A NOTE ON CONSTRUCTION OF PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS WITH PARAMETERS $v = 28, n_1 = 12, n_2 = 15$ AND $p_{11}^2 = 4$[1]

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It is well known [2] that there are exactly four different association schemes of PBIBD with the parameters $v = 28, n_1 = 12, n_2 = 15, p_{11}^2 = 4$. The problem then arises what possibilities does this offer for construction of designs. For $v = b$ one can clearly construct two sets of four distinct designs by forming blocks consisting of all first associates or all second associates of each treatment according to each association scheme. For $v < b$ the problem of construction of designs was not yet investigated. For $v > b$ it is shown here that no other association scheme but the triangular can be used to construct designs.

Let N be the incidence matrix of the design. Connor and Clatworthy [5] showed that:

$$|NN'| = rk\rho_1^{\alpha_1}\rho_2^{\alpha_2} \quad \text{with} \quad \alpha_1 + \alpha_2 = v - 1.$$

If $v \geqq b$ then $\rho_i = 0$ for $i = 1$ or $i = 2$ and $b \geqq v - \alpha_i$ where α_i is the multiplicity of the zero root.

For $n = 8$ let

$$\rho_1 = r + 4\lambda_1 - 5\lambda_2, \qquad \alpha_1 = 7;$$

$$\rho_2 = r - 2\lambda_1 + \lambda_2, \qquad \alpha_2 = 20.$$

Case 1. $\rho_1 = 0$. The first kind of the parameters of the designs have to satisfy the following equations:

(1) $$r = -4\lambda_1 + 5\lambda_2;$$

(2) $$r(k - 1) = 12\lambda_1 + 15\lambda_2;$$

(3) $$28r = bk.$$

Equations 1 and 2 give: $rk = 8\lambda_1 + 20\lambda_2$. Thus rk must be divisible by 4. Eliminating λ_1 from equations (1) and (2) we get $r(k + 2) = 30\lambda_2$. In addition to this $b \geqq 21$ since the multiplicity of ρ_1, $\alpha_1 = 7$. Using this information it is easy to establish that there are just two sets of solutions satisfying the above conditions. They are:

(a) $v = 28; b = 21, r = 15, k = 20, \lambda_1 = 10, \lambda_2 = 11;$

(b) $v = 28, b = 21, r = 6, k = 8, \lambda_1 = 1, \lambda_2 = 2.$

Neither of these sets represents a design. First notice that it suffices to show

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that one of them does not represent a design since they may be obtained from each other by replacing each block of one of them by the block of varieties which do not occur in it. There are several methods apparent to show that (b) does not trepresent a design. One of them will be described presently. By a theorem of Roy and Laha [7] design (b) would have to be a LB (linked block design). Using further a result of Shrikhande. [8] the dual of a LB design is BIBD. This means that the existence of the design (b) would imply the existence of a BIBD with $v = 21, b = 28, r = 8, k = 6, \lambda = 2$. Connor [3] showed that such a design does not exist.

CASE 2. $\rho_2 = 0, \alpha_2 = 20$. Replace now (1) by (1′) and search for values of the first kind of parameter satisfying (1′), (2) and (3).

(1′) and (2) give $rk = 14\lambda_1 + 14\lambda_2$. Hence rk must be even and divisible by 7. Moreover k must be divisible by 7 since otherwise $b \geqq 49$ and thus $\geqq v$. Hence there are just three values for k to be considered, namely $k = 7, 14, 21$. This reduces the consideration of the possible designs to the following:

