GAME VALUE DISTRIBUTIONS II[1]

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1. Summary and introduction. An earlier paper [4] has been concerned with the distribution of the value of perfect information games with random payoffs of a certain very special type: two alternatives were assumed available to each player at every move, and the terminal payoffs were assumed to be iid and uniform. This paper considers a more general class of games, with p and q alternatives available, respectively, for players I and II at every move, and with the terminal payoffs arbitrarily distributed, though still iid. Specifically, consider a two-person zero-sum perfect information game, with player I and player II alternately choosing one of several alternative moves, with n choices to be made in all by each. It is assumed that there are always p and q alternatives available respectively to players I and II. Corresponding to each of the $(pq)^n$ possible sequences of moves, there are $(pq)^n$ payoffs (to player I) $x(i_1, i_2, \cdots, i_{2n})$, where the indices $i_1, i_3, \cdots, i_{2n-1}$, each with range $(1, 2, \cdots, p)$ indicate the successive alternatives chosen by player I, and the indices $(i_2, i_4, \cdots, i_{2n})$, each with range $(1, 2, \cdots, q)$, indicate the successive alternatives chosen by player II. The value $v(\{x(i_1, \cdots, i_{2n})\})$ of such a game is

$$\max_{i_1} \min_{i_2} \max_{i_3} \min_{i_4} \cdots \max_{i_{2n-1}} \min_{i_{2n}} x(i_1, \cdots, i_{2n}).$$

Now replace the $(pq)^n$ numbers $x(i_1, \cdots, i_{2n})$ by independent random variables $X(i_1, \cdots, i_{2n})$, each with cdf F. This paper is concerned with the limiting behavior of the random values $V_n(F) \equiv v(\{X(i_1, \cdots, i_{2n})\})$. The limiting behavior of $V_n(F)$ is investigated in Section 2 for uniform F $(F = U)$. Analogous to the results for $p = q = 2$ obtained in [4], the limiting distribution for the sequence $\{V_n(U)\}$ is everywhere continuous and monotone increasing, and satisfies a certain functional equation. Limiting distributions arising from arbitrary F are considered in Section 3. Section 4 is devoted to some results concerning norming sequences and domains of attraction. The final corollary of Section 4 establishes that all of the common cdf's lead to the same limiting distribution.

This study bears a strong resemblance to Gnedenko's [1] study of extremes. Since, in Gnedenko's case, the limiting distributions for $Z_{(n)} = \max(Z_1, \cdots, Z_n)$, where Z_n are independent identically distributed random variables, must in effect be limiting distributions for $Z_{(k^n)}$ for every positive integer k, Gnedenko's argument involves an infinite sequence of functional equations [1], p. 431, namely, one functional equation for every k. In the present treatment the limiting distributions for $V_n(F)$ must satisfy only the single functional equation (6). It

Received 18 May 1966; revised 1 August 1966.

[1] This research was supported by a National Aeronautical and Space Administration research traineeship, and also in part by National Science Foundation grant No. GP-1801.

may thus be worth noting that even a stronger resemblance would exist between this paper and a study of the limiting distributions for $Z_{(k^n)}$, k fixed.

I would like to acknowledge Professor H. T. David for his many helpful discussions concerning this research.

2. The limit distribution for uniform payoffs. Define, for $0 \leqq z \leqq 1$, $\phi(z) = (1 - (1 - z)^q)^p$. Then the cdf for the random value $V_n(F)$ is $\phi^{(n)}[F(x)]$ where $\phi^{(n)}(z)$ is the nth iterate of $\phi(z)$. Also, let a be the interior fixed-point of $\phi(z)$ in [0, 1] (property (A) below), and let $b \equiv \phi'(a) > 1$ be the slope of ϕ at a. Note that ϕ, and therefore a, b, and L_u defined below, are functions of p and q, the number of choices available respectively to players I and II at each move; however, in order to simplify the notation, this functional dependence on p and q will not be made explicit.

For uniform random payoffs ($F = U$) we have the following result concerning the asymptotic behavior of the game value $V_n(U)$.

