L. S. Bark, L. N. Bolshev, P. I. Kuznetsov and A. P. Cherenkov, *Tables of the Rayleigh-Rice Distributions*, Computation Center USSR, Academy of Sciences USSR 28, 1964. 2.80 r. xxviii + 246 pp.

Review by D. B. Owen

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This volume consists of 28 pages of introduction and 245 pages of tables. The principal function tabulated is

$$Q(u, v) = (2\pi)^{-1} \iint_{x^2+y^2 \geqq u^2} \exp\,[-\tfrac{1}{2}\{(x - v)^2 + y^2)\}]\, dx\, dy.$$

One interpretation of this function is that it gives the probability of falling outside a circle of radius u, centered at the origin when a random observation is taken from a bivariate normal distribution with means v and 0, standard deviations one, and correlation zero. This function is referred to as the Q-function, as the offset circle probabilities for the circular normal distribution, as the circular coverage function, and by other names in the English literature. One Russian name for the Q-function is given in the title of the book. It is a form of non-central chi-square with two degrees of freedom, and is related to the Lommel functions of two variables. Large tables presently available to the Western World have not been widely distributed. The Operations Analysis Group of Bell Aircraft Corporation issued Report No. 02-949-106 during June, 1956, under the title, "Table of Circular Normal Probabilities." The Rand Corporation also issued two or more tables of this function or closely related functions. Aside from these, the tabulations have been small enough to appear in standard journals. The reader is referred to papers by Lowe [5] DiDonato and Jarnagin [1] and [2] and Gilliland [3] for further information. These references also refer to other earlier work. The uses for this function are numerous, including heat flow and ion exchange and in the analysis of transient behavior of transmission lines. It also has uses in steady-state analysis of certain antenna problems as well as in several probability and statistics problems.

The introduction[1] contains an excellent summary of the different probabilistic interpretations which may be applied to the Q-function, including a review of most, but not all, of Western literature on the subject. One omission is the paper by Harold Ruben [6], which deals with the closely related problem of the distribution of quadratic forms. The introduction points out the relationship of the Q-function to the distribution of quadratic forms in two dimensions, but does not exploit it as DiDonato and Jarnagin [3] did. It is very informative and well-written as compared to introductions which usually accompany tables. The computation work also appears to have been expertly done.

The table of $Q(u, v)$ is for $u = 0(0.02) \cdots$ (until Q is zero to 6 decimal places)

[1] The translation was done by Mr. Leslie Cohn (University of Chicago), to whom I express appreciation and thanks.

and $v = 0(0.02)3$ to six decimal places. Second differences in the u and v directions are also given. This covers 200 pages. A spot check of the figures in this table gave agreement with corresponding figures given in the Bell Aircraft Corporation table.

The region of definition of u, v is $0 \leqq u < \infty$ and $0 \leqq v < \infty$. The authors recommend different procedures depending on the particular values of u and v as follows: Region A is defined as $v \leqq 3$ and $u \geqq 0$. Region B is $v > 3$ and $0 \leqq u \leqq 3$. Region C is $u \geqq v > 3$, and Region D is $v > u > 3$.

The authors recommend that the table of $Q(u, v)$ as described above (Table I) be used directly for $v \leqq 3$ and $u \geqq 0$ (Region A). If $v > 3$ and $0 \leqq u \leqq 3$ (Region B), then they recommend that Q be computed by the formula

$$Q(u, v) = 1 - Q(v, u) + Q(v - u, 0)e^{-uv}I_0(uv), \tag{1}$$

where the values of $Q(v, u)$ and $Q(v - u, 0)$ are read from Table I, and $e^{-uv}I_0(uv)$ is read from Table III. Table III consists of 9 pages of the function $e^{-x}I_0(x)$ for $x = 0(0.001)3(0.01)25$ with a quarter page table of $\exp(-y^{-2})I_0(y^{-2})$ for $y = 0(0.001)0.2$ both given to seven decimal places. $I_0(x)$ is the Bessel function of imaginary argument.

