SOME LIMIT THEOREMS FOR NON-HOMOGENEOUS MARKOV CHAINS

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1. Introduction. This paper deals with the problem of finding (necessary or sufficient) conditions for the relative stability and for the strong relative stability of sums of random variables (rv) which form a non-homogeneous Markov chain; we obtain also some results for the sums of arbitrarily dependent rv.

The results obtained in this paper are of classical form, i.e. they come very close to those obtained for mutually independent rv ([1], [3]–[7]); these classical results themselves remain true for a very large class of non-homogeneous Markov chains ($\alpha_i > \lambda > 0$, $i \varepsilon I = (1, 2, \cdots)$) and some of them for arbitrarily dependent rv. In the same way we obtain new results for homogeneous Markov chains ($\alpha_i = \lambda > 0$, $i \varepsilon I$). These results contain as particular cases the analogous results for mutually independent rv ($\alpha_i = 1$, $i \varepsilon I$). This paper contains also some new results for mutually independent rv.

For Markov chains, we express our results by means of the *ergodic coefficient* α of a stochastic transition function [2]; in [9] there can be found several of its definitions and properties that we shall use here; we shall also use the concept of *p-quantile*, the properties of which may be found in [13].

A part of these results were announced in preliminary papers ([11], [12]).

2. Notations. Let $(\mathfrak{A}_i, \Sigma_i)$ be a measurable space, x_i the elements of $\mathfrak{A}_i$, A_i measurable sets, elements of the σ-algebra Σ_i ($i \varepsilon I$). If the sequence of rv $\{\xi_i\}$ is a Markov chain, let us consider that it has the stochastic transition functions $P_i(x_i, A_{i+1})$. Let α_i denote the ergodic coefficient of P_i; that is

$$\alpha_i = 1 - \sup_{x,y \varepsilon \mathfrak{A}_i, A \varepsilon \Sigma_{i+1}} |P_i(x, A) - P_i(y, A)|.$$

Set $\beta_n = \min_{1 \leqq i < n} \alpha_i$. Assume $\alpha_i > 0$ for each $i \varepsilon I$, because in many important formulae ([8]–[10])β_n appears in the denominator.

3. Definitions. $\{\xi_n\}$ is (a) *stable* (S); (b) *strongly stable* (SS); (c) *relatively stable* (RS); (d) *strongly relatively stable* (SRS) if there is some sequence of constants $\{d_n\}$ so that respectively (a) $\{\xi_n - d_n\}$ converges in probability to zero, (b) $\{\xi_n - d_n\}$ converges almost everywhere to zero, (c) $\{\xi_n/d_n\}$ converges in probability to one, (d) $\{\xi_n/d_n\}$ converges almost everywhere to one.

$\{\xi_n\}$ is (a) *normally stable* (NS), (b) *normally strongly stable* (NSS), (c) *normally relatively stable* (NRS), (d) *normally strongly relatively stable* (NSRS) if in the given definitions we may take $d_n = M\xi_n$, the expectation of ξ_n.

We set $\zeta_n = \sum_{i=1}^{n} \xi_i$.

Let us suppose that $\{\zeta_n\}$ is RS with constants $\{d_n\}$; the ξ_i/d_n ($1 \leqq i \leqq n \varepsilon I$)

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are *asymptotically constant* if there exists a sequence of constants $\{a_n\}$ so that for any $\epsilon > 0$,

$$\max_{1 \leqq k \leqq n} P\{|\xi_k - a_k|/d_n \geqq \epsilon\} \to 0 \qquad (n \to \infty). \tag{1}$$

Let us suppose that $\{\zeta_n\}$ is SRS with constants $\{d_n\}$; then ξ_i/d_n $(1 \leqq i \leqq n \varepsilon I)$ are *strongly asymptotically* constant if there exists a sequence of constants $\{a_n\}$ so that

$$P\{\max_{1 \leqq k \leqq n} |\xi_k - a_k|/d_n \to 0, n \to \infty\} = 1. \tag{2}$$

They are *asymptotically infinitesimal,* respectively *strongly asymptotically infinitesimal,* if $a_n = 0$ $(n \varepsilon I)$.

The sequences $\{\xi_{ni}\}$ $(i = 1, 2)$ are *equivalent* if $P\{\xi_{n1} \neq \xi_{n2}\} \to 0$ $(n \to \infty)$ and *strongly equivalent* if $\sum_{n=1}^{\infty} P\{\xi_{n1} \neq \xi_{n2}\} < +\infty$.

We shall exclude the case when all the $\{\xi_n\}$ are constants.

4. Used results. Let us denote by $D\xi$ the variance of the rv ξ and

$$D_n = \sum_{i=1}^{n} D\xi_i .$$

LEMMA L_1. ([8], [9]). *If the* rv $\{\xi_i\}$ *form a Markov chain, they satisfy the inequalities* $C'D_n\beta_n \leqq D\zeta_n \leqq CD_n/\beta_n$, *where* $C = 9 + 8.6^{\frac{1}{2}}$, $C' = 10^{-2}$.

If the rv $\{\xi_i\}$ *form an* m*th order Markov chain, they satisfy the inequality* $D\zeta_n \leqq C \cdot D_n/\beta_n$ *where* $C = 8(1 + 6^{\frac{1}{2}})m + 1$.

Let us denote by $r(\xi; p)$ or (if we do not specify the value of p) by $r\xi$ the p-quantile $(0 < p < 1; q = 1 - p)$ of the rv ξ i.e. the real number for which the two inequalities $P\{\xi \leqq r\xi\} \geqq p$, $P\{\xi \geqq r\xi\} \geqq q$ are simultaneously satisfied.

LEMMA L_2 [13]. *For any real valued* rv ξ *and any real number* a *the inequality* $P\{\xi \leqq a\} \geqq p$ *implies* $r(\xi; p) \leqq a$ *and* $P\{\xi \geqq a\} \geqq q$ *implies* $r(\xi; p) \geqq a$; $\xi \leqq \eta$ *implies* $r\xi \leqq r\eta$.

Let us denote by E' the complement of the event E.

LEMMA L_3 [10]. *If for a sequence* $\{E_i\}$ *of random events in a Markov chain, the series*

$$\sum_{i=1}^{\infty} P(E_i \mid E'_{i-1})$$

diverges, then with probability 1 *an infinite set of the* E_i *occurs.*

5. Auxiliary results. Here $\{\xi_n\}$ is a sequence of arbitrarily dependent rv.

LEMMA 1. *If* $\{\xi_n\}$ *is* S *or* SS *with constants* $\{d_n\}$, *then the necessary and sufficient condition to be so also with the constants* $\{d_n'\}$ *is* $d_n' - d_n \to 0$ $(n \to \infty)$.