THEOREM 1. *The sequence* $b^n[V_n(U) - a]$ *converges in distribution to an everywhere continuous and monotone increasing* cdf L_u, *where* L_u *satisfies the functional equation*

$$(1) \qquad \phi[L_u(y/b)] = L_u(y) \quad \text{for} \quad -\infty < y < \infty.$$

In Section 3 of [4], Theorem 1 was proven for the special case when $p = q = 2$. Essentially the proof consisted of establishing the following: (i) for each y the sequence $\phi^{(n)}(a + y/b^n)$ is eventually non-decreasing which implies that $\lim_{n\to\infty} \phi^{(n)}(a + y/b^n) \equiv L_u(y)$ exists. (ii) The function $L_u(y)$, being the limit of a sequence of functions convex in a neighborhood of $y = 0$, is convex and therefore continuous in a neighborhood of $y = 0$. (iii) The function $L_u(y)$ satisfies $L_u(-\epsilon) < a < L_u(\epsilon)$ for all $\epsilon > 0$. This relation was established by the use of functions λ and μ which bound ϕ, are easily iterated, and tend to simple limit functions under the norming $1/b^n$. With the use of the functional equation (1), results (ii) and (iii) led to the result that $L_u(y)$ is a continuous monotone increasing cdf.

For arbitrary integers $p \geqq 2$, $q \geqq 2$ some modification of the proof given in [4] is required. The modifications are necessary as a result of the following two facts: (I) For arbitrary p and q, the sequence $\phi^{(n)}(a + y/b^n)$ is either eventually non-decreasing in n, as it is for $p = q = 2$, *or* eventually non-increasing; (II) it has only been possible, for arbitrary p and q, to show that the functions μ and λ, defined in (F) of [4], bound ϕ in some *neighborhood* of a fixed-point a.

The remainder of this section is devoted to the necessary modifications of the argument in [4] required in view of I and II. Recall that capital letters ,i.e., (A), (B), $\cdots$, denoted successive steps in the argument of [4]. A starred letter will refer to such a step; this device also is used for equation numbers.

(A) *Define, for all* $n \geqq 1$ *and* $0 \leqq v \leqq 1$, $\phi^{(n+1)}(v) = \phi(\phi^{(n)}(v)) = \phi^{(n)}(\phi(v))$; *then*

(i) $0 \leqq \phi^{(n)}(v) \leqq 1$ *on* [0, 1],

(ii) $\phi^{(n)}(v)$ *is monotone increasing on* [0, 1], *and*

(iii) $\phi^{(n)}(v)$ *is continuous on* [0, 1].

(B) *There is a unique number* a, $0 < a < 1$, *such that, for all* $n \geqq 1$,

(i) $0 < \phi^{(n)}(v) < v$ *for* $0 < v < a$,

(ii) $v < \phi^{(n)}(v) < 1$ *for* $a < v < 1$, *and*

(iii) 0, a, *and* 1 *are the only fixed points of* $\phi^{(n)}$ *on* [0, 1].

(C) *By using* (B)(i) *and* (B)(ii) *it is easily shown that*

(2) $a \lesseqgtr m$ iff $[1 - ((q(p-1))/(pq-1))^p]^q \lesseqgtr 1 - (q(p-1))/(pq-1)$

where m is defined as in (C)*. However, equality cannot hold in (2) for integers $p \geqq 2$, $q \geqq 2$ since the polynomial $(1 - x^p)^q = 1 - x$ has no rational roots in the open interval (0, 1).

For the case when $a < m$ in (2), the only modifications required are given below in (F) and (G). The other case, $a > m$, requires, in addition, the further modifitions below.

(D) In (D)*, note that $b \equiv \phi'(a)$ no longer necessarily equals $4a$; also, reverse the inequalities in (10*), (12*), and (13*) and replace the range $[-a, m - a]$ in (11*) and (12*) by the range $[m - a, 1 - a]$.

(E) As in (E)*, except for the third sentence, which is modified to: then, according to (D), there exists n_0 such that, for $n > n_0$, $\phi^{(n+1)}(a + y/b^{n+1}) \leqq \phi^{(n)}(a + y/b^n)$, i.e., $\phi^{(n)}(a + y/b^n)$ is eventually monotone non-increasing.

