ASYMPTOTIC NORMALITY OF BISPECTRAL ESTIMATES[1]

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1. Summary. The work presented here is a continuation of that presented in an earlier paper, M. Rosenblatt and J. Van Ness [15], in which the basic properties (unbiasedness and consistency) of certain estimates of the bispectrum and bispectral density are discussed. (The bispectrum can be thought of as the Fourier transform of the third-order moment function of the process.) The present paper is concerned with the asymptotic distribution of these estimates. One would expect that under certain regularity conditions these estimates would have distributions which tend to complex normal distributions. The following develops two different sets of such conditions either of which suffice. The first set involves a uniform summability condition on the first six cumulants of a sequence of processes obtained from the original process by projecting it onto a certain sequence of Borel fields. The second, and much more intuitively meaningful, set involves the strong mixing condition (see Rosenblatt [11], [12]; Kolmogorov and Rozanov [6], and Volkonskii and Rozanov [18] e.g.).

The calculations in the earlier paper were carried out for the continuous parameter case; here we restrict ourselves to the discrete parameter case.

For reasons for interest in polyspectra see, for example, Rosenblatt and Van Ness [15]; Hasselmann, Munk and MacDonald [4]; and Brillinger [2].

2. Introduction. Let $\{X_t\}$, $t = 0, \pm 1, \pm 2, \cdots$, be a discrete parameter real-valued random process. For our purposes we take $\{X_t\}$ to have mean zero and finite sixth-order moments and to be sixth-order weakly stationary so that for all t,

$$\begin{aligned} EX_t &= 0, \\ EX_tX_{t+\nu} &= m_2(t, t+\nu) = r(\nu), \qquad (2.1) \\ &\vdots \\ EX_tX_{t+\nu_1} \cdots X_{t+\nu_5} &= m_6(t, t+\nu_1, \cdots, t+\nu_5) = r_6(\nu_1, \cdots, \nu_5). \end{aligned}$$

In addition we require $r(\nu)$ and $r_3(\nu_1, \nu_2)$ to be in l_1 with the spectral distribution function, $F(\lambda)$, of the process absolutely continuous with a continuous density, $f(\lambda)$. Then[2]

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[2] The continuous parameter versions of formulae occurring in that which follows can be found in Rosenblatt and Van Ness [15].

$$r(\nu) = \int_{-\pi}^{\pi} e^{i\nu\lambda} f(\lambda)\, d\lambda; \tag{2.2}$$

$$f(\lambda) = (2\pi)^{-1} \sum_{-\infty}^{\infty} e^{-i\nu\lambda} r(\nu). \tag{2.3}$$

Similarly define the bispectral density function as

$$g(\lambda_1, \lambda_2) = (2\pi)^{-2} \sum_{|\nu_1|,|\nu_2|<\infty} \exp(-i\nu_1\lambda_1 - i\nu_2\lambda_2) r_3(\nu_1, \nu_2) \tag{2.4}$$

and assuming $g(\lambda_1, \lambda_2) \varepsilon L_1$,

$$r_3(\nu_1, \nu_2) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \exp(i\nu_1\lambda_1 + i\nu_2\lambda_2) g(\lambda_1, \lambda_2)\, d\lambda_1\, d\lambda_2. \tag{2.5}$$

This is the harmonic representation of the third-order moment function under the above assumptions.

Since the process is real, the following symmetries occur in the third-order functions:

$$r_3(\nu_1, \nu_2) = r_3(\nu_2, \nu_1) = r_3(-\nu_1, \nu_2 - \nu_1), \tag{2.6}$$

$$g(\lambda_1, \lambda_2) = g(\lambda_2, \lambda_1) = g(\lambda_1, -\lambda_1 - \lambda_2) = \overline{g(-\lambda_1, -\lambda_2)}. \tag{2.7}$$

The symmetries (2.6) imply that $r_3(\nu_1, \nu_2)$ is completely specified over the entire plane by its values in any *one* of six sectors. Similarly $g(\lambda_1, \lambda_2)$ is, by (2.7) and its periodicity, completely specified by its values on any one of the twelve sectors (including boundaries) shown in Figure 1.

It is well known that the random spectral measure $dZ(\lambda)$ associated with the spectral representation of $\{X_t\}$ has its second-order moments related to $F(\lambda)$ and its third-order moments related to $G(\lambda_1, \lambda_2)$:

$$E\, dZ(\lambda_1)\, dZ(\lambda_2)\, dZ(\lambda_3) = \delta(\lambda_1 + \lambda_2 + \lambda_3)\, dG(\lambda_1, \lambda_2).$$

Further, in the real representation of the process,

$$X_t = \int_0^{\pi} \cos t\lambda\, dZ_1(\lambda) + \int_0^{\pi} \sin t\lambda\, dZ_2(\lambda),$$

the expectation of $dZ_i(\lambda_1)\, dZ_j(\lambda_2)\, dZ_k(\lambda_3)$; $i, j, k = 1, 2$; are related to the real and imaginary parts of $G(\lambda_1, \lambda_2)$ (see Rosenblatt and Van Ness [15]).

3. Estimation. The estimate discussed here is the same as that introduced in the earlier paper, i.e., one based on the third-order theory analog of the periodogram. Thus given observations, x_t, for $1 \leqq t \leqq N$ we first choose a natural estimate for $r_3(\nu_1, \nu_2)$

$$\rho_N(\nu_1, \nu_2) = N^{-1} \sum_{t \varepsilon D_N(\nu_1, \nu_2)} x_t x_{t+\nu_1} x_{t+\nu_2} \tag{3.1}$$

where D_N restricts x_t, $x_{t+\nu_1}$ and $x_{t+\nu_2}$ to the domain in which they are known (see [15]). Replacing $r_3(\nu_1, \nu_2)$ in (2.4) by $\rho_N(\nu_1, \nu_2)$, we get an estimate of $g(\lambda_1, \lambda_2)$,

$$g_N(\lambda_1, \lambda_2) = (2\pi)^{-2} \sum_{|\nu_1|,|\nu_2| \leqq N} \exp(-i\nu_1\lambda_1 - i\nu_2\lambda_2) \rho_N(\nu_1, \nu_2). \tag{3.2}$$

However, as discussed in [15], this estimate is not consistent and must therefore be weighted in the form

$$g_N^*(W) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} W(\mu_1, \mu_2) g_N(\mu_1, \mu_2)\, d\mu_1\, d\mu_2 \tag{3.3}$$

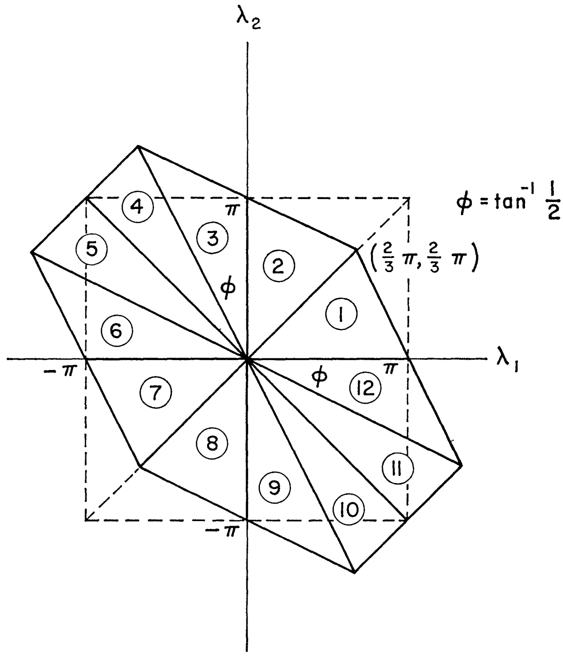

FIG. 1

where $W(\mu_1, \mu_2)$ is for example a "bispectral averaging function" defined in [15] or, if one wishes an asymptotically unbiased estimate, a sequence of weight functions $\{W_N(\mu_1, \mu_2)\}$ can be used so that for example

$$(3.4) \qquad g_N^*(\lambda_1, \lambda_2) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} W_N(\mu_1 - \lambda_1, \mu_2 - \lambda_2) g_N(\mu_1, \mu_2) \, d\mu_1 \, d\mu_2 .$$

As N increases the W_N narrow the region averaged over tending to give asymptotic unbiasedness. The rate at which W_N concentrates must be slow enough to retain the consistency.