If $\{\xi_n\}$ *is* RS *or* SRS *with the constants* $\{d_n\}$, *then the necessary and sufficient condition to be so also with the constants* $\{d_n'\}$ *is* $d_n'/d_n \to 1$ $(n \to \infty)$.

The *proof* follows immediately from the definitions of S, SS, RS, SRS.

Let us write $r(\xi_n ; p) = r_n$.

LEMMA 2. *If* $\{\xi_n\}$ *is* S *or* SS *with the constants* $\{d_n\}$, *then* $r_n - d_n \to 0$ $(n \to \infty)$ *for any* $p(0 < p < 1)$.

If $\{\xi_n\}$ *is* RS *or* SRS *with the constants* $\{d_n\}$, *then* $r_n/d_n \to 1$ $(n \to \infty)$ *for any* $p(0 < p < 1)$.

Proof. Let us consider the rv $\lambda = \xi_n - d_n$, $\mu = (\xi_n/d_n) - 1$ and the random events

$$A_1 = \{|\lambda| \leqq \epsilon\}, \qquad A_2 = \{|\mu| \leqq \epsilon\},$$
$$B_1 = \{\lambda \leqq \epsilon\}, \qquad B_2 = \{\mu \leqq \epsilon\},$$
$$C_1 = \{\lambda \geqq -\epsilon\}, \qquad C_2 = \{\mu \geqq -\epsilon\}.$$

Obviously $A_i \subset B_i$, $A_i \subset C_i$ $(i = 1, 2)$ and from the definitions of S, RS, follow $P(B_i) \to 1$, $P(C_i) \to 1$ $(n \to \infty)$ so that $P(B_i) \geqq p$, $P(C_i) \geqq q$ $(n > N)$ for any given p $(i = 1, 2)$. From Lemma L_2 it follows that $|r_n - d_n| \leqq \epsilon$ $(n > N)$ in the case of S and $|(r_n/d_n) - 1| \leqq \epsilon$ $(n > N)$ in the case of RS. Because SS implies S and SRS implies RS our lemma is proved.

Corollary. *If* $\{\xi_n\}$ *is* S, SS, RS, *or* SRS, *we may consider that the defining constants are* $\{r_n\}$ *for any* $p(0 < p < 1)$.

Let us write $r(\xi_{ni}; p_i) = r_{ni}$.

Lemma 3. *If* $\{\xi_{ni}\}$ $(i = 1, 2)$ *are equivalent, then they are* S *or* RS *only simultaneously and in this case* $r_{n1} - r_{n2} \to 0$ $(n \to \infty)$ *for any* p_i $(0 < p_i < 1, i = 1, 2)$.

If $\{\xi_{ni}\}$ $(i = 1, 2)$ *are strongly equivalent, then they are* SS *or* SRS *only simultaneously and in this case* $r_{n1}/r_{n2} \to 1$ $(n \to \infty)$ *for any* p_i $(0 < p_i < 1, i = 1, 2)$.

Proof. Let us set

$$\eta_{kij} = \xi_{ki} - r_{kj}, \qquad A_{1ij} = \{|\eta_{nij}| > \epsilon\},$$
$$A_{2ij} = \{\sup_{k>n} |\eta_{kij}| > \epsilon\}, \qquad A_{3ij} = \{|\eta_{nij}| > \epsilon \cdot r_{nj}\},$$
$$A_{4ij} = \{\sup_{k>n} r_{kj}^{-1}|\eta_{kij}| > \epsilon\},$$
$$E_{1n} = E_{3n} = \{\xi_{n1} = \xi_{n2}\}, \qquad E_{2n} = E_{4n} = \textstyle\bigcap_{k>n} E_{1k}, \quad (i, j = 1, 2; k, n \in I).$$

If $P(A_{sij} \mid E_{sn})$, $P(A_{sij} \mid E'_{sn})$ are conditional probabilities, then

$$P(A_{sij}) = P(E_{sn})P(A_{sij} \mid E_{sn}) + P(E'_{sn})P(A_{sij} \mid E'_{sn})$$
$$\leqq P(A_{sij} \mid E_{sn}) + P(E'_{sn}) = P(A_{sii}) + P(E'_{sn}),$$

i.e.

$$|P(A_{sii}) - P(A_{sij})| \leqq P(E'_{sn}), \qquad (i, j = 1, 2; 1 \leqq s \leqq 4, n \in I)$$

The proof follows from Lemma 2 and its corollary, together with the fact that $P\{\sup_{k>n} |\omega_k| > \epsilon\} \to 0$ $(n \to \infty)$ for any $\epsilon > 0$ is equivalent with $P\{\omega_n \to 0, n \to \infty\} = 1$.

Remark. Because $\{\xi_n\}$ is obviously equivalent and also strongly equivalent with itself, we obtain the following results: Let us denote $r(\xi_n; p_i) = r_{ni}$. If $\{\xi_n\}$ is S or SS then $r_{n1} - r_{n2} \to 0$ $(n \to \infty)$ and if it is RS or SRS, then $r_{n1}/r_{n2} \to 1$ $(n \to \infty)$ for any p_i $(0 < p_i < 1; i = 1, 2)$. Let us denote $r(\xi_n, p) = r_n$, $M\xi_n = t_n$. If $\{\xi_n\}$ is NS or NSS then $t_n - r_n \to 0$ $(n \to \infty)$ and if it is NRS or NSRS then $t_n/r_n \to 1$ $(n \to \infty)$ for any p $(0 < p < 1)$.

LEMMA 4. *For an increasing sequence of positive numbers* $\{\rho_n\}$ *we denote* $a_n = \rho_n - \rho_{n-1}$. *The conditions* (a) $\rho_n \to \infty$, $\rho_{n-1}/\rho_n \to 1$ $(n \to \infty)$ *and* (b) $\max_{1 \leqq k \leqq n} (a_k/\rho_n) \to 0$ $(n \to \infty)$ *are equivalent.*

The proof of Lemma 4 is left to the reader.

6. Results for arbitrarily dependent rv. Let us denote $r(\xi_k ; p) = r_k$, $M\xi_k = t_k$, $u_{k1} = u_{k3} = r_k$, $u_{k2} = u_{k4} = t_k$ $(k \varepsilon I)$, $\eta_{ni} = \sup_{k>n} |\xi_k - u_{ki}|$, $(i = 1, 2)$, $\eta_{ni} = \sup_{k>n} |(\xi_k/u_{ki}) - 1|$, $(i = 3, 4)$, $P\{\xi_n < x\} = F_n(x)$, $P\{\eta_{ni} < x\} = F_{ni}(x)$, $(1 \leqq i \leqq 4), f(x) = x^2/(1 + x^2)$.