(F) Change (F)(i*) to read $\lambda(v) \leqq \phi(v) \leqq \mu(v)$ in some neighborhood $(\underline{v}, \bar{v})$ of $v = a$. This relation follows by a simple application of the mean value theorem and the following relations:

$$\lambda(a) = \phi(a) = \mu(a) = a,$$
$$\lambda'(a) = \phi'(a) = \mu'(a) = b,$$
$$\lambda''(a) < \phi''(a) < \mu''(a).$$

The first two of the above relations follow directly from the definitions, while the third seems to require considerable algebraic manipulations: By direct substitution of the expressions for $\lambda''(a)$, $\phi''(a)$, and $\psi''(a)$ in the third relation we have that $\lambda''(a) < \phi''(a) < \mu''(a)$ iff

(3) $-b(b-1)/(1-a) < b[q(p-1)a^{-1/p} - pq + 1]/(1-a) < b(b-1)/a.$

By using the relation $b = pqa(a^{-1/p} - 1)/(1-a)$ (and the relation $1 - a^{1/p} = (1-a)^q$ when simplifying the left inequality) the left and right inequalities in (3) can be shown to hold iff

(4) $$a < 1 - (p)^{-(q-1)^{-1}} \quad \text{and} \quad (q)^{-p/(p-1)} < a,$$

respectively. Then, using the property (see (B)) $a \lessgtr x$ iff $x \lessgtr \phi(x)$, the inequalities in (4) are equivalent to

$$[1 - (p)^{-(q-1)^{-1}}]^{1/p} < 1 - (p)^{-q/(q-1)} \quad \text{and} \quad [1 - (q)^{-(p-1)}]^{1/q} < 1 - (q)^{-p/(p-1)}.$$

Finally, these inequalities can be established for $p \geqq 2$, $q \geqq 2$ by expanding the natural logarithm of each side and noticing the term by term dominance by the series on the right side of the inequalities.

(G) In (G)*, define the neighborhood $J = (y^-, y^+) \cap (\underline{v} - a, \bar{v} - a) = (y_l, y_u)$ where y^-, y^+ are defined in (19*). Change the range [0, 1] to (y_l, y_u) in (16*), (17*), (22*), (25*), and (26*).

(I) In (I)*, replace 'convex' by 'concave'; reverse all inequalities, and replace in the last paragraph the interval $[0, m]$ by the interval $[m, 1]$.

(J) In (J)*, replace 'convex' by 'concave', reverse all inequalities except $n > n_0$, and replace 'non-decreasing' by 'non-increasing' in the second sentence.

3. Limit distributions. In this section limit distributions will be considered for game values $V_n(F)$, where, as before, F represents the common distribution of the random payoffs.

For cdf's G, $\{G_n\}$ the symbol $G_n \to G$ will indicate convergence in distribution, i.e. $\lim_{n\to\infty} G_n = G$ for all continuity points of G. Also, all cdf's are taken to be everywhere continuous from the right.

Definition 1. (Gnedenko). The cdf's $G_1(x)$ and $G_2(x)$ are said to be of the same type if, for some constants α and $\beta > 0$, $G_1(x) = G_2(\beta x + \alpha)$ for $-\infty < x < \infty$.

Definition 2. *A* cdf F will be said to belong to the domain of attraction of a non-degenerate cdf L (denoted by $F \varepsilon \mathfrak{D}(L)$) if there exists a sequence of constants $\{a_n, b_n\}(b_n > 0)$, $n = 1, 2, 3, \cdots$, such that $\phi^{(n)}[F(b_n y + a_n)] \to L(y)$.

Definition 3. The non-degenerate cdf L will be said to be a limit distribution if $\mathfrak{D}(L)$ is not empty.

Let $\mathfrak{L}^*$ be a suitable set of non-degenerate cdf's that satisfy

$$L(-\epsilon) < a \leqq L(\epsilon) \quad \text{for all} \quad \epsilon > 0, \tag{5}$$

and contains exactly one member of every possible type. Without loss of generality, consider the problem of characterizing the class $\mathfrak{L}$ of limit distributions contained in $\mathfrak{L}^*$. The actual members of $\mathfrak{L}^*$ are not specified at this point to allow the simplest possible description of $\mathfrak{L}$ in the subsequent argument.

