STATISTICAL PROPERTIES OF THE NUMBER OF POSITIVE SUMS

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1. Introduction and summary. Let $X_1, \cdots, X_n$ be independent random variables having a common, continuous distribution function (df) F and define

$$S_k = S_k(X) = \sum_{j=1}^{k} X_j, \qquad k = 1, \cdots, n, \tag{1.1a}$$

$$S_V = S_V(X) = \sum_{j \varepsilon V} X_j, \qquad \varnothing \neq V \subseteq \{1, \cdots, n\}, \tag{1.1b}$$

$$M_n = M_n(X) = \sum_{k=1}^{n} e(S_k), \tag{1.1c}$$

$$N_n = N_n(X) = \sum_{V \neq \varnothing} e(S_V), \tag{1.1d}$$

where $X = (X_1, \cdots, X_n)$, e is the indicator function of $(0, \infty)$, and $\varnothing$ denotes the empty set. Recently, Kraft and van Eeden [9] have pointed out that since

$$P(M_n = k) = 4^{-n}\binom{2k}{k}\binom{2n-2k}{n-k}, \qquad k = 0, \cdots, n, \tag{1.2a}$$

$$P(N_n = k) = 2^{-n}, \qquad k = 0, \cdots, 2^n - 1, \tag{1.2b}$$

if F is symmetric about zero (in the sense that $F(x) = 1 - F(-x)$, $-\infty < x < \infty$), both M_n and N_n may be used to test the hypothesis H_0 which specifies that F is symmetric about zero. They also considered the consistency of such tests. The present paper gives some further sufficient conditions for the consistency of tests based on M_n and N_n and computes a measure of their asymptotic relative efficiency with respect to each other. The latter, of course, involves finding the asymptotic distributions of M_n and N_n under a sequence of local alternatives. In a final section the asymptotic properties of some confidence intervals and point-estimates based on M_n and N_n are considered.

The alternatives of interest specify that X_1 is stochastically larger than a symmetric random variable in the sense that

$$F(x) \leqq G(x), \qquad -\infty < x < \infty, F \neq G, \tag{1.3}$$

for some G which is symmetric about zero. The tests considered will be denoted by φ_n and δ_n and reject for large values of M_n and N_n respectively.

It should be noted that (1.2a) does not require the continuity of F; in fact, none of our results in Sections 2 and 3 which concern only M_n or φ_{nd} do.

2. Consistency. It is shown in [9] that neither φ_n nor δ_n is consistent when F has derivative

$$F'(x) = 1/\pi(1 + (x - \mu)^2), \qquad -\infty < x < \infty, \tag{2.1}$$

where $\mu > 0$. In this section we remark that the tests will be consistent against

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alternatives F for which $x(1 - F(x) + F(-x)) \to 0$ as $x \to \infty$. Indeed, the condition is necessary and sufficient for the weak law of large numbers to hold—i.e. for $|(1/n)S_n - \mu_n| \to 0$ in probability as $n \to \infty$ with

$$(2.2) \quad \mu_n = \int_{-n}^{n} x\, dF(x) = n(F(n) + F(-n) - 1) - \int_0^n (F(x) + F(-x) - 1)\, dx.$$

(See [5], p. 232.) If, in addition, F satisfies (1.3), then the integral on the right-hand side of (2.2) is negative and bounded away from zero for n sufficiently large so that $\mu_n \geqq \mu > 0$ for n sufficiently large. It follows that $P(S_n > 0) \to 1$ as $n \to \infty$ and therefore that

$$(2.3) \quad (1/n)E(M_n) = (1/n) \sum_{k=1}^{n} P(S_k > 0) \to 1,$$
$$2^{-n}E(N_n) = 2^{-n} \sum_{k=1}^{n} \binom{n}{k} P(S_k > 0) \to 1$$

as $n \to \infty$. Since (2.3) clearly implies the consistency of φ_n and δ_n, the assertion is established.