THEOREM 1. *The relation*

$$\int f(x) \cdot d\Psi_n[\pi_n(x)] \to 0 \qquad (n \to \infty)$$

is a necessary and sufficient condition for $\{\xi_n\}$ *to be:*

(a) S *if* $\Psi_n = F_n$, $\pi_n(x) = x + r_n$;
(b) SS *if* $\Psi_n = F_{n1}$, $\pi_n(x) = x$;
(c) RS *if* $\Psi_n = F_n$, $\pi_n(x) = (1 + x)r_n$;
(d) SRS *if* $\Psi_n = F_{n3}$, $\pi_n(x) = x$;
(a′) NS *if* $\Psi_n = F_n$, $\pi_n(x) = x + t_n$;
(b′) NSS *if* $\Psi_n = F_{n2}$, $\pi_n(x) = x$;
(c′) NRS *if* $\Psi_n = F_n$, $\pi_n(x) = (1 + x)t_n$;
(d′) NSRS *if* $\Psi_n = F_{n4}$, $\pi_n(x) = x$.

For all these cases sufficient conditions are

$$\int x^2 \, d\Psi_n[\pi_n(x)] \to 0 \qquad (n \to \infty).$$

Also $D\xi_n/u_{ni}^2 \to 0$ $(n \to \infty)$ *implies* RS $(i = 1)$ *and* NRS $(i = 2)$.

PROOF. We obtain the first results using the following facts: (1) the corollary of Lemma 2; (2) the remark at the end of the proof of Lemma 3; (3) the necessary and sufficient condition for $\{\xi_n\}$ to be asymptotically infinitesimal is $\sup_{1 \leqq k \leqq n} \int f(x) \, dF_n(x) \to 0$ $(n \to \infty)$, ([3], Section 20, Lemma 2). The other results are consequences of these.

Let us denote $r(\zeta_n ; p) = \rho_n$.

THEOREM 2. *Let us suppose that* $\xi_n \geqq 0$ $(n \varepsilon I)$. *The relation*

$$\rho_n \to \infty \qquad (n \to \infty) \tag{3}$$

(a) *is a necessary and sufficient condition for* ξ_i/ρ_n $(1 \leqq i \leqq n \varepsilon I)$ *to be asymptotically constant if* $\{\zeta_n\}$ *is* RS;

(b) *is a necessary and sufficient condition for* ξ_i/ρ_n $(1 \leqq i \leqq n \varepsilon I)$ *to be strongly asymptotically constant if* $\{\zeta_n\}$ *is* SRS.

PROOF. Let us denote:

$$\varphi_1 = (\zeta_{k_n}/\rho_{k_n}) - 1; \qquad \varphi_2 = (\xi_{k_n} - a_{k_n})/\rho_{k_n};$$

$$\varphi_3 = (\zeta_{k_n-1}/\rho_{k_n-1}) - 1; \qquad \varphi_4 = (\xi_{k_n} - a_{k_n})/\rho_n;$$

$$\omega_1 = \{|\varphi_1| \geqq \delta/2\}; \qquad \omega_2 = \{|\varphi_2| \geqq \delta\};$$

$$\omega_3 = \{|\varphi_3| \leqq \delta/2\}; \qquad \omega_4 = \{|\varphi_4| \geqq \delta\};$$

$$\omega_5 = \{\max_{1 \leqq k \leqq n} (|\xi_k - a_k|/\rho_n) \to 0, n \to \infty\};$$

$$\omega_6 = \{|\varphi_4| \geqq \delta; n > N\}; \qquad \omega_7 = \{|\varphi_2| \geqq \delta; n > N\};$$

$$\omega_8 = \{\varphi_1 \to 0, n \to \infty\}; \qquad \omega_9 = \{\varphi_3 \to 0, n \to \infty\};$$

$$\omega_{10} = \{\varphi_2 \to 0, n \to \infty\}; \qquad a_n = \rho_n - \rho_{n-1}.$$

Let us consider that ξ_k is the first non-identically vanishing rv in $\{\xi_n\}$ and set

$$\omega_{11} = \{|\xi_k - a_k|/\rho_n > \epsilon\}; \qquad \omega_{12} = \{|\xi_k - a_k|/\rho_n \to 0; n \to \infty\}.$$

(a) *Sufficiency.* We shall prove that if $\{\zeta_n\}$ is RS, then (3) implies (1).

Let us suppose that this is not true, i.e. that there exist some numbers $\alpha > 0$, $\delta > 0$, so that for every $n \,\varepsilon\, I$ we may find $k_n \,\varepsilon\, I (k_n \leqq n)$ for which, considering the corollary of Lemma 2, $P(\omega_4) \geqq \alpha$. In this case $k_n \to \infty$ $(n \to \infty)$ because otherwise this inequality would not be true beginning with a definite n.

From Lemma L_2 it follows that $\rho_{k_n-1} \leqq \rho_{k_n} \leqq \rho_n$ so that $\omega_4 \subset \omega_2$. From $\varphi_1 = \varphi_2 + \varphi_3 \cdot \rho_{k_n-1}/\rho_{k_n}$, it follows that $|\varphi_1| \geqq |\varphi_2| - |\varphi_3|$ and therefore $\omega_2 \cap \omega_3 \subset \omega_1$, i.e. $\omega_3 \cap \omega_4 \subset \omega_1$, $P(\omega_1') \leqq P(\omega_3') + P(\omega_4')$. Because $\{\zeta_n\}$ is RS we have $P(\omega_3') < \alpha/2$ $(n > N)$ and consequently $P(\omega_1) \geqq \alpha/2$ $(n > N)$ which is in contradiction with the RS of $\{\zeta_n\}$.

Necessity. From (1) it follows that $P(\omega_{11}) \to 0$ $(n \to \infty)$ i.e. (3).

(b) *Sufficiency.* We shall prove that if $\{\zeta_n\}$ is SRS, then (3) implies (2). Let us suppose that this is not true, i.e. there exists a number $\alpha > 0$ so that $P(\omega_5) < 1 - \alpha$. In this case there exists a number $\delta > 0$, a sequence $k_n \leqq n$ $(k_n \,\varepsilon\, I)$ and a number N so that $P(\omega_6) \geqq \alpha$. As in the previous case $k_n \to \infty$ $(n \to \infty)$; obviously $\omega_6 \subset \omega_7$. Because $\{\zeta_n\}$ is SRS we have $P(\omega_8) = P(\omega_9) = 1$ and from $|\varphi_2| \leqq |\varphi_1| + |\varphi_3|$ it follows that $\omega_8 \cap \omega_9 \subset \omega_{10}$ i.e. $P(\omega_{10}) = 1$, which is in contradiction with $P(\omega_7) \geqq \alpha$.