Theorem 2. *The* cdf $L \varepsilon \mathfrak{L}$ iff *there exists a constant* β, $0 < \beta \leqq 1$, *such that*

$$\phi[L(\beta y)] = L(y) \quad \text{for} \quad -\infty < y < \infty. \tag{6}$$

Proof. First, suppose that $L \varepsilon \mathfrak{L}$. Then by definition there exists a cdf F and a sequence $\{a_n, b_n\}(b_n > 0)$ for which

$$\phi^{(n)}[F(b_n y + a_n)] \to L(y). \tag{7}$$

Define the function $G(y)$ by

$$G(y) = \phi^{-1}[L(y)]. \tag{8}$$

It is easily seen from property (A) of ϕ that $G(y)$ is a non-degenerate cdf. Then, using the continuity of ϕ, it follows from (7) and (8) that

$$\phi^{(n)}[F(b_{n+1} y + a_{n+1})] \to G(y). \tag{9}$$

Then from (7), (9) and Theorem 1 [2], p. 40, it follows that G is of the same type as L, i.e., there are constants $\beta > 0$, α for which $G(y) = L(\beta y + \alpha)$ for $-\infty < y < \infty$. Therefore (8) yields

$$\phi[L(\beta y + \alpha)] = L(y) \quad \text{for} \quad -\infty < y < \infty. \tag{10}$$

Setting $y = 0$ in (10) gives

$$\phi[L(\alpha)] = L(0) \geqq a \Rightarrow L(\alpha) \geqq a \Rightarrow \alpha \geqq 0,$$

where the first implication follows from property (B) of ϕ, and the second from (5). Similarly, setting $y = -\alpha/\beta$ in (10) gives

$$\phi[L(0)] = L(-\alpha/\beta) \geqq a \Rightarrow \alpha \leqq 0,$$

where the implication follows from (5) and $\beta > 0$. Hence, expression (10) reduces to expression (6). Now, suppose that $\beta > 1$ in (6). Then for $y < 0$, $L(\beta y) \leqq L(y)$. Property (B) of ϕ then implies that $L(y) \geqq a$, which is a contradiction of (5).

Conversely, suppose that a cdf L satisfies (6) with $0 < \beta \leqq 1$. Then n-fold iteration of (6) gives $\phi^{(n)}[L(\beta^n y)] = L(y)$ for $-\infty < y < \infty$; $n = 1, 2, 3, \cdots$. Hence, in view of Definition 2, $L \varepsilon \mathfrak{D}(L)$; this is seen by letting $F = L$, $b_n = \beta^n$, and $a_n = 0$.

A more detailed description of $\mathfrak{L}$ is provided by the following two lemmas.

Lemma 1. *Define*

$$\begin{aligned} L^*(y) &= 1 \quad \textit{for} \quad y \geqq 1 \\ &= a \quad \textit{for} \quad 0 \leqq y < 1 \\ &= 0 \quad \textit{for} \quad y < 0. \end{aligned}$$

Then L^* is the only member of $\mathfrak{L}$ which satisfies (6) *for* $\beta = 1$.

Proof. The proof follows immediately since the points 0, a, and 1 are the only fixed-points of $\phi(x)$ for $0 \leqq x \leqq 1$.

Let $\mathfrak{L}' = \mathfrak{L} - \{L^*\}$, and define the following three subclasses of $\mathfrak{L}'$:

$$\begin{aligned} \mathfrak{L}_{\mathrm{I}} &= \{L \varepsilon \mathfrak{L}' \mid 0 < L(y) < a \quad \text{for} \quad -\infty < y < 0, \qquad a < L(y) < 1 \\ &\qquad \text{for} \quad 0 < y < \infty, \qquad L(-0) = a = L(0)\}; \\ \mathfrak{L}_{\mathrm{II}} &= \{L \varepsilon \mathfrak{L}' \mid 0 < L(y) < a \quad \text{for} \quad -\infty < y < 0, \\ &\qquad L(-0) = a, \qquad L(0) = 1\}; \\ \mathfrak{L}_{\mathrm{III}} &= \{L \varepsilon \mathfrak{L}' \mid a < L(y) < 1 \quad \text{for} \quad 0 < y < \infty, \\ &\qquad L(-0) = 0, \qquad L(0) = a\}. \end{aligned} \tag{11}$$

Lemma 2. $\mathfrak{L}' = \mathfrak{L}_{\mathrm{I}} \cup \mathfrak{L}_{\mathrm{II}} \cup \mathfrak{L}_{\mathrm{III}}$.