The problem of asymptotic bias and consistency was treated in [15]. There sufficient conditions on the process and the sequence of weight functions $\{W_N\}$ for the desired properties are obtained. The main conditions employed were summability conditions on the first six cumulant functions of the process. If $s_j(\nu_1, \cdots, \nu_j)$ are the jth order cumulant functions of the process (see [15]), then the following relations hold in our case (see Stratonovich [17], Chapter 1, and Leonov and Shiryaev [7] for a discussion of cumulant functions):

$$m_1(\nu_1) = s_1(\nu_1) = 0;$$

$$m_2(\nu_1, \nu_2) = s_2(\nu_1, \nu_2);$$

$$m_3(\nu_1, \nu_2, \nu_3) = s_3(\nu_1, \nu_2, \nu_3);$$

$$(3.5) \qquad m_4(\nu_1, \cdots, \nu_4) = s_4(\nu_1, \cdots, \nu_4) + \{s_2(\nu_1, \nu_2)s_2(\nu_3, \nu_4)\}_3 ;$$

$$m_6(\nu_1, \cdots, \nu_6) = s_6(\nu_1, \cdots, \nu_6) + \{s_3(\nu_1, \nu_2, \nu_3)s_3(\nu_4, \nu_5, \nu_6)\}_{10} + \{s_2(\nu_1, \nu_2)s_4(\nu_3, \cdots, \nu_6)\}_{15} + \{s_2(\nu_1, \nu_2) \cdot s_2(\nu_3, \nu_4)s_2(\nu_5, \nu_6)\}_{15}$$

where the notation $\{\cdot\}_j$ denotes the sum of all j *different* terms obtained by interchanging the arguments of the terms in brackets (the order of the arguments of the s_j being immaterial). (See Tables I, II and III in [15].) Thus

$$m_4(\nu_1, \cdots, \nu_4) = s_4(\nu_1, \cdots, \nu_4) + s_2(\nu_1, \nu_2)s_2(\nu_3, \nu_4) + s_2(\nu_1, \nu_3)s_2(\nu_2, \nu_4) + s_2(\nu_1, \nu_4)s_2(\nu_2, \nu_3).$$

Due to stationarity, we write

$$(3.6)\qquad \begin{aligned} s_2(t, t+\nu) &= \xi_2(\nu), \\ &\vdots \\ s_6(t, t+\nu_1, \cdots, t+\nu_5) &= \xi_6(\nu_1, \cdots, \nu_5). \end{aligned}$$

Then the conditions used are that $\xi_4(\nu_1, \nu_2, \nu_3) \varepsilon l_1(R_3)$ and $\xi_6(\nu_1, \cdots, \nu_5) \varepsilon l_1(R_5)$. As mentioned in [15], normal processes, linear processes and K-step dependent processes all satisfy this requirement.

4. The bispectral density estimate. The asymptotic distribution will be discussed not for the general form (3.4) but for a subclass of such estimates. To describe these estimates, first define A_N and B_N such that $A_N = B_N^{-1}$ is a positive integer and $B_N \to 0$ and $B_N^2 N \to \infty$ as $N \to \infty$. We should note that we only need that $B_N N \to \infty$ to get the statements of Theorems 1 and 2 (and Theorem 5 of [15]) but if we want the density estimate to be consistent we need $B_N^2 N \to \infty$.

DEFINITION 1. A real function, $w(y_1, y_2)$, is called a symmetric bispectral estimating kernel if[3]

(i) for any $\epsilon > 0$, there is an $M_1(\epsilon)$ such that for all $M > M_1$ and uniformly in $N > M$,

$$B_N^2 \sum\sum_{(\nu_1,\nu_2)\varepsilon\square'_{MA_N}} w^2(B_N\nu_1, B_N\nu_2) < \epsilon;$$

(ii) $w(y_1, y_2) \leqq M_1 < \infty$ for all $-\infty < y_1, y_2 < \infty$;

(iii) $w(y_1, y_2) = w(y_2, y_1) = w(-y_1, y_2 - y_1)$ (same symmetries as those of $r_3(\nu_1, \nu_2)$);

(iv) for any $\epsilon > 0$, there is an $M_2(\epsilon)$ such that for all $M > M_2$ and uniformly in $N > M$ and in ν_1,

$$B_N \sum_{|\nu_2|>MA_N} |w(\nu_1, B_N\nu_2)| < \epsilon;$$

(v) for all fixed numbers a and b, any fixed $M > 0$, and for any $\epsilon > 0$ there is an $N_0(\epsilon, M, a, b)$ such that for all $N > N_0$,

[3] Denote by $\square_M$, the j-dimensional hypercube centered at the origin with sides of length $2M$ parallel to the j axes. The dimension, j, will be obvious from the context. Also let $\square'_M$ denote the complement of $\square_M$ in R_j.

$$B_N^2 \left| \sum_{|\nu_1|,|\nu_2| \geq MA_N} w(B_N\nu_1 + B_N a, B_N\nu_2) w(B_N\nu_1, B_N\nu_2 + B_N b) - \sum_{|\nu_1|,|\nu_2| \leq MA_N} w^2(B_N\nu_1, B_N\nu_2) \right| < \epsilon$$

and

$$B_N \left| \sum_{|\nu_1| \geq MA_N} w(B_N\nu_1, B_N a) - \sum_{|\nu_1| \leq MA_N} w(B_N\nu_1, 0) \right| < \epsilon.$$

This last condition is certainly satisfied if $w(\nu_1, \nu_2)$ is continuous.

The estimates we will consider are of the form

$$g_N^*(\lambda_1, \lambda_2) = (2\pi)^{-2} \sum_{|\nu_1|,|\nu_2| \leq N} \exp(-i\lambda_1\nu_1 - i\lambda_2\nu_2) \cdot w(B_N\nu_1, B_N\nu_2)\rho(\nu_1, \nu_2). \tag{4.1}$$

If we have a function $W_N(\lambda_1, \lambda_2)$ whose Fourier coefficients are $w(B_N\nu_1, B_N\nu_2)$ then, written in terms of W, g_N^* has the form (3.4). The rate at which $B_N \to 0$ governs the rate of concentration of the weight functions.

Condition (iii) of Definition 1 implies that $g_N^*(\lambda_1, \lambda_2)$ has the same symmetries as $g(\lambda_1, \lambda_2)$. It is not a necessary condition but is included solely to permit more compact statements and proofs of theorems. For a discussion of the non-symmetric case see [15].

