# GAME VALUE DISTRIBUTIONS I[1]

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**1. Summary and introduction.** This paper is concerned with the distribution of the value of a game with random payoffs. Two types of games are considered: matrix games with iid matrix elements, and games of perfect information with iid terminal payoffs.

Let $\|x_{ij}\|$, $i$: 1, 2, $\cdots$, $m$; $j$: 1, 2, $\cdots$, $n$, be the matrix of player I's payoffs in a zero-sum two-person game, and let $v(\|x_{ij}\|)$ be its (possibly mixed) value. Consider the random value $V_{m,n}(f) \equiv v(\|X_{ij}\|)$, where the $X_{ij}$ are $mn$ iid random variables, each distributed according to the density $f$. It is pointed out in Section 2 that the conditional distribution of $V_{m,n}$, given that it is pure, is that of the $n$th largest of $m + n - 1$ iid random variables, each distributed according to $f$. For $f$ uniform on (0, 1) (i.e., $f = u$), a method is given for determining the conditional distribution of $V_{2,n}(u)$, given that it is mixed. This leads to an elementary expression for the distribution of $V_{2,2}(u)$ and the asymptotic distribution of $V_{2,n}(u)$.

Consider as well two players alternately choosing one of two alternative moves, with $n$ choices to be made in all by each. Corresponding to each of the $4^n$ possible sequences of moves, there are $4^n$ payoffs $x(i_1, i_2, \cdots, i_{2n})$ for player I, $i_k = 1$ or 2, where the odd and even locations indicate, respectively, the successive alternatives chosen by players I and II. The (pure) value $v(\{x(i_1, \cdots, i_{2n})\})$ of such a game is

$$\max_{i_1} \min_{i_2} \max_{i_3} \min_{i_4} \cdots \max_{i_{2n-1}} \min_{i_{2n}} x(i_1, \cdots, i_{2n}).$$

Now replace the $4^n$ numbers $x(i_1, \cdots, i_n)$ by independent uniformly distributed random variables $X(i_1, \cdots, i_{2n})$. The asymptotic behavior of the random value $V_n \equiv v(\{X(i_1, \cdots, i_{2n})\})$ is investigated in Section 3; it is shown that the asymptotic distribution $L$ of $V_n$ is everywhere continuous and monotone-increasing, and satisfies a certain functional equation; it is also shown that the moments of the normed $V_n$ converge to those of $L$. It is planned, in a subsequent paper, to explore games of perfect information in greater depth.

After this paper was submitted, Thomas M. Cover drew our attention to [3] and [9]. The derivation in [9] of the expected value of a $2 \times n$ game, conditionally on there being a $2 \times 2$ kernel, is based on essentially the geometric considerations leading to our distribution (5); however, since the argument in [9] is not aimed at obtaining distributions, and is thus rather different in detail, a sketch of our derivation of (5) has not been deleted.

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In [3], the probability is computed, in the case of payoff distributions symmetric about zero, that an $m \times n$ game has positive value. Also, the work of Efron [4] and that of Sobel [8] pertain to Section 2, and that of Buehler [1] to Section 3. Finally, closely related to this paper, and indeed the source of our original interest in this area, is the work of Chernoff and Teicher [2].

**2. Matrix games.** For an $(m \times n)$ matrix $\|X_{ij}\|$ of iid random variables $X_{ij}$, each distributed according to the density $f$, let $\pi_{mn}$ denote the event that the corresponding zero-sum two-person game has a pure value. Then [5], p. 79,

$$\Pr[\pi_{mn}] = m!\,n!/(m+n-1)!, \tag{1}$$

and [9], p. 366,

$$\Pr[V_{m,n}(f) \leqq t \mid \pi_{mn}] = \Pr[X_{m+n-1}^{(n)} \leqq t]. \tag{2}$$

where $X_{m+n-1}^{(n)}$ = the $n$th largest of $m + n - 1$ iid random variables, each distributed according to $f$.

