CONVERGENCE PROPERTIES OF CONVERGENCE WITH PROBABILITY ONE

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Convergence in probability, convergence in the sense of Levy or Prohorov, and other familiar types of stochastic convergence are topological in nature. For countably additive probabilities, Thomasian (1957) gives necessary and sufficient conditions that convergence with probability one be equivalent to convergence in a metric or a norm defined on the space of random variables. In this paper we discuss when convergence with probability one is a topological convergence for finitely or countably additive probabilities. For finitely additive probabilities we employ the decomposition theorem of Yosida and Hewitt (1952).

Let $(\mathfrak{X}, \mathfrak{A}, \gamma)$ be a finitely additive probability space: γ is finitely but not necessarily countably additive and $\mathfrak{A}$ is a sigma-field (see Birkhoff, 1948, p. 185). Let $\mathfrak{Y}$ be a nontrivial normed linear space with norm $\|\cdot\|$. Let $\mathfrak{M}$ be a vector space, with the same scalars as $\mathfrak{Y}$, of functions defined on $\mathfrak{X}$ with values in $\mathfrak{Y}$ which contain indicator functions of sets: there is $y \neq 0$ in $\mathfrak{Y}$ so that the function $I_A = 0$sr y as $x \varepsilon A^c$ or $x \varepsilon A$ belongs to $\mathfrak{M}$.

Definitions. A topology $\mathfrak{J}$ for $\mathfrak{M}$ is equivalent to convergence with probability one if for each net $\{f_d : d \varepsilon D\}$ in $\mathfrak{M}$, $\lim_d f_d = f \varepsilon \mathfrak{M}$ relative to $\mathfrak{J}$ if and only if there is $M \varepsilon \mathfrak{A}$ with $\gamma(M) = 0$ and $\lim_d \|f_d(x) - f(x)\| = 0$ for $x \varepsilon M^c$ (we denote the latter by $\gamma(\lim_d f_d = f) = 1$). A topology $\mathfrak{J}$ for $\mathfrak{M}$ is equivalent to sequential convergence with probability one if we replace "net" by "sequence" in the foregoing definition.

We rephrase in probabilistic terminology and slightly specialize in (2), below, two *necessary* conditions [Kelley (1955) p. 69 and 74] for a $\mathfrak{J}$ to be equivalent to convergence with probability one:

(1) If it is false that $\gamma(\lim_d f_d = f) = 1$ then there is a subnet $\{g_e : e \varepsilon E\}$ of $\{f_d\}$ whose domain is a cofinal subset of D and which has no subnet which converges with probability one to f.

(2) Let $\mathfrak{C}$ be a directed set and for each $C \varepsilon \mathfrak{C}$ let $N_C \equiv N$, the nonnegative integers. If $\gamma(\lim_C f_C = f) = 1$ and if $\gamma(\lim_n g_{C,n} = f_C) = 1$, $C \varepsilon \mathfrak{C}$, then there is a directed set D and a function $R: D \rightarrow \mathfrak{C} \times N$ so that $\gamma(\lim_d g \circ R = f) = 1(g(C, n) \equiv g_{C,n})$.

Here are some propositions which help characterize when γ, finitely or countably additive, may fail to satisfy (1) and/or (2). Recall the following definitions: (i) γ is purely finitely additive if the only countably additive nonnegative measure φ on satisfying $\varphi(A) \leqq \gamma(A)$, $A \varepsilon \mathfrak{A}$ is $\varphi \equiv 0$ [Yosida and Hewitt, (1952)]; (ii) $A \varepsilon \mathfrak{A}$ is a positive γ-atom if $\gamma(A) > 0$ and for each $B \subset A$, $B \varepsilon \mathfrak{A}$ either $\gamma(B) = 0$ or $\gamma(B) = \gamma(A)$.

Received 10 January 1966; revised 10 July 1966.

LEMMA 1. *If γ is purely finitely additive and A is a positive γ-atom then there is a sequence $\{f_n : n \varepsilon \mathcal{C} \equiv N\}$ and sequences $\{g_{n,m} : m \varepsilon N\}$, $n \varepsilon N$ so that condition (2) fails.*

PROOF. Since the restriction of γ to A remains purely finitely additive there is a decreasing sequence $\{A_n\} \subset \mathcal{A}$ with $\cap A_n = \emptyset$ and $\gamma(A_n) = \gamma(A)$. The assertion follows upon defining $f_n = I_{A_n^c}$, $f \equiv y$ and $g_{n,m} \equiv I_A c$, $(n, m) \varepsilon N \times N$.

Condition (2) is clearly fulfilled when $\mathcal{C} = N$ and γ is countably additive—but not when $\mathcal{C}$ has cardinal $\mathbf{c}$. For example, let $\gamma = \lambda$ be the Lebesgue measure on the Borel sets of [0, 1] and let $\mathcal{C} = \{C: \lambda(C^c) = 0\}$. Let $\mathcal{Y} = R$ and $y = 1$. Now $\mathcal{C}$ is directed if $C_1 \geqq C_2$ means $C_1 \subset C_2$ and clearly $\lambda(\lim_C I_{C^c} = 1) = 1$. But condition (2) fails if we define $g_{C,n} \equiv 0$, $(C, n) \varepsilon \mathcal{C} \times N$. We mention the following:

PROPOSITION. *Assume that $\gamma(A) > 0$ and each finite subset of A belongs to $\mathcal{A}$ and has probability zero. Then there is a net in $\mathfrak{M}$ so that condition (2) is not fulfilled.*

Because of the above fact we consider the case when $\mathcal{C} = N$, the nonnegative integers.