Necessity. From (2) it follows that $P(\omega_{12}) = 1$, i.e. (3).

THEOREM 3. *Let us suppose that* $\xi_n \geqq 0$ $(n \,\varepsilon\, I)$. *The relations* (3) *and*

$$(4) \qquad \rho_{n-1}/\rho_n \to 1 \qquad (n \to \infty)$$

(a) *are a necessary and sufficient condition for* ξ_i/ρ_n $(1 \leqq i \leqq n \,\varepsilon\, I)$ *to be asymptotically infinitesimal, if* $\{\zeta_n\}$ *is* RS;

(b) *are a necessary and sufficient condition for* ξ_i/ρ_n $(1 \leqq i \leqq n \,\varepsilon\, I)$ *to be strongly asymptotically infinitesimal, if* $\{\zeta_n\}$ *is* SRS.

PROOF. Let us denote:

$$\omega_1 = \{\xi_k/\rho_n < \epsilon\};$$

$$\omega_2 = \{|\xi_k - a_k|/\rho_n < \epsilon/2\},$$

$$\omega_3 = \{\xi_k/\rho_n < \epsilon/2\};$$

$$\omega_4 = \{\max_{1 \leqq k \leqq n} (\xi_k/\rho_n) \to 0, n \to \infty\};$$

$$\omega_5 = \{\max_{1 \leqq k \leqq n} (|\xi_k - a_k|/\rho_n) \to 0, n \to \infty\};$$

$$\omega_6 = \{\max_{1 \leqq k \leqq n} (a_k/\rho_n) \to 0, n \to \infty\}.$$

Let us consider that ξ_i is the first non-identically vanishing rv in $\{\xi_n\}$ and set

$$\omega_7 = \{\xi_i/\rho_n \to 0, n \to \infty\}; \qquad \omega_8 = \{\xi_i/\rho_n < \epsilon/2\}.$$

(a) *Sufficiency.* By our conditions, (1) follows from Theorem 2(a), i.e., $P(\omega_2) \to 1$ $(n \to \infty)$ and from Lemma 4 we obtain $a_k/\rho_n \leqq \max_{1 \leqq k \leqq n} (a_k/\rho_n) \leqq \epsilon/2$ $(1 \leqq k \leqq n > N)$. From $\xi_k - a_k \leqq |\xi_k - a_k|$ it follows that $\omega_2 \subset \omega_1$ $(n > N)$ from which we obtain the desired result.

Necessity. Let us suppose that $\max_{1 \leqq k \leqq n} P(\omega_3) \to 1$ $(n \to \infty)$. From $P(\omega_8) \to 1$ $(n \to \infty)$ it follows (3); from Theorem 2(a) we obtain $P(\omega_2') \to 0$ $(n \to \infty)$ for any $k \leqq n$. The result follows by choosing $k = n$, since the convergence to zero in probability of $(\xi_n - a_n)/\rho_n$ and of ξ_n/ρ_n implies $a_n/\rho_n \to 0$ $(n \to \infty)$, i.e. (4).

(b) *Sufficiency.* By our conditions the relation (2) follows from Theorem 2(b), i.e., $P(\omega_5) = 1$. Using Lemma 4 we obtain $\omega_4 = \omega_5$ from which follows the wanted result.

Necessity. Let us suppose that $\{\zeta_n\}$ is SRS and $P(\omega_4) = 1$. From $P(\omega_8) = 1$ it follows (3); from Theorem 2(b) it follows $P(\omega_5) = 1$. From $\omega_4 \cap \omega_5 \subset \omega_6$ we obtain $\max_{1 \leqq k \leqq n} (a_k/\rho_n) \to 0$ $(n \to \infty)$ and using Lemma 4 it follows (4).

7. The RS of $\{\zeta_n\}$ for Markov chains. Let us set $a_k = \rho_k - \rho_{k-1}$, $A_{kn} = \{|\xi_k - a_k| \geqq \epsilon \cdot \rho_n \beta_n\}$.

THEOREM 4. *If* $\xi_n \geqq 0$ $(n \varepsilon I)$, *for the* RS *of* $\{\zeta_n\}$ *it is sufficient that for any* $\epsilon > 0$,

$$\sum_{k=1}^{n} \int_{A_{kn}} dF_k(x) \to 0 \qquad (n \to \infty); \tag{5}$$

$$\rho_n^{-1} \cdot \sum_{k=1}^{n} \int_{A_{kn}'} x\, dF_k(x) \to 1 \qquad (n \to \infty). \tag{6}$$

PROOF. Let us define the sequence of rv ξ_k' where $\xi_k' = \xi_k$ if $|\xi_k - a_k| < \epsilon \rho_n \beta_n$ and $\xi_k' = a_k$ in the contrary case. Obviously, $A_{kn} = \{\xi_k \neq \xi_k'\}$. We set $\zeta_n' = \sum_{k=1}^{n} \xi_k'$,

$$R_n = \{\zeta_n \neq \zeta_n'\}, \qquad \omega_{kn} = P(A_{kn}),$$
$$\sigma_n = (1/\rho_n) \cdot \sum_{k=1}^{n} a_k \omega_{kn},$$
$$\lambda_n = (1/\rho_n) \cdot \sum_{k=1}^{n} \int_{A_{kn}'} x\, dF_k(x),$$

so that we may express (5), (6) in the form $\sum_{k=1}^{n} \omega_{kn} = o(1)$, $\lambda_n = 1 + o(1)$. From $R \subset \bigcup_{1 \leqq k \leqq n} A_{kn}$, using (5), it follows that $\{\zeta_n\}$, $\{\zeta_n'\}$ are *equivalent.* Now we prove that $\{\zeta_n'\}$ is RS.

Obviously

$$M\xi_k' = \int_{A_{kn}'} x\, dF_k(x) + \int_{A_{kn}} a_k\, dF_k(x) \geqq a_k - \epsilon\rho_n\beta_n,$$
$$M(\xi_k')^2 < (a_k + \epsilon\rho_n\beta_n) \cdot M\xi_k',$$
$$D\xi_k' < 2\epsilon\rho_n\beta_n \cdot M\xi_k', \qquad M(\zeta_n'/\rho_n) = \lambda_n + \sigma_n.$$

From Lemma L_2 it follows that $a_k \leqq \rho_k \leqq \rho_n$ $(1 \leqq k \leqq n)$ and $\sigma_n \leqq \sum_{k=1}^{n} \omega_{kn}$ so that $M(\zeta_n'/\rho_n) = 1 + o(1)$. By means of the first part of Lemma L_1 it follows that $D(\zeta_n'/\rho_n) \leqq C \cdot \sum_{k=1}^{n} D\xi_k'/\rho_n^2\beta_n < 2\epsilon C M(\zeta_n'/\rho_n) = 2\epsilon C + o(1)$ and that $M(\zeta_n'/\rho_n)^2 = D(\zeta_n'/\rho_n) + [M(\zeta_n'/\rho_n)]^2 < 1 + 2\epsilon C + o(1)$. By means of Chebychev's inequality and Lemma 3 the proof is complete.

