ON MOMENTS OF CUMULATIVE SUMS

By Ragnar Ericson

University of Lund

1. Introduction and summary. In the theory of sequential analysis developed by Wald [6] there appear sums of the form $X = \sum_1^N x_i$ where both the x_i and N are random variables. In this note we shall consider conditions for the existence of $E(X^k)$ when the x_i are independent random variables and the event $N \geqq i$ is independent of $x_i, x_{i+1}, \cdots$.

Let $|x_i| = y_i$, $Y = \sum_1^N y_i$. We show that sufficient conditions for $E(Y^k) < \infty$ are that $E(y_i^k) \leqq \beta_k < \infty$, $E(N^k) < \infty$, $k = 1, 2, \cdots$ (proved for $k = 1$ in [7]), and that if we can find a constant $c < \infty$ such that $P(y_i \leqq c)E(y_i \mid y_i \leqq c) \geqq \alpha > 0$ for $i = 1, 2, \cdots$, a necessary condition for $E(Y^k) < \infty$ is $E(N^k) < \infty$. We also show that $E(x_i) = 0$, $E(x_i^{2k}) \leqq \beta_{2k} < \infty$, $E(N^k) < \infty$ imply that $E(X^{2k}) = \lim_{m\to\infty} E(X_m^{2k}) < \infty$ for $k = 1, 2, \cdots$ (proved for $k = 1, 2$ in [1], for $k \geqq 3$ proved independently in [5]).

2. General conditions and notations. In this paper we always assume that the x_i are independent random variables and N is a random variable taking values $1, 2, 3, \cdots, \sum_1^\infty P(N = i) = 1$. The event $N \geqq i$ is independent of $x_i, x_{i+1}, \cdots$.

In order to use this in a convenient way we define, following [4], $n_i = 1\ (0)$ if $N \geqq (<)\ i$. We note that x_i is independent of any n_j with $j \leqq i$. We use the following representation,

$$N = \sum_1^\infty n_i, \quad N_m = \sum_1^m n_i = \min(m, N), \quad X = \sum_1^\infty n_i x_i, \quad X_m = \sum_1^m n_i x_i,$$
$$|x_i| = y_i, \quad Y = \sum_1^\infty n_i y_i, \quad Y_m = \sum_1^m n_i y_i.$$

A useful convention is $X_0 = Y_0 = 0$.

3. Results.

Theorem 1. *The conditions* $E(y_i^k) \leqq \beta_k < \infty$ $(i = 1, 2, \cdots)$, $E(N^k) < \infty$ *imply* $E(Y^k) < \infty$. *If we can find a constant* $c < \infty$ *such that*

$$P(y_i \leqq c)E(y_i \mid y_i \leqq c) \geqq \alpha > 0, \qquad i = 1, 2, \cdots,$$

then the condition $E(Y^k) < \infty$ *implies* $E(N^k) < \infty$.

Proof. Since $n_m^2 = n_m$,

$$E(Y_m^k) - E(Y_{m-1}^k) = \sum_{j=0}^{k-1} \binom{k}{j} E[Y_{m-1}^j (n_m y_m)^{k-j}]$$
$$= \sum_{j=0}^{k-1} \binom{k}{j} E(Y_{m-1}^j n_m) E(y_m^{k-j}).$$

$$E(Y_M^k) = O(1) \sum_{j=0}^{k-1} \sum_{m=1}^{M} E(Y_{m-1}^j n_m) = O(1) \sum_{j=0}^{k-1} E(Y_M^j N_M)$$
$$= O[E(N) + E(Y_M^{k-1} N)] = O[E^{(k-1)/k}(Y_M^k) E^{1/k}(N^k)].$$

Received 30 August 1965; revised 4 April 1966.

Hence $E(Y_M{}^k) = O(1)$ as $M \to \infty$ and $E(Y^k) < \infty$, thereby proving the first part of the theorem. To prove the second part, put $y_{ci} = y_i$ if $y_i \leqq c$, otherwise $y_{ci} = c$. Then $Y_c \leqq Y$, $0 < \alpha \leqq E(y_{ci})$, $y_{ci} \leqq c$; in the sequel we will drop c. Put $x_j = y_j - E(y_j)$; then $|x_j| \leqq 2c$ and

$$|E(X_m{}^k) - E(X_{m-1}^k)| = |\sum_{j=0}^{k-1} \binom{k}{j} E(X_{m-1}^j n_m) E(x_m{}^{k-j})|,$$
$$= O\{\sum_{j=0}^{k-2} E[(2cN_{m-1})^j n_m]\} = O[E(N_m{}^{k-2} n_m)].$$

Hence for $k \geqq 2$, $E(X_m{}^k) = O[E(N_m{}^{k-1})]$. Since $E(X_m) = 0$, this is true for $k = 1$.