5. Asymptotic normality under a uniform summability of cumulants condition. Consider a real, strictly stationary process, $\{X_t\}$, of the following form. Let

$$\eta = (\cdots, \eta_{-1}, \eta_0, \eta_1, \cdots)$$

be a doubly infinite sequence of independent, identically distributed random variables. We could take the η's to be uniformly distributed on [0, 1] or normally distributed. Let T be the shift operator on η, i.e., $T\eta = (\cdots, \eta_0, \eta_1, \eta_2, \cdots)$. Take h to be a Borel measurable function of the doubly infinite vector and define

$$X_t = h(T^t\eta), \qquad t = 0, \pm 1, \cdots. \tag{5.1}$$

Also let

$$X_{t,k} = E[X_t \mid \eta_{t-k}, \cdots, \eta_{t+k}] \tag{5.2}$$

(i.e., $X_{t,k}$ is the projection of X_t onto the Borel field, B_{t-k}^{t+k}, generated by $\eta_{t-k}, \cdots, \eta_{t+k}$) and

$$\begin{aligned} r^{(\infty,k)}(\nu) &= EX_tX_{t+\nu,k}; \\ r^{(k,k)}(\nu) &= EX_{t,k}X_{t+\nu,k}; \\ r_3^{(\infty,\infty,k)}(\nu_1, \nu_2) &= EX_tX_{t+\nu_1}X_{t+\nu_2,k}; \end{aligned} \tag{5.3}$$

and similarly define $r_3^{(\infty,k,k)}$, $r_3^{(k,k,k)}$, $r_4^{(\infty,\infty,\infty,k)}$, $\cdots$, $r_4^{(k,k,k,k)}$, $r_6^{(\infty,\infty,\infty,k,k,k)}$ and $r_6^{(k,k,k,k,k,k)}$. Corresponding to these there are as in (3.5) and (3.6)

$$\begin{aligned} &\xi_2^{(\infty,k)}(\nu), \ \xi_2^{(k,k)}(\nu), \\ &\xi_3^{(\infty,\infty,k)}(\nu_1, \nu_2), \cdots, \xi_3^{(k,k,k)}(\nu_1 . \nu_2). \end{aligned} \tag{5.4}$$

$$\xi_4^{(\infty,\infty,\infty,k)}(\nu_1, \nu_2, \nu_3), \cdots, \xi_4^{(k,k,k,k)}(\nu_1, \nu_2, \nu_3),$$
$$\xi_6^{(\infty,\infty,\infty,k,k,k)}(\nu_1, \cdots, \nu_5), \quad \text{and} \quad \xi_6^{(k,k,k,k,k,k)}(\nu_1, \cdots, \nu_5).$$

The following theorem then holds.

THEOREM 1. *If* (a) $\{X_t\}$ *is a process as defined above,*

(b) $EX_t^{12} < \infty$,

(c) *all cumulant functions* $\xi_2, \cdots, \xi_2^{(j,j)}, \xi_3, \cdots, \xi_3^{(j,j,j)}, \xi_4, \cdots, \xi_4^{(j,j,j,j)}, \xi_6, \cdots, \xi_6^{(j,j,j,j,j,j)}$ *are* $\varepsilon\, l_1$ *of their respective spaces uniformly in* j,

(d) $w(\nu_1, \nu_2)$ *is a symmetric bispectral estimating kernel,*

(e) *for brevity in writing the results,* (μ_1, μ_2) *and* (μ_3, μ_4) *are taken in Section one as shown in Figure 1,*

then

$$N^{\frac{1}{2}}B_N[g_N^*(\mu_1, \mu_2) - Eg_N^*(\mu_1, \mu_2)]$$

converges in distribution to a complex normal random variable, $X + iY$, *where* X *and* Y *have zero mean, are jointly normal, are independent and have the following variances:*

$$\sigma^2(X) = \sigma^2(Y) = \tfrac{1}{2}(w_2/2\pi)f(\mu_1)f(\mu_2)f(\mu_1 + \mu_2)$$

if (μ_1, μ_2) *lies inside the region one and not on its boundaries or if we include the boundaries*

$$\sigma^2(X) = (w_1/2\pi)f(\mu_1)f(\mu_2)f(\mu_1 + \mu_2)[8\delta(\mu_1) + \delta(\mu_2)] + A + B,$$
$$\sigma^2(Y) = A - B,$$

where

$$A = \tfrac{1}{2}(w_2/2\pi)f(\mu_1)f(\mu_2)f(\mu_1 + \mu_2)\{[1 + \delta(\mu_1 - \mu_2)][1 + \delta(\mu_1 + 2\mu_2 - 2\pi) + \delta(2\mu_1 + \mu_2 - 2\pi)] + 4\delta(\mu_1)\},$$

$$B = \tfrac{1}{2}(w_2/2\pi)f(\mu_1)f(\mu_2)f(\mu_1 + \mu_2)\{5\delta(\mu_1) + \delta(\mu_2)[1 + \delta(\mu_1 - \pi)]\}.$$

Here we define

$$w_1 = [\textstyle\int_{-\infty}^{\infty} w(0, \nu)\, d\nu]^2, \qquad w_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w^2(\nu_1, \nu_2)\, d\nu_1\, d\nu_2,$$

and

$$\delta(x) = 1, \quad x = 0,$$
$$= 0, \quad x \neq 0.$$

The proof of this theorem is postponed until Section 7.

6. Asymptotic normality under the strong mixing condition. We now look at the asymptotic distribution of

$$N^{\frac{1}{2}}B_N[g_N^*(\mu_1, \mu_2) - Eg_N^*(\mu_1, \mu_2)] \tag{6.1}$$

under the much more intuitive assumption of strong mixing. Let $\{X_t\}$ be a real 6th-order weakly stationary random process which is strong mixing. This means that the following property holds:

$$\sup_t \sup_{A \varepsilon M_{-\infty}^t, B \varepsilon M_{t+r}^\infty} |P(AB) - P(A)P(B)| = \zeta(r) \to 0 \tag{6.2}$$

as $r \to \infty$. Here M_a^b denotes the σ-algebra generated by the random variables $\{X_t \mid t \varepsilon [a, b]\}$. This can be thought of as a uniform (with respect to time shifts) asymptotic independence condition.

For such a process the analog of Theorem 1 is the following:

THEOREM 2. *If* (a) $\{X_t\}$ *is a real, sixth-order weakly stationary random process,*

(b) $r(\nu)$, $r_3(\nu_1, \nu_2)$, $\xi_4(\nu_1, \nu_2, \nu_3)$, $\xi_6(\nu_1, \cdots, \nu_5)$ $\varepsilon\, l_1$ *of their respective spaces,* $g(\lambda_1, \lambda_2) \,\varepsilon\, L_1$,

(c) $\{X_t\}$ *satisfies the strong mixing condition,*

(d) *hypotheses* (d) *and* (e) *of Theorem* 1 *hold,*

(e) *there is some* $\delta > 0$ *such that for* α_N, β_N *and* σ_λ *as defined in the proof*

$$[(\alpha_N^{\frac{1}{2}}\beta_N^{\frac{1}{2}}\sigma_\lambda)^{2+\delta}]^{-1} \textstyle\sum_1^{\alpha_N} E\, |U_j^{(N)}|^{2+\delta} \to 0$$

then the conclusion of Theorem 1 *holds.*

(The quantity $U_j^{(N)}$ is defined by (6.11). This hypothesis is derived from Liapounov's conditions for a central limit theorem (see Loève [8] p. 275). If $\{X_t\}$ is stationary then hypothesis (e) becomes $E\,|U_1|^{2+\delta}/\alpha_N^{\delta/2} B_N^{1+\delta/2} \to 0$.)