Proof. Let L be any member of $\mathfrak{L}'$. Then by Theorem 2 and Lemma 1, $\phi^{(n)}[L(\beta^n y)] = L(y)$ for $-\infty < y < \infty$ and some β, $0 < \beta < 1$. The fact that (property (B)) $\phi^{(n)}(x) = 0$ or 1 iff $x = 0$ or 1, respectively, then leads to the following relations:

$$L(y) = 1 \quad \text{for some} \quad y > 0 \Rightarrow L(0) = 1; \tag{12}$$
$$L(y) = 0 \quad \text{for some} \quad y < 0 \Rightarrow L(-0) = 0.$$

We also have

$$L(y) > a \quad \text{for} \quad y > 0, \tag{13}$$

for, if not the functional equation $\phi^{(n)}[L(y)] = L(y/\beta^n)$ for $0 < \beta < 1$; $n = 1, 2, 3, \cdots$ implies that $L(\infty) = a < 1$.

From the fact that

$$\begin{aligned} \lim_{n\to\infty} \phi^{(n)}(x) &= 0 \quad \text{for} \quad 0 \leqq x < a \\ &= a \quad \text{for} \quad x = a \\ &= 1 \quad \text{for} \quad a < x \leqq 1, \end{aligned} \tag{14}$$

the functional equations $\phi^{(n)}[L(\beta^n y)] = L(y)$, for some $0 < \beta < 1$; $n = 1, 2, 3, \cdots$ lead to the following relations:

$$L(y) < 1 \quad \text{for some} \quad y > 0 \Rightarrow L(+0) = a; \tag{15}$$
$$L(y) > 0 \quad \text{for some} \quad y < 0 \Rightarrow L(-0) = a.$$

Since $\mathfrak{L}'$ contains only non-degenerate cdf's relations (12), (13), and (15) establish the lemma.

In the remainder of this section a characterization of $\mathfrak{L}$ will be given in terms of the limiting cdf L_u. For any cdf L define the function $C_L(y)$ as

$$C_L(y) = L_u^{-1}[L(y)] \quad \text{for} \quad -\infty < y < \infty. \tag{16}$$

Since L_u is everywhere monotone-increasing and continuous (Theorem 1) $C_L(y)$ is well-defined for all values of y for which $0 < L(y) < 1$. For values of y for which $L(y) = 0$ or 1 define $C_L(y) = -\infty$ or $+\infty$, respectively.

THEOREM 3. *A non-degenerate* cdf $L \varepsilon \mathfrak{L}$ iff *there exists a constant* β, $0 < \beta \leqq 1$, *for which*

$$bC_L(\beta y) = C_L(y) \quad \textit{for} \quad -\infty < y < \infty. \tag{17}$$

PROOF. First, let L be any non-degenerate cdf and suppose there is a constant β, $0 < \beta \leqq 1$, for which (17) holds. Then the functional equations for L_u and L (expressions (1) and (6), respectively) imply that

$$\begin{aligned} \phi[L(\beta y)] &\equiv \phi[L_u(C_L(\beta y))] = \phi[L_u(C_L(y)/b)] \\ &= L_u[C_L(y)] \equiv L(y) \quad \text{for} \quad -\infty < y < \infty. \end{aligned}$$

Hence, from Theorem 2, $L \varepsilon \mathfrak{L}$.

Conversely, suppose that $L \varepsilon \mathfrak{L}$. Then, relation (1) and Theorem 2 give, for

$-\infty < y < \infty$, $\phi[L_u(C_L(y)/b)] = L_u[C_L(y)]$ and $\phi[L_u(C_L(\beta y))] = L_u[C_L(y)]$. Thus, $L_u(C_L(y)/b) = L_u(C_L(\beta y))$ follows from the monotonicity of ϕ. Relation (17) then follows from the monotonicity of L_u.