Let us set

$$B_{kn} = \{\xi_k \geqq \epsilon\rho_n\beta_n\}, \qquad G_{kn} = \{0 \leqq \xi_k < \epsilon\rho_n\beta_n\}.$$

THEOREM 5. *If* $\xi_n \geqq 0$ $(n \varepsilon I)$, $\alpha_i > \lambda > 0$ $(i \varepsilon I)$, *in order that* $\{\zeta_n\}$ *be* RS *and* ξ_i/ρ_n $(1 \leqq i \leqq n \varepsilon I)$ *be asymptotically infinitesimal, it is sufficient that for any* $\epsilon > 0$,

$$\sum_{k=1}^{n} \int_{B_{kn}} dF_k(x) \to 0 \qquad (n \to \infty); \tag{7}$$

$$\rho_n^{-1} \cdot \sum_{k=1}^{n} \int_{G_{kn}} x \, dF_k(x) \to 1 \qquad (n \to \infty). \tag{8}$$

PROOF. (7) *and* (8) *imply* (3). We may remark that (7) and (8) imply $\rho_n\beta_n \to \infty$ $(n \to \infty)$. Indeed, let us suppose that there exists a finite number $d > 0$, so that for any N there is some $n > N$ so that $\rho_n\beta_n < d$. Because $E_k = \{\xi_k \geqq \epsilon \cdot d\} \subset B_{kn}$ $(1 \leqq k \leqq n)$ we obtain

$$\sum_{k=1}^{n} P(E_k) \leqq \sum_{k=1}^{n} P(B_{kn}).$$

From (7) it then follows that $P(E_k) = 0$ $(1 \leqq k \leqq n)$ for any $\epsilon > 0$. From $\xi_k \geqq 0$ $(k \varepsilon I)$ it follows that $P\{\xi_k = 0\} = 1$ $(k \varepsilon I)$ and $\sum_{k=1}^{n} \int_{G_{kn}} x \, dF_k(x) = 0$ which is in contradiction with (8). Consequently $\rho_n\beta_n \to \infty$ $(n \to \infty)$ and from $0 \leqq \beta_n \leqq 1$ we obtain (3).

(7) *and* (8) *imply* (4). Let us suppose that this is not true. In this case we may define a number ω $(0 < \omega < 1)$ so that for any N there is some $n > N$ so that $\rho_{n-1}/\rho_n < 1 - \omega$. Let us consider $\epsilon = \omega/8$ and N sufficiently large so that for $n > N$ the relations

$$\sum_{k=1}^{n} \int_{B_{kn}} dF_k(x) < 1, \tag{9}$$

$$\left|\rho_n^{-1} \cdot \sum_{k=1}^{n} \int_{G_{kn}} x \, dF_k(x) - 1\right| < \omega/4 \tag{10}$$

hold and also those that we obtain from them by taking $n - 1$ instead n. From $\rho_n\beta_n \to \infty$ $(n \to \infty)$ it follows that $G_{k,n-1} \subset G_{kn}$ and let us denote $L_{kn} = G_{kn} - G_{k,n-1} \subset B_{k,n-1}$. Obviously

$$\begin{aligned}(1/\rho_n) \cdot \sum_{k=1}^{n} \int_{G_{kn}} x \, dF_k(x) &= (1/\rho_n) \cdot \int_{G_{nn}} x \, dF_n(x) \\ &+ (\rho_{n-1}/\rho_n)(1/\rho_{n-1}) \cdot \sum_{k=1}^{n-1} \int_{G_{k,n-1}} x \, dF_k(x) \\ &+ (1/\rho_n) \cdot \sum_{k=1}^{n} \int_{L_{kn}} x \, dF_k(x) < (1/\rho_n) \cdot \int_{G_{nn}} \epsilon\rho_n\beta_n \, dF_n(x) \\ &+ (1 - \omega)(1 + \omega/4) + (1/\rho_n) \cdot \sum_{k=1}^{n-1} \int_{L_{kn}} \epsilon\rho_n\beta_n \, dF_k(x) \\ &< 2\epsilon\beta_n + (1 - \omega)(1 + \omega/4) < 1 - \omega/2\end{aligned}$$

which is in contradiction with (10), i.e. it is not possible to define a number ω so that for any N there is some $n > N$ so that $\rho_{n-1}/\rho_n < 1 - \omega$ and consequently (4) is true.

Because (7) and (8) imply (3) and (4), from Lemma 4(b) it follows that they imply $\max_{1 \leqq k \leqq n} (a_k/\rho_n) \to 0$ $(n \to \infty)$. In this case $a_k - \epsilon\rho_n\beta_n \leqq 0$ $(1 \leqq k \leqq n > N)$. Indeed, if for $k = k_0$ this would not be, i.e. $a_{k_0}/\rho_n > \epsilon\beta_n$ $(n > N)$, from

Lemma 4(b) it would follow that $\beta_n \to 0$ $(n \to \infty)$ which is in contradiction with the conditions of our Theorem.

(7) *and* (8) *imply* (5) *and* (6). Let us denote $H_{kn} = \{0 \leqq \xi_k \leqq 2\epsilon\rho_n\beta_n\}$, $T_{kn} = \{0 \leqq \xi_k \leqq a_k + \epsilon\rho_n\beta_n\}$. Obviously, from the previous it follows that $A_{kn} \subset B_{kn}$, $G_{kn} \subset T_{kn} \subset H_{kn}$ so that

$$\int_{A_{kn}} dF_k(x) \leqq \int_{B_{kn}} dF_k(x),$$
$$\int_{G_{kn}} x\, dF_k(x) \leqq \int_{T_{kn}} x\, dF_k(x) \leqq \int_{H_{kn}} x\, dF_k(x) \qquad (1 \leqq k \leqq n)$$

and our theorem is proved.