(a_1) $v = 28, b = 8, r = 2, \lambda_1 = 1, \lambda_2 = 0, k = 7$;
(a_2) $v = 28, b = 16, r = 4, \lambda_1 = 2, \lambda_2 = 0, k = 7$;
(a_3) $v = 28, b = 24, r = 6, \lambda_1 = 3, \lambda_2 = 0, k = 7$;
(b_1) $v = 28, b = 6, r = 3, \lambda_1 = 2, \lambda_2 = 1, k = 14$;
(b_2) $v = 28, b = 12, r = 6, \lambda_1 = 4, \lambda_2 = 2, k = 14$;
(b_3) $v = 28, b = 18, r = 9, \lambda_1 = 6, \lambda_2 = 3, k = 14$;
(b_4) $v = 28, b = 24, r = 12, \lambda_1 = 8, \lambda_2 = 4, k = 14$;
(c_1) $v = 28, b = 8, r = 6, \lambda_1 = 5, \lambda_2 = 4, k = 21$;
(c_2) $v = 28, b = 16, r = 12, \lambda_1 = 10, \lambda_2 = 8, k = 21$;
(c_4) $v = 28, b = 24, r = 18, \lambda_1 = 15, \lambda_2 = 12, k = 21$.

Designs (a_1), (a_2) and (a_3) could not be constructed with the help of another scheme but the triangular association scheme. In these cases $\lambda_2 = 0$ and $k = 7$ and no other scheme but the triangular has seven treatments which are mutually first associates. (a_1) can in fact be constructed using the familiar method of deleting the diagonal entries from the triangular association scheme; (a_2) and (a_3) can be obtained by doubling or tripling of (a_1).

(c_1), (c_2) and (c_3) are duals of the corresponding (a)'s. Hence they too can be constructed using no other association scheme but the triangular.

TABLE 1

α	β					
	1	2	3	4	5	6
1	*	1	1	2	2	2
2	1	*	1	2	2	2
3	1	1	*	1	2	2
4	2	2	1	*	2	2
5	2	2	2	2	*	1
6	2	2	2	2	1	*

TABLE 2

Treatments layouts

Block	I (a)	I (b)	II (a)	II (b)	III (a)
1	1 2 4	3	1 2 4		1 2
2	1 2 4	3	1 2	3	1 2
3	1 2	3	1 2	3	1 2
4	1 2	3	1 2	5	1 2
5	1	5	1 4	3	1 4
6	1	5	1	3 5	1 4
7	2	5	2 4	3	2 4
8	2	5	2	3 5	2 4
9	4	3 5	4	5	4
10	4	3 5	4	5	4
11	4		4		
12	4			5	

TABLE 3

Layouts for (b_3)

Block	I (a)	I (b)	II (a)	II (b)	III (a)	IV (a)
1	1 2 4	3	1 2 4	3	1 2 4	1 2
2	1 2 4	3	1 2 4	3	1 2	1 2
3	1 2 4	3	1 2	3 5	1 2	1 2
4	1 2	3	1 2	3	1 2	1 2
5	1 2	3	1 2	3	1 2	1 2
6	1 2	3	1 2		1 2	1 2
7	1	5	1 4	3	1 4	1 4
8	1	5	1	5	1 4	1 4
9	1	5	1	5	1	1 4
10	2	5	2 4	3	2 4	2 4
11	2	5	2	5	2 4	2 4
12	2	5	2	5	2	2 4
13	4	3 5	4	3 5	4	4
14	4	3 5	4	3 5	4	4
15	4	3 5	4	5	4	4
16	4		4		4	
17	4		4			
18	4			5		

Next it will be shown that designs (b_1), (b_2), (b_3) and (b_4) do not exist.

Since $b \geqq v - \alpha_2$ i.e. $b \geqq 8$, (b_1) cannot be constructed.

No elegant method was found to prove the non-existence of (b_2), (b_3) and (b_4).

Let $(\alpha, \beta) = i$, $i = 1, 2$ denote that treatments α and β are first or second associates respectively. Each of the four association schemes includes five treatments whose association relation can be summarized in Table 1.

TABLE 4

Layouts for (b_4)

