = Q \end{aligned} \tag{5.5}$$

where T_j is the total yield under treatment j.

It is clear, as exploited by Kempthorne in [4], that the interblock and intrablock information matrices have exactly the same set of eigenvectors. If ρ is an eigenvector of $NN' = (\lambda_{ij})$ with eigenvalue 0 or rk then $\rho'\beta$ is b.c.s.w. from one source alone. Restricting attention to only those vectors λ' in $\chi_1 \cap \chi_2$ we may characterize the b.c.s.w. $\lambda'\beta$'s by use of (2.13). Thus $\lambda'\beta$ is b.c.s.w. if and only if λ is the image under NN' of a solution ρ of

$$[(rI - k^{-1}NN') - mNN']\rho = 0 \tag{5.6}$$

for some scalar $m \neq 0$. But (5.6) may be rewritten as

$$(I - cNN')\rho = 0 \tag{5.7}$$

where $c = (1 + km)(rk)^{-1}$. Equation (5.7) is an expression of the ordinary eigenvalue equation of NN' in terms of reciprocals of non-zero eigenvalues. Thus we have the following theorems.

THEOREM 5.1. *In incomplete block designs a treatment contrast $\lambda'\tau$, is best estimated from intrablock information alone if and only if $NN'\lambda = 0$, and from interblock information alone if and only if $NN'\lambda = rk\lambda$.*

THEOREM 5.2. *In incomplete block designs, a treatment contrast, $\lambda'\tau$, estimable from both the intrablock and interblock information, is* b.c.s.w. *if and only if λ is an eigenvector of NN'.*

It can be verified by simple computation that, for the treatment effects, $\gamma_1'\tau = \tau_1 - \bar{\tau} = t_1$ and $\gamma_2'\tau = \tau_2 - \bar{\tau} = t_2$, γ_1 and γ_2 are eigenvectors of NN' if and only if NN' has the form

$$NN' = \left(\begin{array}{cc|cccc} r & \lambda & \lambda & \cdots & & \lambda \\ \lambda & r & \lambda & \cdots & & \lambda \\ \hline \lambda & \lambda & r & & & \lambda_{ji} \\ \vdots & \vdots & & \ddots & & \\ & & & & \ddots & \\ \lambda & \lambda & \lambda_{ij} & & & r \end{array}\right), \tag{5.8}$$

in which case γ_1 and γ_2 have the same eigenvalue and form a two dimensional subspace of b.c.s.w. parametric functions. The extension of the verification to b treatment effects is immediate, and leads to the following theorem.

THEOREM 5.3. *A necessary and sufficient condition for every treatment effect in* the set $\{t_1, \cdots, t_b\}$ *to be* b.c.s.w. *from the interblock and intrablock sources of information is that the matrix* $\Lambda = NN'$ *be of the form*

$$(5.9) \qquad \Lambda = \left(\begin{array}{ccc|ccc} r & & \lambda & & & \\ & \ddots & & & \cdot & \lambda \\ \lambda & & r_b & & & \\ \hline & & & r & & \lambda_{ji} \\ & \lambda & & & \ddots & \\ & & & \lambda_{ij} & & r \end{array}\right),$$

where λ_{ij} *is the number of times treatments* i *and* j *occur together in a block.*

COROLLARY 5.4. *If, in an incomplete block design, the treatment effects* $\{t_1, \cdots, t_b\}$ *are* b.c.s.w. *then so is any linear combination of the set.*

COROLLARY 5.5. *In an incomplete block design all treatment effects are* b.c.s.w. *if and only if the design has a* b.i.b. *structure.*

Every treatment difference $t_i - t_j$, $i, j \leqq k$, is b.c.s.w. if and only if the vector $(0, \cdots, 0, 1, 0, \cdots, 0, -1, 0, \cdots, 0)$, with 1 in the ith position and -1 in the jth position, is an eigenvector of Λ. Again, simple arithmetic will establish the following theorem.

THEOREM 5.6. *In an incomplete block design, every treatment difference* $t_i - t_j$, $i, j \leqq b$, *is* b.c.s.w. *if and only if*

$$(5.10) \qquad \Lambda = \left(\begin{array}{ccc|ccc} r & & \lambda & \lambda_{b+1} & \cdot & \lambda_t \\ & \ddots & & \cdot & \cdot & \cdots \\ \lambda & & r_b & \lambda_{b+1} & \cdot & \lambda_t \\ \hline \lambda_{b+1} & \cdot & \lambda_{b+1} & r & & \lambda_{ji} \\ \cdot & \vdots & \cdot & & \ddots & \\ \lambda_t & \cdot\,\cdot & \lambda_t & \lambda_{ij} & & r \end{array}\right).$$

Proofs of Theorems 5.3 and 5.6, restricted to parametric functions estimable in both sources, were established by Sprott [8] using manipulations of solutions to the normal equations.