PROOF. The random variables we are interested in are

$$V_N = N^{\frac{1}{2}} B_N(g_N^*(\mu_1, \mu_2) - E g_N^*(\mu_1, \mu_2));$$

$$\operatorname{Re} V_N = [B_N/(2\pi)^2 N^{\frac{1}{2}}] \textstyle\sum_{\nu_1,\nu_2,\nu_3=1}^N \cos[(\nu_2 - \nu_1)\mu_1 + (\nu_3 - \nu_1)\mu_2] \tag{6.3}$$
$$\cdot w(B_N(\nu_2 - \nu_1), B_N(\nu_3 - \nu_1))[X_{\nu_1}X_{\nu_2}X_{\nu_3} - r_3(\nu_2 - \nu_1, \nu_3 - \nu_1)],$$

and $\operatorname{Im} V_N$. In a manner analogous to the characteristic way in which strong mixing central limit theorems are proved we will break up the domain of summation into an increasing number of enlarging blocks separated by increasing but comparatively small distances. To this end denote by $[I_1, I_2, I_3]$ the parallelopiped of indices $\{(\nu_1, \nu_2, \nu_3) \mid \nu_1 \varepsilon I_1, \nu_2 \varepsilon I_2, \nu_3 \varepsilon I_3\}$ where I_1, I_2 and I_3 are intervals. Next choose sequences $\{\alpha_N\}$, $\{\beta_N\}$ and $\{\gamma_N\}$ of positive integers so that

$$\text{(i)} \quad \alpha_N[\beta_N + \gamma_N] \sim N,$$
$$\text{(ii)} \quad \alpha_N, \beta_N, \gamma_N \uparrow \infty, \tag{6.4}$$
$$\text{(iii)} \quad \gamma_N = o(\beta_N),$$

then it will be shown that we can replace the sum $\sum_{\nu_1,\nu_2,\nu_3=1}^N$ in (6.3) by

$$\textstyle\sum_{j=1}^{\alpha_N} \sum_{[b_j^{(N)}, b_j^{(N)}, b_j^{(N)}]} \tag{6.5}$$

where $b_j^{(N)} = [(j-1)(\beta_N + \gamma_N) + 1, j(\beta_N + \gamma_N) - \gamma_N]$, $j = 1, \cdots, \alpha_N$, and still get the same asymptotic distribution. Having done this, the sum (6.5) will be shown to be asymptotically normally distributed. It helps to draw a picture here of the domain of summation—a cube with sides parallel to the x, y and z axes and of length $N - 1$. The main diagonal of this cube runs from the point $(1, 1, 1)$ to (N, N, N). Then the sum (6.5) is over α_N smaller cubes whose main diagonals lie on the above diagonal and whose sides are of length $\beta_N - 1$ and are parallel to those of the large cube. These smaller cubes are separated by a distance $\sim\gamma_N$.

We begin the first step by noting that by the properties of w and the summability of the cumulants[4] (see beginning of the proof of Theorem 5 of [15])

$$\begin{aligned} &\operatorname{cov}[V_N(\mu_1, \mu_2), V_N(\mu_3, \mu_4)] \\ (6.6)\qquad &= [B_N^2/(2\pi)^4] \textstyle\sum_{|\nu_1|, \cdots, |\nu_4|, |y| \leqq N} N^{-1} C_N(\nu_1, \cdots, \nu_4, y) \\ &\quad \cdot \exp(-i\mu_1\nu_1 - i\mu_2\nu_2 + i\mu_3\nu_3 + i\mu_4\nu_4) w(B_N\nu_1, B_N\nu_2) w(B_N\nu_3, B_N\nu_4) \\ &\quad \cdot \{m_2(0, \nu_1) m_2(\nu_2, y) m_2(y + \nu_3, y + \nu_4)\}_{15} + O(B_N). \end{aligned}$$

LEMMA 1. *If* (a) *the hypotheses of Theorem* 2 *hold*,
(b) $\gamma_N B_N \to \infty$ *as* $N \to \infty$,
(c) $\alpha_N^2 \gamma_N / N \to 0$ *as* $N \to \infty$,
then

$$\begin{aligned} \sigma^2\{[B_N/(2\pi)^2 N^{\frac{1}{2}}][\textstyle\sum_{v_1, v_2, v_3 = 1}^{N} - \sum_{j=1}^{\alpha_N} \sum_{[b_j(N), b_j(N), b_j(N)]}] \\ \cdot \exp(-i(v_2 - v_1)\mu_1 - i(v_3 - v_1)\mu_2) w(B_N(v_2 - v_1), B_N(v_3 - v_1)) \\ \cdot [X_{v_1} X_{v_2} X_{v_3} - r_3(v_2 - v_1, v_3 - v_1)]\} \to 0 \end{aligned}$$

as $N \to \infty$.

PROOF. For convenience, suppose $\alpha_N[\beta_N + \gamma_N] = N + \gamma_N$. We prove this lemma by proving that the variance of the sums over six sets of indices, $S_1, \cdots, S_6$, tends to zero. The union of these six sets contains all indices contained in the first sum above but not contained in the second thus giving the result. Define

$$S_1 = \{(v_1, v_2, v_3) \mid |v_1 - v_2| > 2^{-\frac{1}{2}}\gamma_N, |v_j| \leqq N \text{ for all } j\},$$

$$S_2 = \{(v_1, v_2, v_3) \mid |v_1 - v_3| > 2^{-\frac{1}{2}}\gamma_N, |v_j| \leqq N \text{ for all } j\},$$

$$S_3 = \{(v_1, v_2, v_3) \mid |v_2 - v_3| > 2^{-\frac{1}{2}}\gamma_N, |v_j| \leqq N \text{ for all } j\},$$

$$S_4 = \{(v_1, v_2, v_3) \mid v_1 \in c_j \text{ for some } j = 1, \cdots, \alpha_N - 1\},$$

[4] Here C_N is defined as follows (see Figure III of [15]):
1. construct the set $D_N(\nu_1, \nu_2) \times D_N(\nu_3, \nu_4) = \hat{D}_N$;
2. take the intersection of this set with the line $\tau - t = y$ and call that segment $\hat{C}_N$;
3. then $C_N(\nu_1, \nu_2, \nu_3, \nu_4, y)$ is equal to the length of the projection of $\hat{C}_N$ onto either axis.