3. Asymptotic distribution of M_n. For each n let $X_{n1}, \cdots, X_{nn}$ be independent random variables having a common df F_n (not necessarily continuous) for which

$$(3.1a) \quad \mu_n = \int x\, dF_n(x) = \mu n^{-\frac{1}{2}} + o(n^{-\frac{1}{2}}),$$

$$(3.1b) \quad \sigma_n^2 = \int x^2\, dF(x + \mu_n) \to \sigma^2, \qquad 0 < \sigma^2 < \infty,$$

$$(3.1c) \quad \int_{|x| \geqq \epsilon n^{\frac{1}{2}}} x^2\, dF(x + \mu_n) \to 0, \quad \text{for all } \epsilon > 0,$$

as $n \to \infty$. For $n = 1, 2, \cdots$ let

$$(3.2) \quad S_{nk} = \sum_{j=1}^{k} X_{nj}, \qquad k = 1, \cdots, n, S_{n0} = 0,$$

$$(3.3) \quad X_n(t) = (1/n^{\frac{1}{2}}\sigma)S_{nk} \quad \text{if} \quad k - 1 \leqq nt < k,\ k = 1, \cdots, n-1$$
$$= (1/n^{\frac{1}{2}}\sigma)S_{nn} \quad \text{if} \quad 1 - (1/n) \leqq t \leqq 1,$$

$$(3.4) \quad X_0(t) = W(t) + \mu t, \qquad 0 \leqq t \leqq 1,$$

where $W(t)$ is a separable Wiener process with parameter 1. Denote by D the complete metric space whose elements are equivalence classes of functions defined on [0, 1] which have discontinuities of the first kind only (the reader is referred to [12] for a discussion of this space and its properties); and let Q_n denote the distribution induced in D by $X_n(t)$ for $n = 0, 1, \cdots$. If we now define a functional L on D by

$$(3.5) \quad L(f) = \int_0^1 e(f(t))\, dt = m(f^{-1}(0, \infty)), \qquad f \varepsilon D,$$

where e is as in (1.1) and m denotes Lebesgue measure, we have

LEMMA 3.1. *L is continuous almost everywhere with respect to Q_0.*

PROOF. Let C denote the subset of D consisting of those $f \varepsilon D$ which are continuous; it is well-known that $Q_0(C) = 1$([3], p. 393). Since $m(f^{-1}(0)) \geqq 0$ for all f and

$$(3.6) \qquad \int m(f^{-1}(0))\, dQ_0 = \int_0^1 Q_0(\{f: f(t) = 0\})\, dt = 0,$$

it is clear that the set $C_1 = \{f \varepsilon C: m(f^{-1}(0)) = 0\}$ also has Q_0-probability one. And if $f_n \varepsilon D$, $n \geqq 1$, and $f_n \to f \varepsilon C_1$ in the topology of D as $n \to \infty$, then by Theorem 4 of Appendix 1 in [12] it follows that $f_n(t) \to f(t)$ as $n \to \infty$ for each $t \varepsilon (0, 1)$. Since $m(f^{-1}(0)) = 0$, $e(f_n(t)) \to e(f(t))$ as $n \to \infty$ a.e. with respect to Lebesgue measure on [0, 1], so that by Lebesgue's bounded convergence theorem $L(f_n) \to L(f)$. QED

LEMMA 3.2. *Let* (3.1) *be satisfied. Then for* $0 < y < 1$,

$$\lim P(M_n(X_{n1}, \cdots, X_{nn}) \leqq yn) = Q_0(\{f: L(f) \leqq y\}).$$

PROOF. (3.1) implies that $(1/n^{\frac{1}{2}}\sigma)S_{nn}$ is asymptotically normal with mean $\mu\sigma^{-1}$ and variance one ([8], pp. 101–103). Thus, in view of the facts that $X_{n1}, \cdots, X_{nn}$ are identically distributed for each n and that the discontinuities of $X_n(t)$ are equally spaced for each n, it follows from Theorem 3.2 of [12] and the ensuing discussion that $Q_n \to Q_0$ weakly as $n \to \infty$. The present lemma is therefore a direct consequence of the previous one and the definition of weak convergence. QED

The significance of Lemma 3.2 is that in order to compute the asymptotic distribution of $(1/n)M_{nn}$ for all sequences of df's satisfying (3.1), it suffices to do so for one. A particularly simple sequence is obtained by letting $Y_{n1} = \pm\sigma$ with probabilities $(1 \pm \mu/n^{\frac{1}{2}}\sigma)/2$ respectively. Moreover, each of the finite sets of random variables $Y_{n1}, \cdots, Y_{nn}$ may be extended to be an entire sequence of independent, identically distributed random variables $Y_{n1}, Y_{n2}, \cdots$ ([3], p. 71). Having done this, let $M_{nk} = M_k(Y_{n1}, \cdots, Y_{nk})$ for $k = 1, 2, \cdots$ and $n > \mu^2/\sigma^2$. Then, as is shown in [2],

$$(3.7) \qquad P(M_{nk} = j) = P(M_{nj} = j)P(M_{n(k-j)} = 0),$$

for $1 \leqq j \leqq k$ and $n > \mu^2/\sigma^2$. The virtue of (3.7) is that the factors on the right may be related to known first-passage time probabilities.