Put $N_m{}^* = \sum_1^m n_i E(y_i)$. Then $\alpha N_m \leqq N_m{}^* \leqq cN_m$ and

$$(-1)^k E(N_m{}^{*k}) - E(X_m{}^k)$$
$$= \sum_{j=0}^{k-1} (-1)^{j+1} \binom{k}{j} E(N_m{}^{*j} Y_m{}^{k-j}) = O[\sum_{j=0}^{k-1} E(N_m{}^j Y_m{}^{k-j})].$$

Hence for $k \geqq 1$

$$E(N_m{}^{*k}) = O[E(N_m^{k-1}) + \sum_{j=0}^{k-1} E^{j/k}(N_m{}^k) E^{(k-j)/k}(Y_m{}^k)] = O[E^{(k-1)/k}(N_m{}^k)].$$

Therefore $E(N_m{}^k) = O(1)$ as $m \to \infty$ and $E(N^k) < \infty$.

THEOREM 2. *If* $E(x_i) = 0$, $E(x_i{}^{2k}) \leqq \beta_{2k} < \infty$ for $i = 1, 2, \cdots$ *and* $E(N^k) < \infty$ *then* $E(X^{2k}) = \lim_{m\to\infty} E(X_m{}^{2k}) < \infty$.

PROOF.

$$E(X_m{}^{2k}) - E(X_{m-1}^{2k}) = \sum_{j=0}^{2k-2} \binom{2k}{j} E(X_{m-1}^j n_m) E(x_m{}^{2k-j})$$
$$= O[E(n_m) + E(X_{m-1}^{2k-2} n_m)].$$
$$E(X_m{}^{2k}) = O[E(N) + \sum_{j=1}^m E(X_{j-1}^{2k-2} n_j)]$$
$$= O[E(N) + \sum_{j=1}^m E(X_m{}^{2k-2} n_j)],$$

since, as $X_1{}^2, X_2{}^2, \cdots$ form a semimartingale, so do $X_1{}^{2k-2}, X_2{}^{2k-2}, \cdots$, for $j \leqq m$, and $E(X_m{}^{2k-2} - X_{j-1}^{2k-2}) = E[(X_m{}^{2k-2} - X_{j-1}^{2k-2})n_j] \geqq 0$. Hence

$$E(X_m{}^{2k}) = O[E(N) + E(X_m{}^{2k-2}N)] = O[E^{(k-1)/k}(X_m{}^{2k})E^{1/k}(N^k)]$$
$$= O[E^{(k-1)/k}(X_m{}^{2k})].$$

Therefore $E(X_m{}^{2k}) = O(1)$ as $m \to \infty$ and, by [2], p. 325,

$$E(X^{2k}) = \lim_{m\to\infty} E(X_m{}^{2k}) < \infty.$$

4. Acknowledgment. I wish to express my great appreciation for the substantial improvement of the proofs made by the referee and for the helpful comments of Professor Gunnar Blom.

REFERENCES

[1] CHOW, Y. S., ROBBINS, HERBERT and TEICHER, HENRY (1965). Moments of randomly stopped sums. *Ann. Math. Statist.* **36** 789–799.

[2] DOOB, J. L. (1953). *Stochastic Processes*. Wiley, New York.

[3] HARRIS, T. E. (1947). Note on differentiation under the expectation sign in the fundamental identity of sequential analysis. *Ann. Math. Statist.* **18** 294–295.

[4] JOHNSON, N. L. (1959). A proof of Wald's theorem on cumulative sums. *Ann. Math. Statist.* **30** 1245–1247.

[5] TEICHER, HENRY (1965). Higher moments of randomly stopped sums. *Ann. Math. Statist.* **36** 1608. (Abstract).

[6] WALD, A. (1944). On cumulative sums of random variables. *Ann. Math. Statist.* **15** 283–296.

[7] WOLFOWITZ, J. (1947). The efficiency of sequential estimates and Wald's equation for sequential processes. *Ann. Math. Statist.* **18** 215–230.