LEMMA 2. *Assume that in condition (2) $\mathcal{C} = N$. A necessary and sufficient condition that (2) always be satisfied is that for each countable class $\{M_i : i \varepsilon N\} \subset \mathcal{A}$ satisfying $\gamma(M_i) = 0$, $i \varepsilon N$, it is true that $\gamma(\bigcup_{i=1}^{\infty} M_i) = 0$.*

PROOF. Sufficiency follows from the fact that if $\gamma(\lim_n f_n = f) = 1$ and $\gamma(\lim_m g_{n,m} = f_n) = 1$ then $f_n \to f$ and $g_{m,n} \to f_n$ in the topology of pointwise convergence on the complement of a fixed null set. This permits us to apply Theorem 4 [Kelley (1955) p. 69] and obtain the result. To verify necessity let $\gamma(M_i) = 0$ and $\gamma(\bigcup_{i=1}^{\infty} M_i) > 0$. The conclusion follows upon defining $B_n = \bigcup_{i=1}^{n} M_i$, $B = \bigcup_{i=1}^{\infty} M_i$, $f_n = I_{B_n}$, $f = I_B$, and $g_{n,m} \equiv 0$.

Of course countable additivity on null sets does not imply countable additivity on $\mathcal{A}$. For example, let $\mathfrak{X}$ be the integers and let $\gamma = \frac{1}{2}(P + \varphi)$ where P is a countably additive probability assigning positive probability to each integer and φ is a purely finitely additive probability. The argument given by Thomasian (1957) in Theorem 1 easily carries over to the next lemma (it is here we require $\mathcal{A}$ to be a sigma-field).

LEMMA 3. *If $(\mathfrak{X}, \mathcal{A}, \gamma)$ is a probability space so that $\{\gamma(A): A \varepsilon \mathcal{A}\}$ is dense in [0, 1] then condition (1) fails for sequences.*

According to a theorem of Yosida and Hewitt (1952) each finitely additive probability γ may be uniquely decomposed into $\gamma = \gamma_1 + \gamma_2$ where γ_1 is a nonnegative countably additive measure and γ_2 is a nonnegative purely additive measure. The following is a partial converse to Lemma 13.

LEMMA 4. *Let $\mathfrak{X} = \bigcup_{i=1}^{\infty} A_i$ where each A_i is a positive γ-atom. Furthermore, for $\gamma = \gamma_1 + \gamma_2$ assume $\gamma_2(A_i) = 0$ and if $\gamma(A_i') = \gamma(A_i)$ then $\gamma_2(\mathfrak{X} \sim \bigcup_{i=1}^{\infty} A_i') = 0$. If $\|f(\cdot)\|$ is measurable then condition (1) is satisfied for sequences.*

PROOF. Let $\gamma(\lim_n f_n = f) < 1$. The assumptions on γ_2 ensure that $\gamma_1(\lim_n f_n = f) < \gamma_1(\mathfrak{X})$. Let $P(A) = \gamma_1(A)/\gamma_1(\mathfrak{X})$. Since convergence in probability is equivalent to convergence with probability one for countably additive probabilities on atomic sigma-fields [Thomasian (1957)] and since convergence in probability is pseudo-metric convergence there is a subsequence $\{f_{n_j}\}$ of $\{f_n\}$ no subsequence of which converges in probability (P) to f. The above mentioned equivalence of the two types of convergence plus the assumptions on γ_2 imply that no subsequence of $\{f_{n_j}\}$ converges with probability one (γ) to f.

THEOREM. *Let* $(\mathfrak{X}, \mathfrak{A}, \gamma)$ *and* $\mathfrak{M}$ *be given with* $\gamma = \gamma_1 + \gamma_2$, γ_1 *countably additive and* γ_2 *purely finitely additive, and let* $\|f(\cdot)\|$ *be measurable,* $f \in \mathfrak{M}$. *The following assertions are equivalent:*

(A) *There is a topology* $\mathfrak{I}$ *for* $\mathfrak{M}$ *equivalent to sequential convergence with probability one.*

(B) *There is a pseudo-metric topology* $\mathfrak{I}$ *for* $\mathfrak{M}$ *equivalent to sequential convergence with probability one.*

(C) $\gamma(f_n \rightarrow f) = 1$ *if and only if* $\gamma_1(f_n \rightarrow f) = \gamma_1(\mathfrak{X})$ *if and only if for each* $\epsilon > 0$, $\gamma_1(\|f_n - f\| > \epsilon) < \epsilon$ *when* $n \geqq n(\epsilon)$.

(D) $\mathfrak{X} = \bigcup_{i=1}^{k} A_i$, $1 \leqq k \leqq \infty$, *each* A_i *is a positive* γ_1*-atom and* $\gamma(M_i) = 0$, $i \in N$, *implies* $\gamma(\bigcup_{i=1}^{\infty} M_i) = 0$.

PROOF. Clearly (B) implies (A) and by the pseudo-metric convergence of convergence in probability under γ_1 (C) implies (B). That (D) implies (C) follows from the hypotheses (and argument) of Lemma 4 and Lemma 2. To verify that (A) implies (D) we first remark that Lemma 3, after a brief argument, implies that $\mathfrak{X} = \bigcup_{i=1}^{k} A_i$, with each A_i a positive γ-atom. On account of Lemma 1 and Theorem 1.19 [Yosida and Hewitt (1952)] $\gamma_2(A_i) = 0$, that is $\gamma_1(A_i) > 0$. Clearly (A) implies countable additivity on null sets because of Lemma 2. This completes the proof.

Acknowledgment. The author thanks the referee for pointing out the paper of Thomasian (1957).

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