Remarks. (1) If $\alpha_i > \lambda > 0$, $(i \,\varepsilon\, I)$, $\tau_n = M\zeta_n$, $M\xi_n < +\infty$ $(n \,\varepsilon\, I)$, then in order that $\{\zeta_n\}$ be NRS and ξ_i/τ_n $(1 \leqq i \leqq n \,\varepsilon\, I)$ asymptotically infinitesimal it is sufficient that for any $\epsilon > 0$,

$$\tau_n^{-1} \cdot \sum_{k=1}^{n} \int_{A_{kn}} x\, dF_k(x) \to 1 \qquad (n \to \infty)$$

where $A_{kn} = \{0 \leqq \xi_k < \epsilon\tau_n\beta_n\}$. If we denote $T_{kn} = \{\xi_k \geqq \epsilon\tau_n\beta_n\}$, obviously

$$\sum_{k=1}^{n} \int_{T_{kn}} dF_k(x) \leqq (\epsilon\lambda\tau_n)^{-1} \cdot \sum_{k=1}^{n} \int_{T_{kn}} x\, dF_k(x)$$
$$= (\epsilon\lambda)^{-1} \cdot \{1 - \tau_n^{-1} \cdot \sum_{k=1}^{n} \int_{A_{kn}} x\, dF_k(x)\}$$

and from Theorem 5 follows our result.

(2) In the case of a stationary and homogeneous Markov chain with positive ergodic coefficient, it follows that $\{\xi_n\}$ are all identically distributed and instead of (7) and (8) we may take $n \cdot \int_{B_n} dF(x) \to 0$, $(n \to \infty)$, and $n\rho_n^{-1} \cdot \int_{G_n} x\, dF(x)] \to 1 (n \to \infty)$ where $F(x) = P\{\xi_k < x\}$, $B_n = \{0 \leqq \xi_k \leqq \epsilon\rho_n\}$, and $G_n = \{\xi_k > \epsilon\rho_n\}$.

Theorem 6. *If* $D_n/\tau_n^2\beta_n \to 0$ $(n \to \infty)$, *then* $\{\zeta_n\}$ *is* NRS.

The proof follows from Theorem 1(c′) using Lemma L_1.

Remark. If $\alpha_k > \lambda > 0$ $(k \,\varepsilon\, I)$ (e.g. in the case of a homogeneous Markov chain or for independent rv) we may take $D_n/\tau_n^2 \to 0$ $(n \to \infty)$ instead of the condition of Theorem 6.

If $D\xi_k < C < +\infty$ $(k \,\varepsilon\, I)$ we may take $\tau_n^2\beta_n/n \to \infty$ $(n \to \infty)$ and if these two conditions are satisfied, we may take $\tau_n^2/n \to \infty$ $(n \to \infty)$.

8. The SRS of $\{\zeta_n\}$ for Markov chains. Let us denote by $\{c_k\}$ a non-increasing sequence of positive constants, and set

$$D_{nm} = \sum_{k=n+1}^{m} c_k^2 \cdot D\xi_k, \qquad U_{nm} = \max_{n \leqq k \leqq m} c_k\, |\zeta_k - M\zeta_k|,$$
$$K = 1 + 6^{\frac{1}{2}}, \qquad K_1 = (20K)^{-\frac{1}{2}}, \qquad \epsilon_1 = K_1\epsilon, \qquad \epsilon_2 = K_1\epsilon/\lambda.$$

Lemma 5.

$$P\{U_{nm} \geqq \epsilon\} \leqq \epsilon_1^{-2}\beta_m^{-1} \cdot (c_n^2 D_n + D_{nm}); \tag{11}$$
$$P\{U_{nm} \geqq \epsilon\} \leqq \epsilon_2^{-2} \cdot (c_n^2 D_n + D_{nm}) \qquad (\alpha_i > \lambda > 0, i \,\varepsilon\, I). \tag{12}$$

Proof. Without any loss of generality we may suppose $M\xi_i = 0$ $(i \,\varepsilon\, I)$. Let us define the constants γ_k $(1 \leqq k \leqq m+1)$ by $\gamma_k = c_k$ $(1 \leqq k \leqq m)$, $\gamma_{m+1} = 0$ and let us denote $\gamma_k^2 - \gamma_{k+1}^2 = \pi_k$ $(r \leqq k \leqq m)$,

$$S_r = \sum_{k=r}^{m} \pi_k \zeta_k^2 \qquad (n \leqq r \leqq m),$$

$$E_r = \{\gamma_s |\zeta_s| < \epsilon; n \leqq s < r; \gamma_r |\zeta_r| \geqq \epsilon\} \qquad (n \leqq r \leqq m),$$

$$E_{n-1} = \{\gamma_s |\zeta_s| < \epsilon; n \leqq s \leqq r\}.$$

Obviously $\{U_{nm} \geqq \epsilon\} = \bigcup_{n \leqq r \leqq m} E_r$ and because the E_r are disjoint,

$$P\{U_{nm} \geqq \epsilon\} = \sum_{r=n}^{m} P(E_r).$$

From $S_n > S_r$ $(n < r < m)$ and from

$$\zeta_k = \zeta_r + \sum_{j=r+1}^{k} \xi_j, \qquad \zeta_k^2 \geqq \zeta_r^2 + 2 \cdot \sum_{i=1}^{r} \sum_{j=r+1}^{k} \xi_i \xi_j \qquad (r \leqq k \leqq m)$$

(if $k = r$, the double sum vanishes) it follows that

$$M(S_n \mid E_r) \geqq M(S_r \mid E_r) \geqq \sum_{k=r}^{m} \pi_k \cdot M(\zeta_k^2 \mid E_r) \qquad (n-1 \leqq r \leqq m),$$

$$M(\zeta_k^2 \mid E_r) \geqq \epsilon^2 \gamma_r^{-2} + 2 \sum_{i=1}^{r} \sum_{j=r+1}^{k} M(\xi_i \xi_j \mid E_r) \qquad (r \leqq k \leqq m)$$

i.e.

$$\epsilon^2 \leqq M(S_n \mid E_r) + 2 \cdot \sum_{k=r}^{m} \pi_k \cdot I_{rk} \qquad (n \leqq r \leqq m),$$

where $I_{rk} = \sum_{i=1}^{r} \sum_{j=r+1}^{k} |M(\xi_i \xi_j \mid E_r)|$, $(n \leqq r \leqq k \leqq m)$.