LEMMA 3.3. *If* $0 \leqq \mu < n^{\frac{1}{2}}\sigma$ *and* $k \geqq 2$, *then*

(i) $P(M_{nk} = k) = \mu/n^{\frac{1}{2}}\sigma + \sum_{2j \geqq k-1} (j + 1)^{-1}\binom{2j}{j}[(1 - \mu^2/n\sigma^2)/4]^{j+1}$,

(ii) $P(M_{nk} = 0) = [(1 + \mu/n^{\frac{1}{2}}\sigma)/2] \cdot \sum_{2j \geqq k} (j + 1)^{-1}\binom{2j}{j}[(1 - \mu^2/n\sigma^2)/4]^{j}$.

PROOF. Let $p(n, k)$ denote the probability that the first passage through $+\sigma$ by the partial sums S_{nj}, $j = 1, 2, \cdots$, takes place at time k and notice that the event $M_{nk} = 0$ occurs iff there is no first passage through $+\sigma$ by time k. Since $\mu \geqq 0$, there is a first passage wp one so that

$$(3.8) \qquad P(M_{nk} = 0) = \sum_{j=k+1}^{\infty} p(n, j).$$

On substituting for $p(n, j)$ its value, as given for example in [4], p. 323, and simplifying, one proves (ii). (i) follows by a similar argument after conditioning on X_1. QED

LEMMA 3.4. *Let $k_n = k_n(y)$ be the greatest integer in yn for $n = 1, 2, \cdots$. Then*

$$(3.9)\quad \lim n^{\frac{1}{2}} \sum_{2j \geqq k_n} (j+1)^{-1}\binom{2j}{j}[(1-\mu^2/n)/4]^j = (\pi)^{-\frac{1}{2}} \int_{y/2}^{\infty} w^{-\frac{3}{2}} \exp(-\mu^2 w)\, dw$$

uniformly in y on compact subsets of $(0, 1)$.

PROOF. Since the detailed proof of Lemma 3.4 is both routine and tedious, it will be omitted. Notice, however, that if $j \sim wn$ as $n \to \infty$, then

$$(3.10)\quad n^{\frac{3}{2}}(j+1)^{-1}\binom{2j}{j}[(1-\mu^2/n)/4]^j \sim (\pi)^{-\frac{1}{2}} w^{-\frac{3}{2}} \exp(-\mu^2 w).$$

For $\mu \geqq 0$ and $0 < y < 1$, define

$$(3.11)\quad \begin{aligned} f(y;\mu) &= (8\pi)^{-\frac{1}{2}} \int_{y/2}^{\infty} w^{-\frac{3}{2}} \exp(-\mu^2 w)\, dw \\ g(y;\mu) &= \mu 2^{\frac{1}{2}} f(1-y;\mu) + f(y;\mu) f(1-y;\mu); \end{aligned}$$

and for $\mu < 0$ let $g(y;\mu) = g(1-y;-\mu)$, $0 < y < 1$.

THEOREM 3.1. *Let* (3.1) *be satisfied. Then for* $0 < y < 1$,

$$\lim P(M_n(X_{n1}, \cdots, X_{nn}) \leqq yn) = \int_0^y g(w;\mu\sigma^{-1})\, dw.$$

PROOF. By Lemma 3.2 it suffices to establish the limiting relation in the special case that $X_{n1} = Y_{n1}$ for $n > \mu^2/\sigma^2$. If $\mu > 0$, then it follows from the preceding two lemmas, (3.7), (3.10), and (3.11) that $\lim_n P(M_{nn} = k_n(y)) = g(y{:}\, \mu\sigma^{-1})$ uniformly in y on compact subsets of $(0, 1)$. Thus we obtain

$$(3.12)\quad \lim P(ny_1 < M_{nn} \leqq ny_2) = \int_{y_1}^{y_2} g(w;\mu\sigma^{-1})\, dw$$

for $0 < y_1 < y_2 < 1$; and by Lemma 3.2 we may let $y_1 \to 0$. For $\mu < 0$ the theorem follows from the obvious analogues of Lemmas 3.3 and 3.4 and the argument given above. QED

