COMPARISON OF COMBINED ESTIMATORS IN BALANCED INCOMPLETE BLOCKS

BY V. SESHADRI

McGill University

0. Introduction. In the analysis of Balanced Incomplete Block designs (BIB), there arise two independent estimates of treatment differences conventionally referred to as the intra-block and the inter-block estimates. Yates [4] showed that an inter-block analysis can be made for the BIB assuming that the block effects are random. He also devised a method for combining this additional information with the customary intra-block information so as to estimate the treatment differences with greater precision than if the intra-block information had been used alone. An alternative combined estimate has been suggested by Graybill and Weeks [1] and shown to be unbiased. In this paper we compare the two estimates and answer the question raised by Graybill and Weeks [1] as to which estimate is better in the sense of smaller variance.

1. Notations and assumptions. We refer the reader for a fuller discussion of the model to Graybill and Weeks [1], and state only those assumptions which pertain to the problem considered in this study.

(i) The $(t - 1) \times 1$ vector $U = (u_i)$ is normally distributed with mean $T = (t_i)$ and covariance matrix $(k/\lambda t)\sigma^2 I$, so that u_i (referred to as the intra-block estimate) is unbiased for t_i and has variance $(k/\lambda t)\sigma^2$.

(ii) The $(t - 1) \times 1$ vector $X = (x_i)$ is normally distributed with mean $T = (t_i)$ and covariance matrix $\{k(\sigma^2 + k\sigma_\beta^2)/(r - \lambda)\}I$, so that x_i (referred to as the inter-block estimate) is unbiased for t_i and has variance $k(\sigma^2 + k\sigma_\beta^2)/(r - \lambda)$.

(iii) s_1^2/σ^2 has a chi-square distribution with $f = (bk - b - t + 1)$ degrees of freedom.

(iv) $s_2^2/(\sigma^2 + k\sigma_\beta^2)$ has a chi-square distribution with $(b - t)$ degrees of freedom, where $b > t$.

(v) $u_1, u_2, \cdots, u_{t-1}, x_1, x_2, \cdots, x_{t-1}, s_1^2, s_2^2$ are all mutually independent.

The following notations will be used in the paper.

(a) $E_x(\cdot)$ denotes the expectation of $(\cdot)$ in the space of x.

(b) $E_{x|y,z}(\cdot)$ denotes the expectation of $(\cdot)$ over fixed values of y and z.

(c) $E_{s_1^2,s_2^2,z_i}(\cdot)$ will be referred to by $E'(\cdot)$.

(d) $V(\cdot)$ denotes the variance of $(\cdot)$.

(e) Yates' estimate is denoted by $\bar{T}_i$, and Graybill and Weeks' estimate by $\hat{T}_i$.

(f) $P(x > a)$ denotes the probability that $x > a$.

2. Statement of the problem. The object of the present study is to compare

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$\bar{T}_i$ with $\hat{T}_i$ on the basis of their variances. We shall first explain briefly what the estimates are and what their qualitative difference is.

The estimates of the various parameters in the BIB are usually obtained from the analysis of variance (Graybill and Weeks, [1], Table I, p. 803). Yates obtains the estimates of σ^2 and σ_β^2 separately and in so doing, considers the situation where negative estimates of σ_β^2 arise. Thus Yates obtains the following estimate of t_i, the ith treatment.

$$\begin{aligned}\bar{T}_i &= u_i + V_1(x_i - u_i), && \text{if } \hat{\sigma}_\beta^2 > 0\\ &= u_i + (r - \lambda)(x_i - u_i)/rk, && \text{if } \hat{\sigma}_\beta^2 \leqq 0\end{aligned}$$

where

$$V_1 = ((r - \lambda)/rk)\{rs_1^2/(rs_1^2 + \lambda ft\hat{\sigma}_\beta^2)\}$$

and

$$\hat{\sigma}_\beta^2 = [\lambda t(r - \lambda) \textstyle\sum_{i=1}^{t-1} (x_i - u_i)^2/rk^2 + s_2^2 - (b - 1)s_1^2/f]/t(r - 1)$$

is an estimate of σ_β^2 obtained by Yates from the ANOVA.

Graybill and Weeks proposed a combined estimate which simply ignores the situation when negative estimates arise. This estimate is given by

$$\hat{T}_i = u_i + V_1(x_i - u_i).$$

Since both estimates are unbiased ([1], [2]), we wish to compare $V(\bar{T}_i)$ with $V(\hat{T}_i)$.

3. Variances of the estimates. Introducing the indicator function $\phi(\hat{\sigma}_\beta^2)$ defined by

$$\begin{aligned}\phi(\hat{\sigma}_\beta^2) &= 0, && \text{if } \hat{\sigma}_\beta^2 > 0,\\ &= 1, && \text{if } \hat{\sigma}_\beta^2 \leqq 0,\end{aligned}$$

$$\bar{T}_i = \{u_i + V_1(x_i - u_i)\} + \phi(\hat{\sigma}_\beta^2)\{(r - \lambda)/rk - V_1\}(x_i - u_i).$$

Therefore $V(\bar{T}_i)$ can now be expressed as

$$\begin{aligned}V(\bar{T}_i) = E(u_i - t_i)^2 &+ E\{(x_i - u_i)^2[V_1^2 + \phi(\hat{\sigma}_\beta^2)\{((r - \lambda)/rk)^2 - V_1^2\}]\}\\ &+ 2E\{(x_i - u_i)(u_i - t_i)[V_1 + \phi(\hat{\sigma}_\beta^2)\{(r - \lambda)/rk - V_1\}]\}.\end{aligned}$$