Using the same methods as in the proof of Lemma 1 from [9] it is easy to obtain $I_{rk} < 2K\beta_m^{-1} \cdot \sum_{i=1}^{k} D(\xi_i \mid E_r)$. $(r \leqq k \leqq m)$; i.e.

$$\epsilon^2 < M(S_n \mid E_r) + 4K\beta_m^{-1} \cdot \sum_{k=r}^{m} \{\pi_k \cdot \sum_{i=1}^{k} D(\xi_i \mid E_r)\}$$

$$= M(S_n \mid E_r) + 4K\beta_m^{-1} \cdot \{c_r^2 \cdot \sum_{i=1}^{r} D(\xi_i \mid E_r) + \sum_{i=r+1}^{m} c_i^2 D(\xi_i \mid E_r)\}.$$

Since for any rv ξ,

$$\sum_{r=n-1}^{m} P(E_r) D(\xi \mid E_r) < D\xi,$$

it follows, by means of Lemma L_1, that $\epsilon^2 \cdot \sum_{r=n}^{m} P(E_r) \leqq MS_n + 4K\beta_m^{-1} \cdot (c_n^2 \cdot D_n + D_{nm})$ and $MS_n \leqq 16K\beta_m^{-1} \cdot (c_n^2 D_n + D_{nm})$ from which we obtain the proof of (11); (12) is a consequence of (11).

REMARK. This lemma contains $(c_k = 1, k \in I; n = 1)$ Lemma 1 from [10].

If $1 < z \in I$, we denote $z^m = s$, $z^{m+1} = v$, $z^u = \sigma$, $z^{u+1} = h$, $\mu_m = \tau_s^2 \beta_v$,

$$\omega_i^{-2} = \sum_{m=u}^{\infty} \mu_m^{-1} \qquad (\sigma \leqq i < h; u, m \in I).$$

THEOREM 7. *If for some* z,

$$\sum_{n=1}^{\infty} \omega_n^{-2} \cdot D\xi_n < +\infty \tag{13}$$

then $\{\zeta_n\}$ *is* NSRS.

If $\xi_n \geqq 0$ $(n \in I)$ *and if there are some numbers* a, b, c *so that* $a\beta_s \leqq \beta_v$, $b\tau_s \leqq \tau_v \leqq c\tau_s$ $(m \geqq m_0, 1 < b < c, 1/b^2 < a < 1)$ *then instead of* (13) *we may take*

(14) $$\sum_{n=1}^{\infty} \tau_n^{-2} \beta_n^{-1} \cdot D\xi_n < +\infty .$$

PROOF. Let us denote $\eta_n = \zeta_n - M\zeta_n$,

$$A_m = \{\max_{s \leqq n < v} |\eta_n| \tau_n^{-1} \geqq \epsilon\},$$

$$B_m = \{\max_{s \leqq n < v} |\eta_n| \geqq \epsilon\tau_s\}.$$

The proof of (13) runs analogous to that of ([10], Theorem 1). (14) follows from (13) because it is easy to obtain the relations

$$\mu_m \geqq (ab^2)^{m-u} \cdot \mu_u \ (m \geqq u); \qquad \omega_i^2 \geqq (1 - a^{-1}b^{-2}) \cdot \mu_u ;$$

$$\mu_u \geqq ac^{-2} \cdot \tau_i^2 \beta_i \qquad (\sigma \leqq i < h).$$

REMARK. The condition imposed on β_n in (14) is satisfied in a large class of cases ([10], Remark (a_1) to the Theorem 1).

THEOREM 8. *Let us consider* $M\xi_n \geqq 0, \alpha_n > \lambda > 0 \ (n \ \varepsilon \ I)$ *and* $\tau_n \rightarrow \infty \ (n \rightarrow \infty)$.

(a) *In order that* $\{\zeta_n\}$ *be* NSRS *it is sufficient that*

(15) $$\sum_{n=1}^{\infty} \tau_n^{-2} \cdot D\xi_n < +\infty$$

and if in this case $\xi_n \geqq 0$, *the* rv $\xi_i/\tau_n \ (1 \leqq i \leqq n \ \varepsilon \ I)$ *are strongly asymptotically constant.*

(b) *This condition is the best in the sense that if for some sequences of non-negative constants* $\{b_n\}$, $\{\tau_n\}$ *the series*

(16) $$\sum_{n=1}^{\infty} \tau_n^{-2} b_n$$

diverges and τ_n *is monotonically increasing to infinity, it is possible to construct a Markov chain* $\{\xi_n\}$ *(non-degenerated into a sequence of mutually independent* rv) with $M\xi_n > 0$, $M\zeta_n = \tau_n$, $D\xi_n = b_n$, $\alpha_n > \lambda > 0 \ (n \ \varepsilon \ I)$ *and for which* $\{\zeta_n\}$ *is not* NSRS.

PROOF. Let us denote $U_n = \sup_{k \geqq n} |(\zeta_k/\tau_n) - 1|$.

(a) If in (12) we take $c_k = 1/\tau_k \ (k \ \varepsilon \ I)$, $m = \infty$, it follows that

$$P\{U_n \geqq \epsilon\} \leqq \epsilon_2^{-2} \cdot \{\tau_n^{-2} \cdot D_n + \sum_{k=n+1}^{\infty} \tau_k^{-2} \cdot D\xi_k\}.$$

By our conditions, obviously $D_n/\tau_n^2 \rightarrow 0 \ (n \rightarrow \infty)$ i.e. $P\{U_n \geqq \epsilon\} \rightarrow 0 \ (n \rightarrow \infty)$ for any $\epsilon > 0$ which is equivalent to the NSRS of $\{\zeta_n\}$. The last result follows from Theorem 2.

(b) The proof runs analogously to that of ([10] Theorem 2). For this we retain all the notations and all the constructions used there with the following exceptions:

(1) We must define I_1 as the set of $r \ \varepsilon \ I$, for which $b_r < \tau_r^2$.

(2) In the definition of the auxiliary sequence of constants $\{\delta_r\}$ we must consider $a_r = \frac{1}{2} \cdot \min (\tau_r^{-2} b_r \ ; 1 - \tau_r^{-2} b_r) \leqq \frac{1}{4} \ (r \ \varepsilon \ I_1)$.

(3) In the definition of the stochastic matrices we must consider: $p_{i1r} = p_{i3r} = \tau_r^{-2} b_r/2 + \pi_i \delta_r \ (i = 1, 2, 3; r \ \varepsilon \ I_{11})$, $p_{i1r} = p_{i3r} = \tau_r^{-2} b_r/2 + \theta_i \delta_r \ (i = 1, 2; r \ \varepsilon \ I_{21})$, $(\pi_1 = \theta_1 = -\pi_3 = -\theta_2 = -1; \pi_2 = 0)$.

(4) For the absolute probabilities we must take $P_{1r} = P_{3r} = \tau_r^{-2}b_r/2$ $(r \varepsilon I_1)$.

(5) If $a_r' = \tau_r - \tau_{r-1}$ $(r \varepsilon I)$, then the rv ξ_r must be defined by $\xi_r(\omega_{ir}) = a_r' + \pi_i\tau_r$ $(i = 1, 2, 3; r \varepsilon I_1)$, $\xi_r(\omega_{ir}) = a_r' + \theta_i b_r^{\frac{1}{2}}$ $(i = 1, 2; r \varepsilon I_2)$.