Block	I		II		III		IV	V
	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(a)
1	1 2 4	3	1 2 4	3	1 2 4	3	1 2 4	1 2
2	1 2 4	3	1 2 4	3	1 2 4	3	1 2	1 2
3	1 2 4	3	1 2 4	3	1 2	3 5	1 2	1 2
4	1 2 4	3	1 2	3	1 2	3 5	1 2	1 2
5	1 2	3	1 2	3	1 2	3	1 2	1 2
6	1 2	3	1 2	3	1 2	3	1 2	1 2
7	1 2	3	1 2	3 5	1 2		1 2	1 2
8	1 2	3	1 2		1 2		1 2	1 2
9	1	5	1 4	3	1 4	3	1 4	1 4
10	1	5	1	5	1 4	3	1 4	1 4
11	1	5	1	5	1	5	1 4	1 4
12	1	5	1	5	1	5	1	1 4
13	2	5	2 4	3	2 4	3	2 4	2 4
14	2	5	2	5	2 4	3	2 4	2 4
15	2	5	2	5	2	5	2 4	2 4
16	2	5	2	5	2	5	2	2 4
17	4	3 5	4	3 5	4	3 5	4	4
18	4	3 5	4	3 5	4	3 5	4	4
19	4	3 5	4	3 5	4	5	4	4
20	4	3 5	4	5	4	5	4	4
21	4		4		4		4	
22	4		4		4			
23	4		4			5		
24	4			5		5		

The value in row α and column β gives the value of (α, β), where $\alpha \neq \beta$ and $*$ along the diagonal indicates that no treatment is either 1-associate or 2-associate of itself. The proof that designs (b_2), (b_3) and (b_4) do not exist will be carried out in two steps.

STEP 1. If an association scheme includes six treatments satisfying Table 1, then one cannot construct either of the designs of type (b).

STEP 2. Each of the four distinct association schemes does include at least one sixtuple of treatments satisfying Table 1. To establish the first step it will be shown that for each possible layout of the treatments 1, 2, and 4 the positions of 3, 5 are uniquely determined and 6 cannot be accommodated without violating the conditions regarding the values of λ_1 and λ_2.

Consider first (b_2) and all possible layouts for treatments 1, 2 and 4 and their supplements including the treatments 3 and 5. They are displayed in Table 2.

Notice that III(a) does not have to be considered since it can be obtained from I(a) replacing each block by its complement.

It is easy to see that the positions of treatments 3 and 5 are determined by parts (a) of the layouts. Hence treatment 6, which bears the same relation to

TABLE 5
Table of first association scheme

Treatments	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
f.a.	2	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	3	3	4	4	4	5	5	6	8	8
	3	3	2	2	2	2	2	2	3	3	3	4	5	3	3	3	6	6	9	9	5	6	6	7	7	7	11	11
	4	4	4	3	3	4	5	6	4	5	8	6	7	4	4	8	7	7	10	10	12	9	9	10	10	9	12	12
	5	5	5	5	4	7	6	7	6	7	9	8	8	5	5	11	8	8	11	11	13	12	12	13	13	10	13	13
	6	6	9	6	7	8	8	11	10	9	10	9	10	15	14	14	14	15	14	15	14	14	15	14	15	17	14	15
	7	7	10	9	10	9	10	12	11	11	12	11	11	16	16	15	16	16	16	16	15	17	18	17	18	18	16	16
	8	8	11	12	13	12	13	13	12	13	13	13	12	17	18	17	18	17	17	18	22	19	20	19	20	19	17	18
	9	14	14	14	14	17	17	16	19	19	16	21	21	19	20	18	19	20	20	19	23	21	21	21	21	20	19	20
	10	15	15	15	15	18	18	17	20	20	19	22	24	21	21	19	22	23	22	23	24	23	22	22	23	22	21	21
	11	16	16	21	21	22	24	18	22	24	20	23	25	22	23	20	24	25	24	25	25	24	25	25	24	23	22	23
	12	17	19	22	24	23	25	27	23	25	27	27	27	24	25	27	26	26	26	26	27	26	26	26	26	24	24	25
	13	18	20	23	25	26	26	28	26	26	28	28	28	27	28	28	27	28	27	28	28	27	28	27	28	25	28	27