By setting $\beta = (1/b)^{1/\gamma}$ in expression (17) of Theorem 3, the following two parameter family ($\gamma > 0$, $\tau > 0$) of cdf's is seen to belong to $\mathfrak{L}_{\mathrm{I}}$:

$$\left\{L_{\gamma,\tau}(y) = L_u[C_{\gamma,\tau}(y)] \mid C_{\gamma,\tau}(y) = \begin{matrix} \tau y^\gamma & \text{for} & y > 0 \\ -|y|^\gamma & \text{for} & y < 0 \end{matrix}\right\}.$$

4. Domains of attraction. The results in this section concerning domains of attraction for limiting distributions of $V_n(F)$ are similar to results obtained by Gnedenko [1] in his study of extremes; Lemma 3 and Theorem 5 given below being the analogues respectively of Lemma 4 and the sufficiency part of Theorem 5 in [1].

LEMMA 3. *The* cdf $F \varepsilon \mathfrak{D}(L)$, *for* $L \varepsilon \mathfrak{L}$, iff *there exists a sequence of constants* $\{a_n, b_n\}(b_n > 0); n = 1, 2, 3, \cdots$, *for which*

$$b^n[F(b_n y + a_n) - a] \to C_L(y), \quad as \quad n \to \infty, \tag{18}$$

at all continuity points of $C_L(y)$, *where, as before*, $C_L(y) \equiv L_u^{-1}[L(y)]$.

PROOF. Let y be any continuity point of $C_L(y)$, or equivalently any continuity point of $L(y)$ since L_u is continuous.

CASE 1. $0 < L(y) < 1$. For this case $-\infty < C_L(y) < \infty$. Then $L_u$continuous and monotone-increasing implies that for any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$L_u[C_L(y)] - \epsilon < L_u[C_L(y) - \delta] < L_u[C_L(y) + \delta] < L_u[C_L(y)] + \epsilon. \tag{19}$$

Suppose now that (18) holds for some cdf F and some sequence $\{a_n, b_n\}$ ($b_n > 0$). Then there exists a number n_0 such that

$$a + (C_L(y) - \delta)/b^n < F(b_n y + a_n) < a + (C_L(y) + \delta)/b^n \quad \text{for} \quad n \geqq n_0$$

which implies that

$$\phi^{(n)}[a + (C_L(y) - \delta)/b^n] < \phi^{(n)}[F(b_n y + a_n)] \leqq \phi^{(n)}[a + (C_L(y) + \delta)/b^n]$$

for $n \geqq n_0$, since $\phi^{(n)}$ is monotone-increasing. Therefore, from the definition of L_u and (19)

$$\begin{aligned} L_u[C_L(y)] - \epsilon < L_u[C_L(y) - \delta] &\leqq \liminf \phi^{(n)}[F(b_n y + a_n)] \\ &\leqq \limsup \phi^{(n)}[F(b_n y + a_n)] \leqq L_u[C_L(y) + \delta] < L_u[C_L(y)] + \epsilon. \end{aligned}$$

Hence, $\phi^{(n)}[F(b_n y + a_n)] \to L_u[C_L(y)] \equiv L(y)$, i.e. $F \varepsilon \mathfrak{D}(L)$.

Conversely, suppose that $F \varepsilon \mathfrak{D}(L)$, i.e., there exists a sequence $\{a_n, b_n\}(b_n > 0)$ for which $\lim_{n\to\infty} \phi^{(n)}[F(b_n y + a_n)] = L(y) \equiv L_u[C_L(y)]$. Now suppose on the contrary that (18) does not hold for the point y. Then there exists an $\epsilon > 0$ and an infinite subsequence $\{n_i\}$ of the positive integers for which

$$\begin{gathered} b^{n_i}[F(b_{n_i} y + a_{n_i}) - a] \leqq C_L(y) - \epsilon \\ \text{or} \\ b^{n_i}[F(b_{n_i} y + a_{n_i}) - a] \geqq C_L(y) + \epsilon. \end{gathered} \tag{20}$$

If the first relation of (20) holds, then $\phi^{(n)}$ monotone increasing implies that

$$\phi^{(n_i)}[F(b_{n_i}y + a_{n_i})] \leqq \phi^{(n_i)}(a + (C_L(y) - \epsilon)/b^{n_i})$$

which further implies that $L_u[C_L(y)] \leqq L_u[C_L(y) - \epsilon] < L_u[C_L(y)]$, since L_u in monotone increasing. Similarly, the second relation of (20) leads to the contradiction that (18) does not hold when $F \varepsilon \mathfrak{D}(L)$.

