A CHARACTERISTIC PROPERTY OF THE MULTIVARIATE NORMAL DISTRIBUTION

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1. Introduction. Suppose that X and Y are two independent $(n \times 1)$ random vectors and the conditional distribution of X given $X + Y$ is known to be multivariate normal. What can we conclude about the distributions of X and Y? The case when X and Y are scalar random variables has been treated in [2].

If X and Y are two independent $(n \times 1)$ random vectors having multivariate normal distributions $N(0, A)$ and $N(0, B)$ respectively (that is with 0 means and symmetric positive definite covariance matrices A, B respectively), it can be easily shown that the conditional distribution of X given $(X + Y)$ is $N[A(A + B)^{-1}(X + Y), \{I - A(A + B)^{-1}\}A]$. Denoting by V the matrix $(I - C)A$ where $C = A(A + B)^{-1}$, it is easy to see that (a) both $V^{-1}C$ and $V^{-1} - V^{-1}C$ are symmetric positive definite and (b) the eigenvalues of C are in $(0, 1)$. These properties are used in establishing the characterization theorem.

2. A characterization theorem. In this section we prove the following theorem.

Theorem. *Let X and Y be two $(n \times 1)$ independent random vectors with continuous probability density functions $f(x)$ and $g(y)$, which are non-vanishing at $x = 0$ and $y = 0$ (0 being the null vector). Let V be a $(n \times n)$ symmetric positive definite matrix and C a non-singular $(n \times n)$ matrix satisfying one of the following two conditions:*

(i) *both $V^{-1}C$ and $V^{-1} - V^{-1}C$ are positive definite, and $V^{-1}C$ is symmetric;*

(ii) *$V^{-1}C$ is symmetric and the eigenvalues of C lie in $(0, 1)$. If the conditional distribution of X given $(X + Y)$ is multivariate normal with mean $C(X + Y)$ and covariance matrix V, then both X and Y are multivariate normal.*

Proof. Denoting the density of $(X + Y)$ by $h(x + y)$, we can write

$$(1) \quad f(x)g(y) = kh(x + y) \exp -\tfrac{1}{2}\{x - C(x + y)\}'V^{-1}\{x - C(x + y)\}$$

where $k = 1/[(2\pi)^{\frac{1}{2}}]^n|V|^{\frac{1}{2}}$.

If in (1) we successively let $x = 0$, $y = 0$, we obtain

$$(2) \quad f(0)g(y) = kh(y) \exp -\tfrac{1}{2}\{Cy\}'V^{-1}\{Cy\},$$

$$(3) \quad f(x)g(0) = kh(x) \exp -\tfrac{1}{2}\{(I - C)x\}'V^{-1}\{(I - C)x\}.$$

From (1), (2) and (3) after some simplification we have

$$(4) \quad h(x + y) = k'h(x)h(y) \exp -\{(I - C)x\}'V^{-1}\{Cy\},$$

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where $k' = k/f(0)g(0)$. Writing $(I - C)'V^{-1}C = P$ and multiplying both sides of (4) by $k' \exp \frac{1}{2}\{x'Px + y'Py\}$ yields

$$k'h(x + y) \exp \tfrac{1}{2}\{x'Px + y'Py + 2x'Py\} = [k'h(x) \exp \tfrac{1}{2}\{x'Px\}][k'h(y) \exp \tfrac{1}{2}\{y'Py\}]$$

which reduces to

$$(5) \quad k'h(x + y) \exp \tfrac{1}{2}\{x + y\}'P\{x + y\} = [k'h(x) \exp \tfrac{1}{2}(x'Px)][k'h(y) \exp \tfrac{1}{2}(y'Py)]\cdot$$

Setting $\psi(x + y) = k'h(x + y) \exp \frac{1}{2}\{x + y\}'P\{x + y\}$ in (5) yields the functional equation $\psi(x + y) = \psi(x)\psi(y)$ whose solution is $\psi(x) = \exp x'Q$, Q being an arbitrary $(n \times 1)$ vector (Aczél [1]). We then have

$$(6) \qquad k'h(x) = \exp -\tfrac{1}{2}\{x'Px - 2x'Q\}$$

and

$$(7) \qquad k'h(y) = \exp -\tfrac{1}{2}\{y'Py - 2y'Q\}.$$

From (2) and (7) we obtain

$$(8) \qquad g(y) = g(0) \exp [-\tfrac{1}{2}\{Cy\}'V^{-1}\{Cy\} - \tfrac{1}{2}y'Py + y'Q].$$

Similarly (3) and (6) yield

$$(9) \quad f(x) = f(0) \exp [-\tfrac{1}{2}\{(I - C)x\}'V^{-1}\{(I - C)x\} - \tfrac{1}{2}x'Px + x'Q].$$

Simplifying (8) and (9) we have

$$(10) \qquad f(x) = f(0) \exp [-\tfrac{1}{2}\{x - T^{-1}Q\}'T\{x - T^{-1}Q\} + \tfrac{1}{2}Q'T^{-1}Q]$$

and

$$(11) \quad g(y) = g(0) \exp [-\tfrac{1}{2}\{y - R^{-1}Q\}'R\{y - R^{-1}Q\} + \tfrac{1}{2}Q'R^{-1}Q]$$

where $T = (I - C)'V^{-1}$ and $R = V^{-1}C$.

By condition (i) of the theorem both T and R are positive definite and T is also symmetric. The equivalence of conditions (i) and (ii) can be established by the following simple lemma.

LEMMA. *Let A be a real symmetric positive definite matrix and B a real symmetric matrix. The matrices B and $A - B$ are positive definite if and only if the eigenvalues of $A^{-1}B$ lie in the open unit interval.*

It is readily seen that for $f(x)$ and $g(y)$ to be probability distributions

$$f(0) \exp \tfrac{1}{2}Q'T^{-1}Q = |T|^{\frac{1}{2}}/[(2\pi)^{\frac{1}{2}}]^n$$

and

$$g(0) \exp \tfrac{1}{2}Q'R^{-1}Q = |R|^{\frac{1}{2}}/[(2\pi)^{\frac{1}{2}}]^n.$$

This concludes the proof.

3. Remarks. The matrix C of the theorem is the matrix of partial regression coefficients in the conditional distribution of X given $(X + Y)$. In order that multivariate normal distributions for X and Y to exist, the matrix C has to satisfy either condition (i) or (ii). In the case when X and Y are scalar random variables the condition on C reduces to $0 < C < 1$.

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REFERENCES

[1] Aczél, Janos (1965). *Lectures on Functional Equations and Their Applications*. Academic Press, New York.

[2] Patil, G. P. and Seshadri, V. (1964). Characterization theorems for some univariate probability distributions. *J. Roy. Statist. Soc. Ser. B* **26** 286–292.