Note that C_N can be written out analytically in terms of its arguments but the expression is cumbersome and not as enlightening as Figure III. Also, $0 \leqq C_N/N \leqq 1$ and $C_N/N \to 1$ as $N \to \infty$.

$$S_5 = \{(v_1, v_2, v_3) \mid v_2 \varepsilon c_j \text{ for some } j = 1, \cdots, \alpha_N - 1\},$$

$$S_6 = \{(v_1, v_2, v_3) \mid v_3 \varepsilon c_j \text{ for some } j = 1, \cdots, \alpha_N - 1\},$$

where $c_j = [j(\beta_N + \gamma_N) - \gamma_N, j(\beta_N + \gamma_N)]$. For the first set we have

$$E|[B_N/(2\pi)^2 N^{\frac{1}{2}}] \textstyle\sum_{(\nu_1,\nu_2,\nu_3)\varepsilon S_1} [\cdots]|^2$$

$$(6.7) \quad = [B_N^2/(2\pi)^4] \textstyle\sum_{y=-N}^{N} \sum_{(\nu_1,\nu_3)\varepsilon R_1} \sum_{|\nu_2|,|\nu_4| \leqq N}$$

$$\cdot [w(B_N\nu_1, B_N\nu_2)w(B_N\nu_3, B_N\nu_4) \exp(i\nu_1\mu_1 + i\nu_2\mu_2 - i\nu_3\mu_3 - \nu_4\mu_4)$$

$$\cdot (C_N/N)\{m_2(0, \nu_1)m_2(\nu_2, y)m_2(y + \nu_3, y + \nu_4)\}_{15}] + O(B_N)$$

where $R_1 = \{(\nu_1, \nu_3) \mid (\nu_1, \nu_3) \varepsilon [-N, N] - [-2^{-\frac{1}{2}}\gamma_N, 2^{-\frac{1}{2}}\gamma_N]\}$. By hypothesis (b) it is readily seen that this tends to zero.

Exactly the same argument holds for the sum over S_2.

The variance of the sum over S_3 is

$$(6.8) \quad \leqq [B_N^2/(2\pi)^4] \textstyle\sum_{|y| \leqq N} \sum_{(\nu_1,\nu_2)\varepsilon R_2} \sum_{(\nu_3,\nu_4)\varepsilon R_3} |w(B_N\nu_1, B_N\nu_2)w(B_N\nu_3, B_N\nu_4)$$

$$\cdot \{m_2(0, \nu_1)m_2(\nu_2, y)m_2(y + \nu_3, y + \nu_4)\}_{15}| + O(B_N)$$

where

$$R_2 = \{(\nu_1, \nu_2) \mid |\nu_1| \leqq N, |\nu_2| \leqq N, |\nu_1 - \nu_2| > 2^{-\frac{1}{2}}\gamma_N\},$$

$$R_3 = \{(\nu_3, \nu_4) \mid |\nu_3| \leqq N, |\nu_4| \leqq N, |\nu_3 - \nu_4| > 2^{-\frac{1}{2}}\gamma_N\}$$

and as above this also tends to zero. We demonstrate the calculations using the twelfth term as listed in Table III of [15]. The contribution of this term to (6.8) is (letting $\hat{y} = y - \nu_1$, $\hat{\nu}_1 = \nu_1 + \nu_3$ and $\hat{\nu}_2 = \nu_2 - \nu_1 + \nu_3 - \nu_4$)

$$\leqq B_N^2 \textstyle\sum_{|\hat{y}|,|\hat{\nu}_1|,|\hat{\nu}_2| \leqq 6N} \sum_{|\nu_3-\nu_4|>2^{-\frac{1}{2}}\gamma_N}$$

$$\cdot |w(B_N\hat{\nu}_3 - B_N\nu_1, B_N\hat{\nu}_2 + B_N\nu_4)w(B_N\nu_3, B_N\nu_4)|\, |r(\hat{y} + \hat{\nu}_1)r(\hat{y})r(\hat{y} - \hat{\nu}_2)|$$

$$\leqq \textstyle\sum_{|y|,|\nu_1|,|\nu_2| \leqq 6N} |r(y + \nu_1)r(y)r(y - \nu_2)|$$

$$\cdot \{[B_N^2 \textstyle\sum_{|\nu_3-\nu_4|>2^{-\frac{1}{2}}\gamma_N} w^2(B_N\nu_3 - B_N\nu_1, B_N\nu_2 + B_N\nu_4)]$$

$$\cdot [B_N^2 \textstyle\sum_{|\nu_3-\nu_4|>2^{-\frac{1}{2}}\gamma_N} w^2(B_N\nu_3, B_N\nu_4)]\}^{\frac{1}{2}}$$

which tends to zero by the properties of w, r and γ_N.

The next domain of summation to be considered is S_4. Define

$$(6.9) \quad Q_j^{(1)} = [B_N/(2\pi)^2 N^{\frac{1}{2}}] \textstyle\sum_{[c_j,[1,N],[1,N]]} \{\cdots\}, \qquad j = 1, \cdots, \alpha_N - 1,$$

then

$$(6.10) \quad E|[B_N/(2\pi)^2 N^{\frac{1}{2}}] \textstyle\sum_{S_4} \{\cdots\}|^2 = E|\sum_{j=1}^{\alpha_N - 1} Q_j^{(1)}|^2 \leqq [\sum_1^{\alpha_N-1} \{E|Q_j^{(1)}|^2\}^{\frac{1}{2}}]^2.$$

Next

$$E|Q_j^{(1)}|^2 \leqq (\gamma_N/N)B_N^2 \sum_{|\nu_1|,\cdots,|\nu_4| \leqq N} |w(B_N\nu_1, B_N\nu_2)w(B_N\nu_3, B_N\nu_4)| \cdot \sum_{y=-\gamma_N}^{\gamma_N} |\{m_2(0_1\nu_1)m_2(\nu_2, y)m_2(y+\nu_3, y+\nu_4)\}_{15}| \sim K\gamma_N/N$$

(K some constant).

Therefore

$$\sum_1^{\alpha_N - 1} \{E|Q_j^{(1)}|^2\}^{\frac{1}{2}} \leqq \alpha_N(\gamma_N/N)^{\frac{1}{2}}K^{\frac{1}{2}}$$

which tends to zero by hypothesis (c).

Sums over S_5 and S_6 behave the same thus giving the lemma. Q.E.D.

By Lemma 1, it remains to be shown that sums of the form (6.5) tend to a complex normal distribution in distribution. To do this define

$$\begin{aligned} U_j^{(N)} = {} & [B_N/(2\pi)^2] \sum_{[b_j^{(N)}; b_j^{(N)}; b_j^{(N)}]} \{\lambda_1 \cos[(v_2 - v_1)\mu_1 + (v_3 - v_1)\mu_2] \\ & + \lambda_2 \sin[(v_2 - v_1)\mu_1 + (v_3 - v_1)\mu_2]\} w(B_N(v_2 - v_1), B_N(v_3 - v_1)) \\ & \cdot [X_{v_1}X_{v_2}X_{v_3} - r_3(v_2 - v_1, v_3 - v_1)] \end{aligned} \tag{6.11}$$

where λ_1 and λ_2 are any 2 real parameters. By previous results (Theorem 1 and [15]) we know that since $B_N\beta_N \to \infty$,

$$\lim_{N\to\infty} \sigma^2(U_j^{(N)}/\beta_N^{\frac{1}{2}}) = \sigma_\lambda^2 \tag{6.12}$$

for $\sigma_\lambda^2 = \lambda_1^2\sigma_R^2 + \lambda_2^2\sigma_I^2$ where σ_R^2 and σ_I^2 are defined as the variances of the real and imaginary parts in the previous section. Then we show that

$$\sum_1^{\alpha_N} U_r^{(N)}/N^{\frac{1}{2}}\sigma_\lambda \to \sum_1^{\alpha_N} U_r^{(N)}/(\alpha_N\beta_N)^{\frac{1}{2}}\sigma_\lambda \to N(0, 1)$$

in distribution.