COROLLARY 3.1. *Let $w_\alpha = 1 - (\sin(\pi\alpha/2))^2$ where α is the limiting size of φ_n. Then $\beta_\alpha(\mu\sigma^{-1}) =$ (def) $\lim E(\varphi_n(X_{n1}, \cdots, X_{nn})) = \int_{w_\alpha}^1 g(w;\mu\sigma^{-1})\, dw$.*

PROOF. Let a_n' be the least positive integer for which $\varphi_n = 1$ if $M_n > a_n'$. Then it follows easily from the arc sine law, which is a special case of Theorem 3.1, that $a_n'/n \to w_\alpha$ as $n \to \infty$. Corollary 3.1 is an easy consequence. QED

4. Asymptotic distribution of N_n. The main result of this section, Theorem 4.1, depends on a combinatorial lemma which is an easy extention of the lemma in [9]. The following notation will be used. Let $t_1, \cdots, t_n$ be rationally independent (i.e. linearly independent with respect to rational coefficients), positive real numbers. Then the subsets of $\{1, \cdots, n\}$ may be so labelled that

$$(4.1)\quad 0 = S_{V(1)}(t) < \cdots < S_{V(2^n)}(t)$$

where $S_V(t) = \sum_{j \in V} t_j$ for $V \neq \varnothing$ and $S_\varnothing(t) = 0$. Define vectors $\epsilon_i = (\epsilon_{i1}, \cdots, \epsilon_{in})$, $i = 1, \cdots, 2^n$, by

$$(4.2)\qquad \begin{aligned} \epsilon_{ij} &= +1 \quad \text{if} \quad j \in V(i), \quad j = 1, \cdots, n, \\ &= -1 \quad \text{if} \quad j \notin V(i), \quad j = 1, \cdots, n, \end{aligned}$$

and let $t\epsilon_i = (\epsilon_{i1} t_1, \cdots, \epsilon_{in} t_n)$.

LEMMA 4.1. *Let* $S_n(t\epsilon_i) = \sum_{j=1}^n \epsilon_{ij} t_j$ *and let* $N_n(t\epsilon_i) = \sum_{V \neq \varnothing} e(S_V(t\epsilon_i))$ *for* $i = 1, \cdots, 2^n$. *Then* $S_n(t\epsilon_1) < \cdots < S_n(t\epsilon_{2^n})$, *and* $N_n(t\epsilon_i) = i - 1, i = 1, \cdots, 2^n$.

PROOF. The first assertion follows from

$$(4.3)\qquad S_n(t\epsilon_i) = 2S_{V(i)}(t) - S_n(t), \qquad i = 1, \cdots, 2^n.$$

The second may be established by defining a one-one correspondence $V \leftrightarrow V'$ of the power set of $\{1, \cdots, n\}$ with itself for which $S_{V(p)'}(t\epsilon_i) > 0$ iff $p = 2, \cdots, i$. Such a correspondence is

$$(4.4)\qquad \begin{aligned} V' &= V \qquad &\text{if} \quad V \subseteq V(i) \\ V' &= V \bigtriangleup V(i) \qquad &\text{otherwise} \end{aligned}$$

where $\bigtriangleup$ denotes symmetric difference. Its properties are easily checked. QED

Lemma 4.1 will be used as follows: Let (3.1) be satisfied with each F_n continuous and let

$$(4.5)\qquad T_n = T_n(X_{n1}, \cdots, X_{nn}) = (|X_{n1}|, \cdots, |X_{nn}|)$$

for $n = 1, 2, \cdots$. Then given $T_n = t$, the co-ordinates of which will be rationally independent wp one, N_n will be k or more iff S_{nn} exceeds its kth largest possible value. Thus δ_n is equivalent to a test which has been known long enough to be in advanced textbooks on statistics and is known to be most powerful against normal shift alternatives among all unbiased tests of H_0. (See, for example, [6], pp. 203 and 281, and [10], p. 206.) The following theorem extends results given in [6]:

THEOREM 4.1. *Let* (3.1) *be satisfied with each* F_n *continuous. Then* $\lim P(N_n(X_{n1}, \cdots, X_{nn}) \leqq y2^n) = \Phi(z_y - \mu\sigma^{-1})$, $0 < y < 1$, *where* Φ *denotes the standardized normal* df *and* $z_y = \Phi^{-1}(y)$.