Writing $\binom{w_i}{z_i} = \binom{u_i}{x_i - u_i}$ it follows from the assumptions in Section 1, that the vector $\binom{w_i}{z_i}$ has a bivariate normal distribution with mean $\binom{t_i}{0}$ and covariance matrix

$$\begin{pmatrix} k\sigma^2/\lambda t & -k\sigma^2/\lambda t \\ -k\sigma^2/\lambda t & k^2(r\sigma^2 + \lambda t\sigma_\beta^2)/\lambda t(r - \lambda) \end{pmatrix}.$$

Since V_1 is a function of s_1^2, s_2^2, and z_i only, we can show that

$$E\{(x_i - u_i)(u_i - t_i)V_1\} = E_{s_1^2, s_2^2, z_i}\{E_{\omega_i | s_1^2, s_2^2, z_i} z_i(w_i - t_i)V_1\}$$
$$= -\{(r-\lambda)\sigma^2/k(r\sigma^2 + \lambda t\sigma_\beta^2)\}E'(V_1 z_i^2)$$

Similarly since $\phi(\hat{\sigma}_\beta^2)$ is a function of s_1^2, s_2^2, and z_i,

$$E(x_i - u_i)^2[V_1^2(1-\phi(\hat{\sigma}_\beta^2)) + ((r-\lambda)/rk)^2\phi(\hat{\sigma}_\beta^2)]$$
$$= E'[V_1^2 z_i^2(1-\phi(\hat{\sigma}_\beta^2)) + ((r-\lambda)/rk)^2 z_i^2\phi(\hat{\sigma}_\beta^2)].$$

Therefore

$$(1)\quad V(\bar{T}_i) = k\sigma^2/\lambda t + E'[V_1^2 z_i^2(1-\phi(\hat{\sigma}_\beta^2)) + ((r-\lambda)/rk)^2 z_i^2\phi(\hat{\sigma}_\beta^2)]$$
$$- 2\{(r-\lambda)\sigma^2/k(r\sigma^2 + \lambda t\sigma_\beta^2)\}E'[V_1 z_i^2(1-\phi(\hat{\sigma}_\beta^2)) + ((r-\lambda)/rk)z_i^2\phi(\hat{\sigma}_\beta^2)].$$

Using the definition of $\phi(\hat{\sigma}_\beta^2)$, (1) reduces to

$$(k/\lambda t)\sigma^2 + E'[V_1^2 z_i^2 - 2(r-\lambda)\sigma^2 V_1 z_i^2/k(r\sigma^2 + \lambda t\sigma_\beta^2) \mid \hat{\sigma}_\beta^2 > 0]p$$
$$+ E'[((r-\lambda)/rk)^2 z_i^2 - 2(r-\lambda)\sigma^2 z_i^2/k(r\sigma^2 + \lambda t\sigma_\beta^2) \mid \hat{\sigma}_\beta^2 \leqq 0](1-p)$$

where $p = P(\hat{\sigma}_\beta^2 > 0)$.

By a similar analysis we obtain the variance of Graybill and Weeks' estimate. Thus

$$(2)\qquad V(\hat{T}_i) = k\sigma^2/\lambda t + E'[V_1^2 z_i^2 - 2(r-\lambda)\sigma^2 V_1 z_i^2/k(r\sigma^2 + \lambda t\sigma_\beta^2)].$$

Therefore from (1) and (2) we have

$$V(\hat{T}_i) - V(\bar{T}_i) = E'[V_1^2 z_i^2 - 2(r-\lambda)\sigma^2 V_1 z_i^2/k(r\sigma^2 + \lambda t\sigma_\beta^2) \mid \hat{\sigma}_\beta^2 \leqq 0](1-p)$$
$$- E'[((r-\lambda)/rk)^2 z_i^2 - 2(r-\lambda)\sigma^2 z_i^2/k(r\sigma^2 + \lambda t\sigma_\beta^2) \mid \hat{\sigma}_\beta^2 \leqq 0](1-p).$$

Since the distribution of $\hat{\sigma}_\beta^2$ is not known, the evaluation of the expectations is a formidable problem and therefore the question of how big is the difference in the variances cannot be answered.

Writing $V_1 = ((r-\lambda)/rk)y$ where $y = rs_1^2/(rs_1^2 + \lambda ft\hat{\sigma}_\beta^2)$ and noting that $y \geqq 1$ whenever $\hat{\sigma}_\beta^2 \leqq 0$, and letting $rk/(r-\lambda) = \gamma$ and $\theta = r\sigma^2/(r\sigma^2 + \lambda t\sigma_\beta^2)$, we have

$$V(\hat{T}_i) - V(\bar{T}_i) = (1/\gamma^2)E'[(y^2 - 2\theta y - 1 + 2\gamma\theta)z_i^2 \mid y \geqq 1](1-p)$$

which can be simplified to,

$$[(1-p)/\gamma^2]E'[(y-1)(y+1-2\theta) + 2\theta(\gamma-1)z_i^2 \mid y \geqq 1].$$

We note that $\theta > 0$, $\gamma > 1$ implies $2\theta(\gamma - 1) > 0$ and since $E'(z_i^2) \neq 0$, it follows that $V(\hat{T}_i) - V(\bar{T}_i) > 0$. Therefore Yates' estimate is superior to Graybill and Weeks' estimate.

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