Set

$$G_{r,N}(x) = P\{U_r^{(N)}/(\alpha_N\beta_N)^{\frac{1}{2}}\sigma_\lambda \leqq x\} \tag{6.13}$$

then by arguments almost exactly the same as those of Rosenblatt [11], pp. 45–46, we see that the distribution we are interested in tends to the convolution,

$$G_{1,N} * \cdots * G_{\alpha_N,N}(x) \tag{6.14}$$

provided

$$\alpha_N(2t_\alpha/\delta_N)^{\alpha_N}\zeta(\gamma_N) \to 0 \tag{6.15}$$

as $N \to \infty$ where $t_\alpha = (\alpha_N/\epsilon)^{\frac{1}{2}}$, $\{\delta_N\}$ is a decreasing sequence of positive numbers with $\alpha_N\delta_N \to 0$, and ϵ is any constant > 0.

Thus if we can choose sequences $\{\alpha_N\}$, $\{\beta_N\}$, $\{\gamma_N\}$ and $\{\delta_N\}$ such that hypotheses (b) and (c) of Lemma 1 and (6.15) hold, we are finished since (6.14) tends to $N(0, 1)$ provided Liapounov's conditions are satisfied (see Loéve [8], p. 275). Hypothesis (e) guarantees these conditions.

Therefore we are finally left with the task of choosing the five sequences

$$B_N, \delta_N \downarrow 0 \quad \text{as} \quad N \to \infty,$$

$$\alpha_N, \beta_N, \gamma_N \uparrow \infty \quad \text{as} \quad N \to \infty$$

such that

(i) $B_N^2 N \to \infty$,

(ii) $\alpha_N(\beta_N + \gamma_N) \sim N$,

(iii) $\gamma_N = o(\beta_N)$,

(iv) $\gamma_N B_N \to \infty$,

(v) $\alpha_N^2 \gamma_N / N \to 0$,

(vi) $\alpha_N (2t_\alpha/\delta_N)^{\alpha_N} \zeta(\gamma_N) \to 0$,

(vii) $\alpha_N \delta_N \to 0$.

First choose $\{B_N\}$ satisfying (i). Now for example choose $\delta_N = \alpha_N^{-2}$ then (vii) is satisfied. Next choose $\alpha_N \leqq [-\log \zeta(\gamma_N)]^{\frac{1}{2}}$ for $\zeta(\gamma_N) < 1$ then $\alpha_N \to \infty$ and (vi) is satisfied (see Rosenblatt [11]). We can suppose that $\zeta(\gamma_N) > \gamma_N^{-1}$ for all N. Choosing $\gamma_N = B_N^{-1-\epsilon_1}$ for some $0 < \epsilon_1 < \frac{1}{2}$ (iv) is certainly satisfied and so is (v) since

$$-\log \zeta(\gamma_N) B_N^{-1-\epsilon_1}/N \leqq \log (B_N^{-1-\epsilon_1}) B_N^{-1-\epsilon_1}/N.$$

Finally choose β_N so that (ii) is satisfied and (iii) will be automatically satisfied and the theorem is proved. Q.E.D.

7. Proof of Theorem 1. Define

$$(7.1) \quad V_{NM} = [B_N/(2\pi)^2 N^{\frac{1}{2}}] \sum_{|\nu_1|, |\nu_2| \leqq MA_N} \exp(-i\mu_1\nu_1 - i\mu_2\nu_2) w(B_N\nu_1, B_N\nu_2) \cdot \sum_{t=1}^{N} (X_t X_{t+\nu_1} X_{t+\nu_2} - r_3(\nu_1, \nu_2)).$$

LEMMA 2. *The hypotheses of Theorem* 1 *imply that for any* $\epsilon > 0$, *there is an* $M_0(\epsilon)$ *such that for all* $M > M_0$, $N > M$;

$$\sigma^2(V_N - V_{NM}) < \epsilon.$$

PROOF. This lemma is readily proved using the bounds and summability conditions on w and the summability conditions on the cumulants (see Lemma 7 of [14] for the detail). Recall the remarks leading to (6.6) imply that most terms are $O(B_N)$. Q.E.D.

Lemma 2 indicates that the asymptotic distribution in question is the same as that of V_{NM}. The second modification is to replace the X_j in V_{NM} by $X_{j,k}$ to get $V_{NM}^{(k)}$. The next lemma shows that this can be done—that $\sigma^2(V_{NM} - V_{NM}^{(k)})$ can be made smaller than any previously chosen $\epsilon > 0$ uniformly in N for k sufficiently large (M being fixed).

LEMMA 3. *The hypotheses of Theorem* 1 *imply that for every* M, $\epsilon > 0$, *there is a* $k_0(\epsilon, M)$ *independent of* N *and a constant* K *independent of both* M *and* N *such that for all* $k > k_0(\epsilon, M)$;

$$\sigma^2(V_{NM} - V_{NM}^{(k)}) \leqq \epsilon + KB_N.$$

(Note that the conditions are sufficient to remove the KB_N term on the bound.)

PROOF. Due to the uniform summability conditions on the cumulants, we can uniformly approximate the appropriate sums in this difference by finite sums. Then the fact that $r^{(\infty,k)}(\nu) \to r(\nu)$ and $r^{(k,k)}(\nu) \to r(\nu)$ as $k \to \infty$ gives the desired result. See, for example, Chapter VII of Doob [3] on martingales. Q.E.D.

To apply a central limit theorem, write

$$U_R^{(N)} = \text{Re } V_{MN}^{(k)} = N^{-\frac{1}{2}} \sum_1^N Y_t^{(k,N,M)},$$
$$U_I^{(N)} = \text{Im } V_{NM}^{(k)} = N^{-\frac{1}{2}} \sum_1^N Z_t^{(k,N,M)},$$

where

$$Y_t^{(k,N,M)} = [B_N/(2\pi)^2] \sum_{|\nu_1|,|\nu_2| \leq MA_N} \cos(\mu_1\nu_1 + \mu_2\nu_2) \cdot w(B_N\nu_1, B_N\nu_2)[X_{t,k}X_{t+\nu_1,k}X_{t+\nu_2,k} - r_3^{(k,k,k)}(\nu_1, \nu_2)]$$

and $Z_t^{(k,M,N)}$ is as above except with a sine instead of cosine. For any two real parameters, λ_1 and λ_2, form

$$U_N(\lambda_1, \lambda_2) = \lambda_1 U_R^{(N)} + \lambda_2 U_I^{(N)} \tag{7.2}$$

with $U_t^{(k,N,M)} = N^{-\frac{1}{2}}(\lambda_1 Y_t^{(k,N,M)} + \lambda_2 Z_t^{(k,N,M)})$. Note that the $\{U_t^{(k,N,M)}\}$ sequence is a $2MA_N + 2k$ step dependent process. (See Hoeffding and Robbins [5].) This prompts one to use the following lemma from Rosenblatt [13], p. 262.