6. Symmetric factorials in incomplete blocks. Since symmetric factorial designs are resolvable into uncorrelated replicates, the set of individual replicates may be regarded as a set of r sources of information for the estimation of treatment contrasts, and thus yield a total of $2r$ uncorrelated interblock and intrablock sources of information. Each replicate may be viewed as an incomplete block design of Section 5 with the addition of a replicate effect in the model, i.e., $y_{hij} = r_h + t_j + b_i + e_{ij}$. The only modification occurs in $W_{1h} = k^{-\frac{1}{2}}B_h = k^{\frac{1}{2}}J_b^{\,1}r_h + k^{-\frac{1}{2}}N_h{}'\tau + \xi$, so that the corresponding normal equations, similar to (5.4), become

$$(6.1a) \qquad k^2 b r_h + kJ_1^{\,b}N_h{}'\tau = kG_h$$

$$(6.1b) \qquad kN_hJ_b^{\,1}r_h + N_hN_h{}'\tau = R_h$$

where G_h is the total of all yields in replicate h. Eliminating r_h by multiplying (6.1a) by $(kb)^{-1}N_hJ_b^1$ and subtracting the product from (6.1b) we obtain the equivalent system (6.1a) and

$$(6.2) \qquad (N_hN_h' - b^{-1}J_t^t)\tau = R_h - b^{-1}G_h .$$

We note that whenever the right-hand side of (6.1a) is used nontrivially in forming a linear estimator, there cannot result an unbiased estimator of a treatment contrast. Equation (6.1a) is irrelevant for estimating treatment contrasts. Since any contrast $\gamma'\tau$ is an eigenvector of J_t^t with eigenvalue zero, a contrast is an eigenvector of the interblock information matrix in (6.2) if and only if it is an eigenvector of N_hN_h'. For this reason we shall simply refer to N_hN_h' as the interblock information matrix.

A treatment contrast, $\gamma'\tau$, is said to be completely confounded in a replicate if the components of γ, corresponding to each group of treatments occurring together in a block, are equal within each group. Again, a treatment contrast, $\gamma'\tau$, is said to be completely unconfounded in a replicate if the sum of the components of γ, corresponding to a group of treatments occurring together in a block, is zero for every block.

The interblock information matrix, $N_hN_h' = \lambda_{uv}^h$ for the hth replicate consists of ones on the main diagonal and zeros and ones off the main diagonal in such a way that the sum of elements of any row is the block size k. For row u, corresponding to τ_u, λ_{uv}^h, $v = 1, \cdots, t$, is unity if and only if the vth treatment occurs in the block containing the uth treatment. Thus the unit elements in the rows of N_hN_h' correspond to the groups of treatment combinations appearing in a block. If $\gamma'\tau$ is completely confounded, the product of the uth row of N_hN_h' and γ is equal to kl_u, where l_u is the component of γ corresponding to the uth treatment in τ. Hence $N_hN_h'\gamma = k\gamma$. If $\gamma'\tau$ is completely unconfounded the product of the uth row of N_hN_h' and γ is zero, so $N_hN_h'\gamma = 0$. Hence, in either case γ is an eigenvector of N_hN_h'.

We note that if the confounding scheme in a symmetrical factorial design is by full sets of effect and interaction degrees of freedom then any effect or interaction degree of freedom contrast, $\gamma'\tau$, is either completely confounded or unconfounded in a replicate. Thus for any replicate, γ is an eigenvector of N_hN_h'. Since the interblock information matrix, NN', for the whole design is the sum of the individual replicate matrices, $N_hN_h'\cdot\gamma$ is an eigenvector of NN'.

By the extension of Theorem 4.1 to the $2r$ interblock and intrablock sources, any effect or interaction degree of freedom contrast is b.c.s.w. from all $2r$ sources in the design. Thus we have derived the following desired result.

THEOREM 6.1. *In a symmetric factorial design employing complete confounding of full sets of effect or interaction degrees of freedom within each replicate, the coefficient vector of any effect or interaction degree of freedom contrast, $\gamma'\tau$, is an eigenvector of the interblock information matrix, N_hN_h', of each replicate, and $\gamma'\tau$ is* b.c.s.w. *from all $2r$ sources of interblock and intrablock information.*

The above theorem is applicable to the case where the block size, although constant within each replicate, may vary from one replicate to another.

It is interesting to note that Theorem 6.1 can also be deduced from notions developed by Jones in [3], wherein he was investigating the resolvability of observed contrasts into intrablock and interblock observed contrasts.

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