If $u \geqq v > 3$ (Region C) the authors recommend that Q be computed from

$$Q(u, v) = q - R(q, \epsilon) \tag{2}$$

where $q = 1 - \Phi(w)$, $w = u - v - (2v)^{-1}$, $\epsilon = (1 + v^2)^{-1}$, $\Phi(w)$ is given in Table IV, and $R(q, \epsilon)$ is given in Table II. The function $\Phi(w)$ is the standardized univariate normal cumulative and is given for $w = 0(0.001)3(0.005)4(0.01)5$ to seven places and covers 11 pages. The function $R(q, \epsilon)$ is given for $q = 0(0.0001)0.001(0.001)0.1(0.01)0.57$ and $\epsilon = 0(0.005)0.1$ to six decimal places with first differences in each direction included in the table. $R(q, \epsilon)$ is the difference between $Q(u, v)$ and the approximation $q = 1 - \Phi(w)$. Table II of $R(q, \epsilon)$ covers 15 pages.

If $v > u > 3$ (Region D) then $Q(u, v)$ is obtained from (1) where $Q(v, u)$ in (1) requires the use of (2) and where $Q(v - u, 0)$ and $e^{-uv}I_0(uv)$ are given in Tables I and III, respectively.

The authors recommend a second order Newton interpolation formula for interpolation on one argument in the table of $Q(u, v)$. A modified Bessel interpolation formula is recommended for interpolation in both arguments of $Q(u, v)$. Table V gives values of $\theta(1 - \theta)/2$ for $\theta = 0(0.001)1$ to five decimal places to aid in this interpolation. Linear interpolation is recommended for the table of $R(q, \epsilon)$.

There are two additional tables. The first (Table VI) is a one-page table taken from DiDonato and Jarnagin [3] of the value of u such that $Q(u, v) = 0.99, 0.95(0.05)0.05, 0.03, 0.01, 0.005, 0.001, 0.0001, 0.00001, 0.000001$ for $v = 0.1, 0.5(0.5)2.0(1.0)6.0(2)10(10)30, 50, 80$ and 120. The second table covers two partial pages and is taken from Harter [4]. It is a table of $V(K, c)$ for $K = 0.1(0.1)6$ and $c = 0.0(0.1)1.0$. The $V(K, c)$ function is related to the $Q(u, v)$ function by the following formula:

$$Q(u, v) = \tfrac{1}{2}[-1 + \exp\{-(u^2 + v^2)/2\} I_0(uv) \pm V(|u - v|, |u - v|/(u + v))],$$

where $+$ is taken if $u < v$ and $-$ if $u > v$.

The function $V(K, c)$ can be interpreted as follows: Let x, y be two normally and independently distributed random variables (viewed as a point in a plane) each with mean zero and standard deviations σ_x and σ_y, respectively, where for convenience one labels x and y so that $\sigma_x \geqq \sigma_y$. If $c = \sigma_y/\sigma_x$, the probability that a point lies within a circle centered at the means (the origin) and with radius $K\sigma_x$ is $V(K, c)$. An integral form for $V(K, c)$ is

$$V(K, c) = c^{-1} \int_0^K \exp\{-[(1 + c^2)/4c^2]x^2\} I_0\{[(1 - c^2)/4c^2]x^2\} x \, dx.$$

The reviewer strongly recommends that an English translation of this work be made available or, as an alternative, that someone publish a definitive table of this Q-function in the West. Its uses are manifold and have only been broadly mentioned above. One or two applications would make it desirable to tabulate to 10 decimal places, which would make even the Russian version under review somewhat less useful than it might be. Several points could be added to the Russian introduction, but the Q-function has such a wide variety of applications that it will be difficult, if not impossible, to be sure that all applications have been mentioned.

P. I. Kuznetsov, one of the authors, pointed out in a letter to W. H. Kruskal dated August 18, 1966, that on page XXII, the eighth line from the top, $Q(2.65; 2.153)$ should be $Q(2.655; 2.153)$. This correction was originally made by S. O. Rice.

Kuznetsov also said, "If it is necessary to have seven decimal places, the following formula for $Q(u, v)$ should be used:

$$Q(u, v) = 1 - e^{-\frac{1}{2}(u^2+v^2)}\{\Upsilon_1(u^2, uv) + \Upsilon_2(u^2, uv)\},$$

and the value of function $\Upsilon_n(x, y)$ can be taken from the book by L. S. Bark and P. I. Kuznetsov, *Tables of Lommel's Functions of Two Pure Imaginary Variables*, Pergamon Press, 1965."

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