LEMMA 4. *If* (a) $\{V_t^{(N)}\}$ *is a sequence of* $d(N)$*-dependent strictly stationary random variables,*

(b) $d(N) \to \infty$ *as* $N \to \infty$,

(c) $d(N)/N \to 0$ *as* $N \to \infty$,

(d) $E|V_t^{(N)}|^{2+\delta} < \infty$ *for some* $\delta > 0$,

(e) $t(N)$ *is an integer-valued function*

(i) $t(N) \to 0$,

(ii) $d(N) = o(t(N))$,

(iii) $t(N) = o(N)$,

(f) *for* $\{r^{(N)}\}$ *the covariance sequence of* $\{V_t^{(N)}\}$, $\sum_{|\nu| \leq t(N)} |\nu| r_\nu^{(N)} = o(\sum_{|\nu| \leq t(N)} r_\nu^{(N)} t(N))$ *as* $N \to \infty$,

(g) $E|\sum_1^{t(N)} V_t^{(N)}|^{2+\delta}/N^{\delta/2} t(N)(\sum_{-d(N)}^{d(N)} r_\nu^{(N)})(1 + \delta/2) \to 0$ *as* $N \to \infty$,

then $\sum_1^N V_t^{(N)}$ *is asymptotically normally distributed with mean zero and variance* $2\pi N h_N(0)$, *where* $h_N(\lambda)$ *is the spectral density of* $\{V_t^{(N)}\}$.

To apply this lemma to (7.2), put

$$V_t^{(N)} = U_t^{(k,N,M)},$$
$$d(N) = 2MA_N + 2k,$$
$$t(N) = MA_N^2,$$
$$\delta = 2.$$

Conditions (a), (b), (c), and (e) of Lemma 4 are certainly satisfied. Condition

(d) is satisfied since there is a constant K so that

$$E|V_t^{(N)}|^4 \leqq (B_N^4/N^2) \sum_{|\nu_1|,\cdots,|\nu_8| \leqq MA_N} |w(B_N\nu_1, B_N\nu_2) \cdots w(B_N\nu_7, B_N\nu_8)| \cdot KEX_t^{12}$$
$$\leqq \hat{w}^4 KEX_t^{12}\, 2/(NB_N^2)^2 < \infty.$$

Condition (f) involves

$$(7.3) \quad \sum_{|\nu| \leqq MA_N^2} |\nu| r_\nu^{(N)} / \sum_{|\nu| \leqq MA_N^2} r_\nu^{(N)} MA_N^2 = N \sum_{|\nu| \leqq MA_N^2} (|\nu|/MA_N^2) r_\nu^{(N)} / N \sum_{|\nu| \leqq MA_N^2} r_\nu^{(N)}.$$

But,

$$(7.4) \quad \begin{aligned} N \sum_{|\nu| \leqq MA_N^2} r_\nu^{(N)} &= N \sum_{|\nu| \leqq MA_N^2} EV_0^{(N)} V_\nu^{(N)} \\ &= [B_N^2/(2\pi)^4] \sum_{|\nu| \leqq MA_N^2} \sum_{|\nu_1|, |\nu_2| \leqq MA_N} \\ &\quad \cdot [\lambda_1 \cos(\mu_1\nu_1 + \mu_2\nu_2) + \lambda_2 \sin(\mu_1\nu_1 + \mu_2\nu_2)] \\ &\quad \cdot w(B_N\nu_1, B_N\nu_2) \sum_{|\nu_3|, |\nu_4| \leqq MA_N} \\ &\quad \cdot [\lambda_1 \cos(\mu_1\nu_3 + \mu_2\nu_4) + \lambda_2 \sin(\mu_1\nu_3 + \mu_2\nu_4)] \\ &\quad \cdot w(B_N\nu_3, B_N\nu_4) \\ &\quad \cdot [r_6(\nu_1, \nu_2, \nu, \nu + \nu_3, \nu + \nu_4) - r_3(\nu_1, \nu_2) r_3(\nu_3, \nu_4)] \end{aligned}$$

and this, from earlier results, converges absolutely uniformly in N. Therefore provided $Nh_N(0) \neq 0$ and since $|\nu|/MA_N^2$ converges to zero pointwise, (7.3) tends to zero. Finally condition (g) leads to

$$(7.5) \qquad E[\textstyle\sum_{t=1}^{MA_N^2} V_t^{(N)}]^4 / NMA_N^2 (\sum r_\nu^{(N)})^2.$$

The denominator $\sim N^{-1} B_N^{-2}$ by (7.4). Define

$$D_j = \sum_{t=(j-1)d(N)+1}^{jd(N)} V_t^{(N)}, \qquad 1 \leqq j \leqq 2u_0;$$

$$2u_0 = \text{largest even integer} \leqq MA_N^2/d(N),$$

$$\begin{aligned} D_{2u_0+1} &= 0, & MA_N^2 &= 2u_0\, d(N); \\ &= \sum_{2u_0 d(N)+1}^{\min[MA_N^2, (2u_0+1)d(N)]} V_t^{(N)} & MA_N^2 &> 2u_0\, d(N); \\ D_{2u_0+2} &= 0, & MA_N^2 &\leqq (2u_0 + 1)\, d(N), \\ &= \sum_{(2u_0+1)d(N)+1}^{MA_N^2} V_t^{(N)}, & MA_N^2 &> (2u_0 + 1)\, d(N); \end{aligned}$$

then

$$\sum_{t=1}^{MA_N^2} V_t^{(N)} = \sum_{j=1}^{u_0+1} D_{2j-1} + \sum_{j=1}^{u_0+1} D_{2j}.$$

By Minkowski's inequality, the fourth root of the numerator of (7.5) is

$$(7.6) \qquad \leqq [E(\textstyle\sum_{j=1}^{u_0+1} D_{2j-1})^4]^{\frac{1}{4}} + [E(\sum_{j=1}^{u_0+1} D_{2j})^4]^{\frac{1}{4}}.$$

Note that the D_j's in the first term are independent of one another and have zero mean so that

$$(7.7)\quad E\left(\sum_{j=1}^{u_0+1} D_{2j-1}\right)^4 = \sum_{j=1}^{u_0+1} ED_{2j-1}^4 + 3\sum_{j_1,j_2=1}^{u_0+1} ED_{2j_1-1}^2 ED_{2j_2-1}^2 - 3\sum_{j=1}^{u_0+1} ED_{2j-1}^2 ED_{2j-1}^2 .$$

Further

$$(7.8)\quad \begin{aligned} E\left(\sum_1^{d(N)} V_t\right)^4 \leqq (B_N{}^4/N^2) \sum_{t_1,\cdots,t_4=1}^{d(N)} \sum_{|\nu_1|,\cdots,|\nu_8| \leqq MA_N} M_1{}^4 \\ \cdot \left|E\{[X_{t_1,k}X_{t_1+\nu_1,k}X_{t_1+\nu_2,k} - r_3{}^{(k,k)}(\nu_1, \nu_2)] \right. \\ \left. \cdots [X_{t_4,k}X_{t_4+\nu_7,k}X_{t_4+\nu_8,k} - r_3{}^{(k,k)}(\nu_3, \nu_4)]\}\right|. \end{aligned}$$

To get a bound on this recall that the $X_{j,k}$ are k-step dependent and have mean zero. This implies that each $X_{j,k}$ must be within k steps of at least one other $X_{j,k}$ otherwise the expected value of the product is zero. Hence the sum in (7.8) has many zero terms. The fact that $EX_t^{12} < \infty$ says that all non-zero terms are bounded. Thus we can get a bound by enumerating the non-zero terms. Look first at the terms with indices such that all X_j's are "tied together," that is there is no way of dividing the X_j's into 2 groups so that one group of random variables is independent of the other. It is easily seen that there can be only $O(d(N)k^{11})$ number of such terms. One could proceed down the line and look at those groups of indices such that the X_j's divide into 2 and only 2 independent groups of 2 in the first group and 10 in the second or of 3 in the first and 9 in the second, etc.; then go on to the various combinations of 3 groups, 4 groups, 5 groups, and lastly 6 groups. It is seen that the highest order number of non-zero terms occurs when there are 6 independent groups (implies two X_j's in each group). Here there are $O(d^6(N)k^6)$ such non-zero terms. This means that the first term of (7.7) has $O(u_0 A_N{}^6)$ non-zero terms all of which are $O(B_N{}^4/N^2)$, so that the contribution of it to the numerator of (7.5) is $O(A_N{}^3/N^2)$ and comparing this with the denominator obtain $O(A_N/N) \to 0$.