TABLE 6
Table of second association scheme

Treatments	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
f.a.	2	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	3	3	4	4	5	5	6	6	7	8
	3	3	2	2	2	2	2	2	3	3	3	4	4	3	3	3	5	6	5	9	6	9	7	8	7	8	9	9
	4	4	4	3	3	4	5	5	4	5	5	6	6	4	4	4	7	7	10	10	12	12	10	11	10	11	10	11
	5	5	5	6	7	7	6	6	10	7	8	7	8	5	6	9	8	8	11	11	13	13	12	13	12	13	12	13
	6	6	9	9	8	8	8	7	11	9	9	9	9	15	14	14	14	15	14	15	14	14	14	14	15	15	16	16
	7	7	10	12	10	12	10	11	12	11	10	10	11	16	16	15	16	16	15	16	15	16	17	17	18	18	17	17
	8	8	11	13	11	13	12	13	13	12	13	13	12	17	18	17	18	17	20	18	19	17	19	19	19	19	18	18
	9	14	14	14	14	15	17	17	16	19	19	21	21	19	19	18	22	20	21	19	22	21	21	21	20	20	20	20
	10	15	15	15	17	18	18	18	20	20	20	22	22	21	20	20	23	25	23	25	23	23	22	22	21	21	22	22
	11	16	16	16	19	21	23	24	22	23	24	23	24	22	21	22	24	26	24	26	24	24	24	23	23	24	23	24
	12	17	19	21	23	25	25	26	27	25	26	25	26	23	25	27	27	27	25	27	25	27	25	26	26	25	25	26
	13	18	20	22	24	26	27	28	28	27	28	27	28	24	26	28	28	28	26	28	26	28	27	28	27	28	28	27

TABLE 7

Table of third association scheme

Treatments	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
	2	1	1	1	1	1	1	1	1	1	1	1	1	2	2	2	2	2	3	3	3	4	4	4	5	6	7	8
	3	3	2	2	2	2	2	2	3	3	4	6	7	3	3	4	5	7	5	6	9	5	6	9	7	8	11	9
	4	4	4	3	3	3	5	6	4	6	5	7	8	4	5	6	7	8	9	10	10	9	11	11	11	10	12	10
	5	5	5	5	4	4	8	7	5	8	7	8	9	6	7	8	8	13	10	12	13	11	12	13	12	12	13	13
	6	6	6	6	7	8	11	10	10	9	9	10	10	15	14	14	15	14	15	14	14	16	14	14	15	16	14	16
f.a.	7	7	9	9	9	10	12	12	11	12	12	11	11	16	17	17	16	15	17	15	15	17	16	16	17	17	15	17
	8	8	10	11	11	12	13	13	13	13	13	13	12	18	18	18	18	16	20	19	18	19	20	18	19	19	18	18
	9	14	14	14	15	14	15	16	19	19	22	20	18	20	19	22	19	17	21	21	19	23	22	21	20	20	20	19
	10	15	15	16	17	16	17	17	21	20	23	23	21	21	20	23	22	21	22	23	20	24	24	22	22	22	21	21
	11	16	19	22	19	20	18	18	22	21	24	25	24	23	22	24	25	24	25	25	24	25	25	23	23	23	23	22
	12	17	20	23	22	23	25	26	24	26	25	26	27	24	25	26	26	27	26	26	27	26	26	27	26	25	24	24
	13	18	21	24	25	26	27	28	28	28	27	27	28	27	27	28	28	28	28	27	28	28	27	28	27	28	25	26

treatments 1 through 4 as 5 would have to coincide in the layout I six times and in the layout II at least five times contradicting $\lambda = 4$. The same reasoning applies to the possibility of designs (b_3) and (b_4). Here the corresponding designs have the form as given in Tables 3 and 4.

As before the consideration of III and IV can be omitted. In I, 6 would have to coincide with 5 nine times and in II at least seven times contradicting $\lambda_1 = 6$.

Again IV and V need not to be considered. In I, II and III 6 would have to coincide with 5 at least twelve, eleven, ten times respectively thus violating $\lambda_1 = 9$.

Six treatments satisfying Table 1 were found in each of the four association schemes. They are exhibited in the following table:

Association scheme	Treatments					
1	1	2	3	19	21	23
2	1	2	3	19	22	27
3	1	2	3	19	23	27
4	1	2	4	19	26	27

Since the three non-triangular schemes formed by Chang are not easily accessible it seems worthwhile to reprint them here in order that the reader may check their properties asserted here and examine them further if he so desires. The fourth scheme is the triangular scheme.

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