CASE 2. $L(y) = 0$ or $L(y) = 1$, i.e., $C_L(y) = -\infty$ or $C_L(y) = +\infty$. Then it must be shown that $\lim_{n\to\infty} b^n[F(b_ny + a_n) - a] = -\infty$ or $+\infty$ iff $\lim_{n\to\infty} \phi^{(n)}[F(b_ny + a_n)] = 0$ or 1. Since the equivalence of these relations follow from a straightforward modification of the proof given for Case 1 the details will not be given.

The following lemma establishes that, after a certain translation, scale norming alone is sufficient to reach any limit distribution $L \varepsilon \mathfrak{L}'$.

LEMMA 4. *If $F \varepsilon \mathfrak{D}(L)$ for $L \varepsilon \mathfrak{L}'$, then there exists a unique x_a satisfying*

$$(21) \qquad F(x_a - \epsilon) < a < F(x_a + \epsilon) \quad \textit{for all} \quad \epsilon > 0$$

and there exists a positive sequence $\{b_n\}$ for which

$$(22) \qquad \phi^{(n)}[F(b_ny + x_a)] \to L(y).$$

PROOF. If $F \varepsilon \mathfrak{D}(L)$ for $L \varepsilon \mathfrak{L}'$, i.e., there exists a sequence $\{a_n, b_n\}(b_n > 0)$ for which

$$(23) \qquad \phi^{(n)}[F(b_ny + a_n)] \to L(y),$$

then there must exist a point x_a which satisfies (21); for if not, there would be some interval $[x_l, x_u)$ on which $F(x) = a$. Then it would follow from (14) that $\phi^{(n)}[F(x)] \to L'(x)$, where L' is a cdf of the same type as L^* (defined in Lemma 1). However, $L^* \not\varepsilon \mathfrak{L}'$ which implies that $F \not\varepsilon \mathfrak{D}(L)$ for any $L \varepsilon \mathfrak{L}'$ since a non-degenerate limiting type is unique [2], p. 40.

By Theorem 2 [2], p. 42, all that needs to be established to prove that (23) implies (22) is that

$$(24) \qquad \lim_{n\to\infty} y_n = 0,$$

where $y_n = (x_a - a_n)/b_n$ and the constants $\{b_n\}$ are taken to be the same in (22) as in (23). From Lemma 2 we have

$$(25) \qquad L(-\epsilon) < a < L(\epsilon) \quad \text{for all} \quad \epsilon > 0 \quad \text{and all} \quad L \varepsilon \mathfrak{L}'.$$

Suppose that for some $\delta > 0$

$$(26) \qquad \liminf (y_n) = -\delta.$$

Let ϵ be chosen such that $0 < \epsilon < \delta$ and $-\epsilon$ is a continuity point of L. Then from (21), (23), (25) and (26)

$$a \leqq \liminf \phi^{(n)}[F(x_a)] \equiv \liminf \phi^{(n)}[F(b_ny_n + a_n)]$$
$$\leqq \liminf \phi^{(n)}[F(b_n(-\epsilon) + a_n)] = L(-\epsilon) < a;$$

thus establishing that

$$\liminf (y_n) \geqq 0. \tag{27}$$

Suppose now that for some $\delta > 0$

$$\limsup (y_n) = \delta. \tag{28}$$

Choose ϵ such that $0 < \epsilon < \delta/2$ and ϵ is a continuity point of L. Then from (21), (23), (25) and (28)

$$\begin{aligned} a &\geqq \limsup \phi^{(n)}[F(x_a - (\delta b_n)/2)] \\ &\equiv \limsup \phi^{(n)}[F(b_n(y_n - \delta/2) + a_n)] \\ &\geqq \limsup \phi^{(n)}[F(b_n\epsilon + a_n)] = L(\epsilon) > a; \end{aligned}$$

thus establishing that

$$\limsup (y_n) \leqq 0. \tag{29}$$

Expressions (27) and (29) imply (24), and therefore (22).

Sufficient conditions for a cdf F to belong to the domain of attraction of a limit law are given in the following theorem.