The next term in (7.7) is

$$3\left(\sum_{j=1}^{u_0} ED_{2j-1}^2\right)^2 = O\left(u_0\, d(N) \sum_{-d(N)}^{d(N)} r_\nu{}^{(N)}\right)^2.$$

Comparing this with the denominator get

$$O\left(A_N{}^4\left(\sum r_\nu{}^{(N)}\right)^2 / NA_N{}^2\left(\sum r_\nu{}^{(N)}\right)^2\right) = O((NB_N{}^2)^{-1}) \to 0.$$

The third term of (7.7) is contained in the case just discussed. Thus Lemma 4 applies.

The proof of the theorem is complete except for the evaluation of the variances and covariances of the real and imaginary parts. Lemma 4 states that

$$\operatorname{Re} V_{NM}^{(k)} + i \operatorname{Im} V_{NM}^{(k)} \to_{\text{dist}} X_M{}^{(k)} + iY_M{}^{(k)} \quad \text{as} \quad N \to \infty$$

where $X_M{}^{(k)}$ and $Y_M{}^{(k)}$ are jointly normal with mean zero and

$$E(X_M{}^{(k)})^2 = \sigma_{kMR}^2 ,$$

$$E(Y_M^{(k)})^2 = \sigma_{kMI}^2,$$

$$E(X_M^{(k)} Y_M^{(k)}) = r_{kM}.$$

As $k \to \infty$, $\sigma_{kMR}^2 \to \sigma_{MR}^2$, $\sigma_{kMI}^2 \to \sigma_{MI}^2$, and $r_{kM} \to r_M$. A lemma is needed to evaluate σ_{MR}^2, σ_{MI}^2, and r_M.

LEMMA 5. *If* (a) $h(x)$, $h(x, y)$ *are bounded and their respective sets*, D_1 *and* D_2, *of points of discontinuity have measure zero,*

(b) A_N *and* B_N *as in Theorem* 1,

(c) μ, μ_1, μ_2 *are real constants*

then (1) $B_N \sum_{|\nu| \leqq MA_N} \sin \nu\mu h(B_N \nu) \to 0$ *as* $N \to \infty$,

(2) $B_N \sum_{|\nu| \leqq MA_N} \cos \nu\mu h(B_N \nu) \to 0$ *as* $N \to \infty$ *provided* $\mu \neq 0, \pm 2\pi, \cdots$,

(3) $B_N^2 \sum_{|\nu_1|,|\nu_2| \leqq MA_N} \sin \nu_1\mu_1 \sin \nu_2\mu_2 h(B_N\nu_1, B_N\nu_2) \to 0$ *as* $N \to \infty$,

(4) $B_N^2 \sum_{|\nu_1|,|\nu_2| \leqq MA_N} \sin \nu_1\mu_1 \cos \nu_2\mu_2 h(B_N\nu_1, B_N\nu_2) \to 0$ *as* $N \to \infty$,

(5) $B_N^2 \sum_{|\nu_1|,|\nu_2| \leqq MA_N} \cos \nu_1\mu_1 \cos \nu_2\mu_2 h(B_N\nu_1, B_N\nu_2) \to 0$ *as* $N \to \infty$ *unless* $\mu_1 = 0, \pm 2\pi, \cdots$ *and* $\mu_2 = 0, \pm 2\pi, \cdots$.

PROOF. First prove for step functions and then use the fact that any $h(x)$ satisfying the conditions of the lemma can be approximated in mean by step functions. Q.E.D.

The three quantities σ_{MR}^2, σ_{MI}^2 and r_M must be evaluated separately for each of the fifteen terms corresponding to the fifteen terms, $\{m_2(0, \nu_1) m_2(\nu_2, y) m_2(y + \nu_3, y + \nu_4)\}_{15}$ (see Table III of [15]). This is a very tedious task and will be illustrated by just one such calculation. More detail may be found in [14]. Consider the δ-function as being periodic in 2π. Looking at the calculation of σ_{MR}^2 for the first of the fifteen terms as listed in Table III of [15] we have the expression

$$[B_N^2/(2\pi)^4] \sum_{|\nu_1|,\cdots,|\nu_4| \leqq MA_N} \sum_{|y| \leqq N} [(N - |y|)/N] \cos(\nu_1\mu_1 + \nu_2\mu_2)$$
$$\cdot \cos(\nu_3\mu_1 + \nu_4\mu_2) w(B_N\nu_1, B_N\nu_2) w(B_N\nu_3, B_N\nu_4) r(\nu_1) r(y - \nu_2) r(\nu_4 - \nu_3).$$

This behaves like

$$[f(0)/(2\pi)^3] B_N^2 \sum_{|\nu_1|,\cdots,|\nu_4| \leqq MA_N} \cos(\nu_1\mu_1 + \nu_2\mu_2) \cos(\nu_3(\mu_1 + \mu_2) + \mu_2\nu_4)$$
$$\cdot w(0, B_N\nu_2) w(-B_N\nu_3, 0) r(\nu_1) r(\nu_4)$$

using the modified continuity conditions, (v). Using trigonometric identities this is

$$[f(0)/(2\pi)^3] \sum_{|\nu_1|,\cdots,|\nu_4| \leqq MA_N} [\cos \nu_1\mu_1 \cos \nu_2\mu_2 - \sin \nu_1\mu_1 \sin \nu_2\mu_2]$$
$$\cdot [\cos \nu_3(\mu_1 + \mu_2) \cos \nu_4\mu_2 - \sin \nu_3(\mu_1 + \mu_2) \sin \nu_4\mu_4]$$
$$\cdot B_N^2 w(0, B_N\nu_2) w(-B_N\nu_3, 0) r(\nu_1) r(\nu_4)$$
$$\to [f(0)/(2\pi)^3][\textstyle\int_{-M}^{M} w(0, \nu)\, d\nu]^2 \sum_{\nu_1,\nu_2=-\infty}^{+\infty} \cos \nu_1\mu_1 \cos \nu_2\mu_2 r(\nu_1) r(\nu_2)$$
$$\cdot \delta(\mu_1 + \mu_2) \delta(\mu_2)$$
$$= (w_1/2\pi) f(0) f(\mu_1) f(\mu_2) \delta(\mu_1 + \mu_2) \delta(\mu_2)$$

by Lemma 5. Q.E.D.

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