PROOF. Let $[\cdot]$ denote the greatest integer function and let $c_n(t)$ be the $[y2^n]$th largest value of $(1/n^{\frac{1}{2}}\sigma)S_{nn}$ given $T_n = t$. Then by Lemma 4.1

$$(4.6)\qquad P(N_n(X_{n1}, \cdots, X_{nn}) < [y2^n]) = P((1/n^{\frac{1}{2}}\sigma)S_{nn} \leqq c_n(T_n))$$

for $n = 1, 2, \cdots$. Since (3.1) implies the asymptotic normality of $(1/n^{\frac{1}{2}}\sigma)S_{nn}$ with mean $\mu\sigma^{-1}$ and variance one ([8], pp. 101–103), it clearly suffices to show that $c_n(T_n) \to z_y$ in probability as $n \to \infty$. This is established in [6] under more restrictive regularity conditions than we are assuming. That these regularity conditions are unnecessary follows from Theorems 4.1 and 4.2 of [6], Chapter 7 and

LEMMA 4.2. *Let* $W_1, W_2, \cdots$ *and* $W_1', W_2', \cdots$ *be independent sequences of mutually independent, identically distributed random variables which are also independent of* $X_{n1}, \cdots, X_{nn}$ *for every* n *and let* $P(W_1 = \pm 1) = \frac{1}{2} = P(W_1' = \pm 1)$. *Then* (3.1) *implies that for* $-\infty < w, w' < \infty$

$$\lim P((1/n^{\frac{1}{2}}\sigma) \textstyle\sum_{j=1}^n W_j X_{nj} \leqq w, \quad (1/n^{\frac{1}{2}}\sigma) \sum_{j=1}^n W_j' X_{nj} \leqq w') = \Phi(w)\Phi(w').$$

PROOF. Let a and b be real numbers, $a^2 + b^2 \neq 0$, and let $Y_{nj} = aW_jX_{nj} + bW_j'X_{nj}$, $j = 1, \cdots, n$, $n = 1, 2, \cdots$. Then $Y_{n1}, \cdots, Y_{nn}$ are independent and identically distributed for each n and

$$(4.7a) \qquad E(Y_{n1}) = 0, \qquad n = 1, 2, \cdots,$$

$$(4.7b) \qquad \mathrm{Var}\,(Y_{n1}) = (a^2 + b^2)(\sigma^2 + o(1)) \quad \text{as} \quad n \to \infty.$$

Moreover, if $\epsilon > 0$ and we let $\epsilon' = \epsilon/(|a| + |b|)$, then

$$(4.8) \qquad \int_{|y| \geqq \epsilon n^{\frac{1}{2}}} y^2 \, dP(Y_{n1} \leqq y) \leqq (|a| + |b|)^2 \int_{|x| \geqq \epsilon' n^{\frac{1}{2}}} x^2 \, dF_n(x)$$

which is $o(1)$ as $n \to \infty$ by (3.1c). Since (4.7) and (4.8) imply the asymptotic normality of $(1/n^{\frac{1}{2}}\sigma) \sum_{j=1}^{n} Y_{nj}$ with mean zero and variance $a^2 + b^2$, the lemma follows from the arbitrariness of a and b. QED

COROLLARY 4.1. *Let α denote the limiting size of δ_n. Then* $\gamma_\alpha(\mu\sigma^{-1}) =$ (def) $\lim E(\delta_n(X_{n1}, \cdots, X_{nn})) = 1 - \Phi(z_{1-\alpha} - \mu\sigma^{-1})$.

5. Asymptotic efficiency. In this section we compute a measure of the asymptotic relative efficiency (ARE) of φ_n with respect to δ_n. This measure is not the usual Pitman ARE; it is the square of the ratio of the slopes of the limiting power functions β_α and γ_α at zero. This ratio tends in a rather imprecise way to measure the same limiting ratio of sample sizes as does Pitman ARE. Formally, the measure is

$$(5.1) \qquad \mathrm{eff}\,(\alpha) = [(\partial/\partial\mu)\beta_\alpha(\mu)|_{\mu=0}]^2/[(\partial/\partial\mu)\gamma_\alpha(\mu)|_{\mu=0}]^2$$

where β_α and γ_α are as defined in Corollaries 3.1 and 4.1 respectively and the right-hand derivative is understood in the numerator.