THEOREM 4. *The* cdf $F \varepsilon \mathfrak{D}(L)$, *for* $L \varepsilon \mathfrak{L}'$, *if the following conditions are satisfied*:

(i) *there exists a point* x_a *for which*

$$F(x_a - \epsilon) < a < F(x_a + \epsilon) \quad \textit{for all} \quad \epsilon > 0,$$

and

(ii) *there exists a point* $y_0 \neq 0$ *for which*

$$C_L(y_0) \quad \textit{is finite,} \tag{30}$$

$$F(y_0 z + x_a) \to a \quad \textit{as} \quad z \to +0, \tag{31}$$

$$[F(yz + x_a) - a]/[F(y_0 z + x_a) - a] \to C_L(y)/C_L(y_0), \quad \textit{as} \quad z \to +0, \tag{32}$$

for all continuity points y *of* $C_L(y)$.

PROOF. Let y_0 be any point for which conditions (i) and (ii) hold, and define

$$b_n = \min \{x \mid F(y_0 x + x_a) - a \geqq C_L(y_0)/b^n\}, \tag{33}$$

for $n = 1, 2, 3, \cdots$. Then (31) implies that

$$b_n > 0 \quad \text{for} \quad n = 1, 2, 3, \cdots, \tag{34}$$

and condition (i) implies that

$$b_n \to +0 \quad \text{as} \quad n \to \infty. \tag{35}$$

Let y be any continuity point of $C_L(y)$ and ϵ any positive number. Then there exists a number δ, $0 < \delta < 1$, for which

$$|C_L(y/(1 - \delta)) - C_L(y)| < \epsilon \tag{36}$$

and $y/(1 - \delta)$ is a continuity point of C_L.

CASE 1. $y_0 > 0$, i.e. $C_L(y_0) > 0$. For this case, (30), (33) and (34) imply that

$$(37) \quad F[y_0 b_n(1 - \delta) + x_a] - a \leqq C_L(y_0)/b^n \leqq F[y_0 b_n + x_a] - a$$

for $n = 1, 2, 3, \cdots$, which gives

$$(38) \quad \begin{aligned} &[F(b_n y + x_a) - a]/[F(b_n y_0 + x_a) - a] \\ &\qquad \gtreqless b^n[F(b_n y + x_a) - a]/C_L(y_0) \\ &\qquad \gtreqless [F(b_n y + x_a) - a]/[F(b_n y_0(1 - \delta) + x_a) - a] \end{aligned}$$

according to $y \lesseqgtr 0$. Then it follows from (32) and (35) that

$$(39) \quad \begin{aligned} &|\limsup \{b^n[F(b_n y + x_a) - a]\} - \liminf \{b^n[F(b_n y + x_a) - a]\}| \\ &\qquad \leqq |C_L(y/(1 - \delta)) - C_L(y)| < \epsilon. \end{aligned}$$

CASE 2. $y_0 < 0$, i.e. $C_L(y_0) < 0$. Relation (39) also follows for this case since only the inequalities in (37) and (38) will be reversed when $y_0 < 0$.

Hence, it has been established that the sequence $\{b_n > 0\}$ defined in (33) satisfies

$$\lim_{n\to\infty} \{b^n[F(b_n y + x_a) - a]\} = C_L(y)$$

for all continuity points of $C_L(y)$. Then by Lemma 3 it follows that $F \varepsilon \mathfrak{D}(L)$.

COROLLARY. *If the* cdf $F(x)$ *satisfies condition* (i) *of Theorem* 4 *and the density* $f(x)$ *exists and is larger than zero in some neighborhood of the point* x_a *(defined by condition* (i)), *then* $F \varepsilon \mathfrak{D}(L_u)$.

PROOF. Since $F(yz + x_a) \to a$, as $z \to 0$, for $-\infty < y < \infty$, and the derivative of F exists in some neighborhood of x_a, L'Hospital's rule may be used, which yields

$$\lim_{z\to 0} [F(yz + x_a) - a]/[F(z + x_a) - a] = \lim_{z\to 0} yf(yz + x_a)/f(z + x_a) = y.$$

Hence, $F \varepsilon \mathfrak{D}(L_u)$ by Theorem 4.

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