In this manner we obtain $M\xi_r = a_r'$, $M\zeta_n = \tau_n$, $(n \varepsilon I)$, $D\xi_r = b_r$ $(r \varepsilon I)$.

(6) In the proof that $\{\xi_r\}$ is not NSRS we must take $E_r = \{|\xi_r - M\xi_r| > \epsilon\tau_r\}$, $P(E_r \mid E_{r-1}') = b_r/\tau_r^2$ $(r \varepsilon I_{11})$.

Remark. Let us denote by $\{n_k\}$ a sequence of natural numbers, increasing to infinity $(n_0 = 1)$, $I_k = \{n_k, n_k + 1, \cdots, n_{k+1} - 1\}$ and $\eta_k = \sum_{i\varepsilon I_k} \xi_i\tau_i^{-1}$ $(k \varepsilon I)$.

(a) If $\alpha_i > \lambda > 0$ $(i \varepsilon I)$, $\tau_n \to \infty$ $(n \to \infty)$, then $\sum_{k=1}^{n} D\eta_k < +\infty$ is sufficient for the NSRS of $\{\zeta_n\}$.

(b) This condition is the best in the terms of $\{\eta_k\}$ in the sense that it is possible to construct a Markov chain $\{\xi_n\}$ (non-degenerated into a sequence of mutually independent rv) for which this series diverges, $\alpha_i > \lambda > 0$ $(i \varepsilon I)$ and for which $\{\zeta_n\}$ is not NSRS.

The proof of (a) follows from the fact that by means of Lemma L_1 we obtain

$$C' \sum_{i\varepsilon I_k} \tau_i^{-2} D\xi_i \leqq D\eta_k \leqq C \cdot \sum_{i\varepsilon I_k} \tau_i^{-2} D\xi_i \qquad (k \varepsilon I)$$

i.e. our series and (15) converge only simultaneously.

The proof of (b) follows using the Markov chain that we have constructed in the proof of Theorem 8(b). Indeed, the sums ζ_n are not NSRS and from the previous part (a) we obtain that (15) and our series diverge only simultaneously.

Let us set $E_n = \{|\xi_n - r_n| > \epsilon\}$, $r_n = r\xi_n$.

Theorem 9. *If* $\{\xi_n\}$ *is* SS *then for any* $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P(E_n \mid E_{n-1}') < +\infty. \tag{17}$$

The proof follows from Lemma 3 using Lemma L_3.

Let us consider a real valued function $\varphi(x) = o(x)$.

Theorem 10. (a) *If* $E_n = \{|\xi_n - r_n| > \epsilon\rho_n\}$, *then* (17) *is a necessary condition of the* SRS *of* $\{\zeta_n\}$.

(b) *If* $E_n = \{|\xi_n - t_n| > \epsilon\cdot\varphi(\tau_n)\}$, $\sum_{n=1}^{\infty} \tau_n^{-2} < +\infty$, *then* (17) *is not a necessary condition of the* NSRS *of* $\{\zeta_n\}$.

Proof. (a) Let us set $\sigma_n = (\zeta_n/\rho_n) - 1$, $\lambda_n = \xi_n/\rho_n$, $a_n = \rho_n - \rho_{n-1}$, $q_n = a_n/\rho_n$,

$$A = \{\sigma_n \to 0, n \to \infty\}, \qquad A_1 = \{\sigma_{n-1} \to 0, n \to \infty\},$$

$$B = \{\lambda_n - q_n \to 0, n \to \infty\}, \quad E_n = \{|\lambda_n - r\lambda_n| > \epsilon\} = \{|\xi_n - r\xi_n| > \epsilon\rho_n\}.$$

If $\{\zeta_n\}$ is SRS from $\sigma_n - \rho_{n-1} \cdot \sigma_{n-1}/\rho_n \doteq \lambda_n - q_n$ we obtain $A \cap A_1 \subset B$, $P(A) = P(A_1) = 1$, from which it follows $P(B) = 1$, i.e. the Markov chain $\{\lambda_n\}$ is SS. From Theorem 9 it follows the convergence of (17).

(b) The proof runs analogously to that of ([10], Theorem 4). For this we must conserve all the notations and all the constructions used there with the following exceptions:

(1) For the definition of I_1, I_2 we must take in ([10], (22), (23)) τ_n instead of n.

(2) We must define the function g_n as equal to τ_n^{-2} for $n \varepsilon I_2$ and to τ_s^{-2} for $n = n_s \varepsilon I_1$.

(3) We must take $2v_n = \tau_n g_n / \varphi(\tau_n) < 1$.

(4) We must define the rv ξ_r by $\xi_r(\omega_{ir}) = a_r + c\pi_i\varphi(\tau_r)$ $(i = 1, 2, 3; r \varepsilon I)$. In this way we obtain $M\xi_r = a_r$, $M\zeta_n = \tau_n$, $D\xi_r = c^2\tau_r g_r \cdot \varphi(\tau_r)$.

(5) Instead of ([10], Theorem 2, (8)) we must use Theorem 8(a) of the present paper.

9. General remarks. (1) For an mth order Markov chain, Theorems 4–6 (and therefore the remarks to them) remain true. Indeed this follows easily if we consider the manner in which these theorems are proved. Theorem 5 follows from Theorem 4; if in the proofs of Theorems 4 and 6 we use the second part of Lemma L_1 instead of the first part, our result is proved.

(2) In Theorems 2–5 we do not suppose the existence of the variances of the rv.

(3) In the case of mutually independent rv; (a) Theorem 2 contains the results from [1] i.e. that in the conditions of RS the relation $d_n \to \infty$ $(n \to \infty)$ implies that ξ_i/d_n $(1 \leqq i \leqq n \varepsilon I)$ are asymptotically constant; (b) in the case of RS with constants $\{d_n\}$, Theorem 3 contains the analogous results obtained in [1]; (c) Theorem 8(a) implies the results from ([1], [4], [5]) and Lemma 5 from ([4], [7]); (d) the rest of the results in Theorems 2, 3, 7, and 8 are new, even for mutually independent rv; the same thing also for Theorem 10, where in this case instead of (17) we must take $\sum_{n=1}^{\infty} P(E_n) < +\infty$; (e) Theorems 4 and 5 contain the analogous results obtained for independent rv in [1]. Lemma 5 implies the analogous result obtained for the same case in [4].

(4) We may obtain results analogous to these obtained in this paper, if we consider a sequence of series of rv ξ_{nk} $(1 \leqq k \leqq k_n)$ which for any $n \varepsilon I$ is a Markov chain.

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