LEMMA 5.1. *For $0 \leqq \mu \leqq 1$ and $0 < y < 1$, $(\partial/\partial v)f(y; v)|_{v=\mu}$ exists and is bounded in absolute value by one. The right-hand derivative is understood at $\mu = 0$.*

PROOF. For $0 < \mu \leqq 1$ the lemma is obvious. At $\mu = 0$ a change of variables shows that as $v \to 0$,

$$(5.2) \qquad \begin{aligned} (f(y; v) - f(y; 0))/v &= (8\pi)^{-\frac{1}{2}} \int_{yv^2/2}^{\infty} w^{-\frac{3}{2}}(\exp(-w) - 1)\, dw \\ &\to (8\pi)^{-\frac{1}{2}} \int_0^\infty (\exp(-w) - 1) w^{-\frac{3}{2}}\, dw \\ &= -(2\pi)^{-\frac{1}{2}} \int_0^\infty w^{-\frac{1}{2}} \exp(-w)\, dw. \quad \text{QED} \end{aligned}$$

An application of the product rule to (3.11) now yields the facts that $(\partial/\partial v)g(y; v)|_{v=\mu}$ exists for $0 \leqq \mu \leqq 1$ and $0 < y < 1$, is dominated by an integrable function of y, and at $\mu = 0$ assumes the values

$$(5.3) \qquad (\partial/\partial v)g(y; v)|_{v=0} = (2\pi)^{-\frac{1}{2}}((1 - y)^{-\frac{1}{2}} - y^{-\frac{1}{2}})$$

for $0 < y < 1$. Thus we may calculate $(\partial/\partial\mu)\beta_\alpha(\mu)|_{\mu=0}$ by simply integrating (5.3) from w_α to one. Since the denominator in (5.1) is clearly $\Phi'(z_{1-\alpha})^2$, we have proved

THEOREM 5.1. *Let w_α and z_α be as in Corollary 3.1 and Theorem 4.1 respectively.*

Then

$$\text{eff}(\alpha) = (2/\pi)((1 - w_\alpha)^{\frac{1}{2}} - (1 - w_\alpha^{\frac{1}{2}}))^2/\Phi'(z_{1-\alpha})^2.$$

REMARK. eff (α) may be calculated from readily available tables. Some typical values are eff $(0.01) = 0.22$, eff $(0.05) = 0.34$, and eff $(0.10) = 0.43$.

6. Confidence sets and estimation. An hypothesis testing problem which is different from that considered in the Introduction is the following: For μ real let $\mathfrak{F}_\mu$ be the class of all continuous df's G which are symmetric about μ (in the sense that $G(\mu + x) = 1 - G(\mu - x), -\infty < x < \infty$) and suppose that F, the common df of the independent random variables $X_1, \cdots, X_n$ is known to be in $\mathfrak{F} = U\{\mathfrak{F}_\mu : -\infty < \mu < \infty\}$. Define $\bar{X}_k = S_k/k$, $k = 1, \cdots, n$, and $\bar{X}_V = S_V/c(V)$, $V \neq \emptyset$, where $c(V)$ denotes the cardinality of V and let

$$\bar{X}_{(1)} > \cdots > \bar{X}_{(n)}, \tag{6.1a}$$

$$\bar{X}_{V(1)} > \cdots > \bar{X}_{V(2^n-1)}, \tag{6.1b}$$

be their ordered values. (Notice that the definition of $V(i)$ in (6.1b) differs from that in (4.1).) It is merely a matter of translation to see that the random variables

$$M_n(\mu) = M_n(X; \mu) = \textstyle\sum_{k=1}^{n} e(\bar{X}_k - \mu), \tag{6.2a}$$

$$N_n(\mu) = N_n(X; \mu) = \textstyle\sum_{V\neq\emptyset} e(\bar{X}_V - \mu), \tag{6.2b}$$

may be used to test the hypothesis H_μ which specifies that $F \varepsilon \mathfrak{F}_\mu$ against alternatives which specify that $F \varepsilon \mathfrak{F}_\mu^c = U\{\mathfrak{F}_v : v \neq \mu\}$. Specifically, the tests, which will be denoted by $\varphi_n(\mu)$ and $\delta_n(\mu)$, accept H_μ iff $a_n < M_n(\mu) < n - a_n$ and $b_n < N_n(\mu) < 2^n - (b_n + 1)$ respectively. We require a_n and b_n to be integers and consider only non-randomized tests. Thus if α is the size of $\varphi_n(\mu)$, we find

$$1 - \alpha = P(a_n < M_n(\mu) < n - a_n) = P(\bar{X}_{(n-a_n+1)} \leqq \mu < \bar{X}_{(a_n)}) \tag{6.3a}$$

if $F \varepsilon \mathfrak{F}_\mu$, so that $[\bar{X}_{(n-a_n+1)}, \bar{X}_{(a_n)})$ is a $1 - \alpha$ confidence interval for μ. Similarly, if α' is the size of $\delta_n(\mu)$, then

$$\begin{aligned} 1 - \alpha' &= P(b_n < N_n(\mu) < 2^n - (b_n + 1)) \\ &= P(\bar{X}_{V(2^n-b_n)} \leqq \mu < \bar{X}_{V(b_n)}) \end{aligned} \tag{6.3b}$$

if $F \varepsilon \mathfrak{F}_\mu$.

While the procedure for obtaining confidence intervals from hypothesis tests has been known for a long time, a method for obtaining point-estimates from non-parametric tests has only recently been proposed by Hodges and Lehmann in [7]. When applied to φ_n and δ_n, their method produces the estimates

$$\hat{\mu}_n = \text{med}(\bar{X}_1, \cdots, \bar{X}_n), \tag{6.4a}$$

$$\hat{\mu}_n' = \bar{X}_{V(2^{n-1})}, \tag{6.4b}$$

respectively.

The asymptotic distribution of the confidence intervals' end-points (6.3) and of the estimates (6.4) may be obtained from Theorems 3.1 and 4.1 provided that $F \varepsilon \mathfrak{F}_\mu$ and

$$\int (x - \mu)^2 \, dF(x) = \sigma^2 < \infty. \tag{6.5}$$

THEOREM 6.1. *Let $j \sim \beta n$ as $n \to \infty$ and let $F \varepsilon \mathfrak{F}_\mu$ satisfy (6.5). Then for $-\infty < y < \infty$,*

$$\lim P(n^{\frac{1}{2}}(\bar{X}_{(j)} - \mu) \leqq y) = \int_0^\beta g(w; y\sigma^{-1}) \, dw.$$

PROOF. There is clearly no loss of generality in assuming that $\mu = 0$. In this case the hypotheses of Theorem 3.1 are satisfied with $X_{nk} = X_k - yn^{-\frac{1}{2}}$, $k = 1, \cdots, n$, $n = 1, 2, \cdots$, for any fixed y. Therefore

$$\begin{aligned} P(n^{\frac{1}{2}}\bar{X}_{(j)} \leqq y) &= P(M_n(yn^{-\frac{1}{2}}) < \beta n) + o(1) \\ &= P(M_n(X_{n1}, \cdots, X_{nn}) < \beta n) + o(1) \\ &\to \int_0^\beta g(w; y\sigma^{-1}) \, dw \end{aligned} \tag{6.6}$$

as $n \to \infty$. QED

COROLLARY 6.1. $\lim P(n^{\frac{1}{2}}(\hat{\mu}_n - \mu) \leqq y) = \int_0^{\frac{1}{2}} g(w: y\sigma^{-1}) \, dw$.

PROOF. Let $n = 2m$; then it suffices to show that $n^{\frac{1}{2}}(\bar{X}_{(m)} - \bar{X}_{(m+1)}) \to 0$ in probability as $m \to \infty$, and as above, there is no loss of generality in assuming that $\mu = 0$. If $\epsilon > 0$ is given, then by Theorem 6.1 there exist x_0 and n_0 for which $P(n^{\frac{1}{2}} |\bar{X}_{(k)}| \geqq x_0) < \epsilon/2$ for $n \geqq n_0$ and $k = m, m + 1$. Thus letting $x_j = -x_0 + j\epsilon/3$ for $j = 1, \cdots, [6x_0\epsilon^{-1}] + 1$, we find

$$\begin{aligned} P(n^{\frac{1}{2}}(\bar{X}_{(m)} - \bar{X}_{(m+1)}) > \epsilon) &< \sum_j P(\bar{X}_{(m+1)} < x_j n^{-\frac{1}{2}} < \bar{X}_{(m)}) + \epsilon \\ &\leqq \sum_j P(M_n(x_j n^{-\frac{1}{2}}) = m) + \epsilon. \end{aligned} \tag{6.7}$$

Since $(1/n)M_n(x_j n^{-\frac{1}{2}})$ has a continuous limiting distribution for each j (see (6.6)), the corollary follows. QED

Similar considerations lead to

THEOREM 6.2. *Let $j \sim \beta 2^n$ as $n \to \infty$ and let $F \varepsilon \mathfrak{F}_\mu$ satisfy (6.5). Then $n^{\frac{1}{2}}(\bar{X}_{V(j)} - \mu)$ is asymptotically normal with mean $\sigma z_{1-\beta}$ and variance σ^2.*

COROLLARY 6.2. *$n^{\frac{1}{2}}(\hat{\mu}_n' - \mu)$ is asymptotically normal with mean zero and variance σ^2.*

The final topic to be considered in this paper is the asymptotic behavior of the length $L_n = (\bar{X}_{V(b_n)} - \bar{X}_{V(2^n - b_n)})$ of the confidence intervals (6.3b). It will be shown that under regularity conditions $n^{\frac{1}{2}}L_n$ converges in probability to $2\sigma z_{1-\alpha/2}$ as $n \to \infty$ where $\alpha = \lim b_n 2^{-n+1}$ is the limiting size of $\delta_n(\mu)$. A similar analysis has been made by Lehmann [11] for confidence intervals obtained from the Wilcoxon Test; the methods employed here, however, are essentially different from those of [11].

The regularity conditions which will be needed are (1) that $F \varepsilon \mathfrak{F}_\mu$ and

$$\int (x - \mu)^4 \, dF(x) = \tau < \infty \tag{6.8}$$

and (2) that the sequence of sample means be asymptotically efficient among all unbiased estimates with respect to the family of measures induced by the translates of F. More specifically, (2) states that if

$$\int W_n(x_1, \cdots, x_n)\, dF(x_1 - \theta) \cdots dF(x_n - \theta) = \mu + \theta \tag{6.9a}$$

for all θ and all n, then

$$\liminf n \operatorname{Var}(W_n(X_1, \cdots, X_n)) \geqq \sigma^2 \tag{6.9b}$$

where σ^2 is given by (6.5).

LEMMA 6.1. *Let $j \sim \beta n$ as $n \to \infty \cdots$. Then* (6.8) *implies the existence of a constant B for which $P(n^{\frac{1}{2}}|\bar{X}_{V(j)} - \mu| > y) \leqq By^{-4}$ uniformly in $y > 0$ and n sufficiently large.*

PROOF. Again there is no loss of generality in assuming that $\mu = 0$. In this case we have for n sufficiently large

$$\begin{aligned} P(n^{\frac{1}{2}}\bar{X}_{V(j)} > y) &\leqq P(N_n(yn^{-\frac{1}{2}}) \geqq \beta 2^{n-1} + 1) \\ &\leqq (1/\beta 2^{-n+1})E(N_n(yn^{-\frac{1}{2}})) \\ &\leqq (1/\beta 2^{-n+1}) \textstyle\sum_{k=1}^{n} \binom{n}{k} P(S_k > ykn^{-\frac{1}{2}}) \\ &\leqq (1/\beta 2^{-n+1}) \textstyle\sum_{k=1}^{n} \binom{n}{k} (k\tau + k^2\sigma^4)y^{-4}k^{-4}n^2 \end{aligned} \tag{6.10}$$

which does not exceed $48(\tau + \sigma^4)/\beta y^4$. A similar argument yields the same bound for $P(n^{\frac{1}{2}}\bar{X}_{V(j)} < -y)$. QED

COROLLARY 6.3. $\lim n^{\frac{1}{2}}E(\bar{X}_{V(j)} - \mu) = \sigma z_{1-\beta}$, *and* $\lim n \operatorname{Var}(\bar{X}_{V(j)}) = \sigma^2$.

THEOREM 6.3. *Let* (6.8) *and* (6.9) *be satisfied and let $b_n \sim \alpha 2^{n-1}$ as $n \to \infty$. Then $n^{\frac{1}{2}}L_n$ converges in probability to $2\sigma z_{1-\alpha/2}$ as $n \to \infty$.*

PROOF. It is easily seen that $(\bar{X}_{V(b_n)} + \bar{X}_{V(2^n - b_n)})/2$ satisfies (6.9a); therefore

$$\liminf n \operatorname{Var}(\bar{X}_{V(b_n)} + \bar{X}_{V(2^n-b_n)}) \geqq 4\sigma^2. \tag{6.11}$$

Expanding the left-hand side of (6.11), we find from Corollary 6.3 that $\lim n \operatorname{Cov}(\bar{X}_{V(b_n)}, \bar{X}_{V(2^n-b_n)}) = \sigma^2$, thus implying that $\lim n \operatorname{Var}(L_n) = 0$; and since $b_n \sim \alpha 2^{n-1}$ as $n \to \infty$ by assumption, Corollary 6.3 implies $\lim n^{\frac{1}{2}}E(L_n) = 2\sigma z_{1-\alpha/2}$. QED

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