We turn next to conditioning on the complement $\bar{\pi}_{mn}$ of $\pi_{mn}$, and specialize the discussion initially to the case $m = 2$ and $f = u$; accordingly, we abbreviate $V_{m,n}(f)$ to $V_n$. We lean now on the usual geometric construction ([6], p. 405) based on the convex hull $CH_n$ of the $n$ points $p_j$: $(X_{1j}, X_{2j})$ and the right-angular wedge $W_n$, with apex on the equiangular line, touching $CH_n$. Conditionally on $\bar{\pi}_{2n}$, the following will obtain with probability one: (i) $CH_n \cap W_n$ will contain exactly one point $Q_n$, $Q_n$ lying on the equiangular line; (ii) $Q_n$ will lie as well on precisely one of the edges, say $E_n$, of $CH_n$, and $E_n$ will connect two extreme points $P_{1,n}$ and $P_{2,n}$ of $CH_n$, respectively above and below the equiangular line; (iii) there will be a unique "separating" line $L_n$ for $CH_n$ and $W_n$, namely the line through $P_{1,n}$ and $P_{2,n}$; $L_n$ will have negative slope, and its intercepts $A_n$ and $B_n$ with the horizontal and vertical axes will determine $V_n$ in accordance with

$$V_n = A_n B_n/(A_n + B_n). \tag{3}$$

Our approach has been to compute the conditional (on $\bar{\pi}_{2n}$) distribution of $V_n$ through (3) and the joint distribution of $(A_n, B_n)$. To this end, for any positive $a$, $b$ and $\Delta$, let $l_1$ be the line through $(a, 0)$ and $(0, b)$, $l_2$ the line through $(a + a\Delta/b, 0)$ and $(0, b + \Delta)$, and let $U$ and $L$ be, respectively, the regions in the positive quadrant bounded by the equiangular line, the vertical axis, $l_1$ and $l_2$, and the region bounded by the equiangular line, the horizontal axis, $l_1$ and $l_2$. Also, for any two points $p$ and $q$ in the positive quadrant, let $h(p, q)$ and $v(p, q)$ be the respective horizontal- and vertical-axis intercepts of the line $l(p, q)$ through $p$ and $q$. Now define the event $\tau_{IJ}(a, b, \Delta)$: $[p_I \in U; p_J \in L; b \leqq v(p_I, p_J) \leqq b + \Delta; a \leqq h(p_I, p_J) \leqq a + a\Delta/b; p_j, j \neq I, J$, above $l(p_I, p_J)]$. Then it is clear that, excepting an event of zero probability, the event $[\bar{\pi}_{2n}; a \leqq A_n \leqq a + a\Delta/b; b \leqq B_n \leqq b + \Delta]$ is the sum of the $(n)(n-1)$ mutually exclusive events $\tau_{IJ}(a, b, \Delta)$. Moreover, symmetry implies that $\Pr[\tau_{IJ}(a, b, \Delta)]$ does not depend on

$(I, J)$, so that

$$\Pr[a \leqq A_n \leqq a + a\Delta/b;\ b \leqq B_n \leqq b + \Delta \mid \bar{\pi}_{2n}] = n(n-1)\Pr[\tau_{12}(a, b, \Delta)]/\Pr[\bar{\pi}_{2n}]$$

and the conditional density $g_n(a, b)$ of $(A_n, B_n)$ is given by

$$n(n-1)/\Pr[\bar{\pi}_{2n}] \cdot \lim_{\Delta \to 0} (b/a\Delta^2) \Pr[\tau_{12}(a, b, \Delta)]. \tag{4}$$

The integration and limit for the second factor of (4), denoted, say, by $h_n(a, b)$, are routine, and yield the following expressions (where it has been convenient to set $v \equiv ab/(a + b)$):

$$\begin{aligned} &\text{for } 0 < b \leqq a \leqq 1,\ h_n(a, b) = (v^2/2)(1 - ab/2)^{n-2}; \\ (5)\quad &\text{for } 0 < b \leqq 1 < a,\ h_n(a, b) = (bv/2a^2)(1 - v)(1 - b(1 - \tfrac{1}{2}a))^{n-2}; \\ &\text{for } 0 < 1 < b \leqq a,\ ab/(a + b) \leqq 1,\ h_n(a, b) \\ &\qquad = (1 - v)^3/2v(1 - a - b + (a^2 + b^2 + a^2b^2))/2ab)^{n-2}. \end{aligned}$$

Since $h_n(a, b)$ clearly is symmetric, relations (5) determine $h_n$ as well for $b > a$.

To obtain the conditional (on $\bar{\pi}_{2n}$) density $g_n(v)$ of $V_n$, one must integrate the conditional density $g_n(a, b)$ of $(A_n, B_n)$, as given by (4) and (5), in accordance with (3). This can be done in closed form when $n = 2$, and yields a density symmetric about $\frac{1}{2}$, given on $(0, \frac{1}{2}]$ by:

$$g_2(v) = 3(4v^2 - v^3/(1 - v) + 4v^3 \ln((1 - v)/v)). \tag{6}$$

Finally, combining the two conditional distributions (2) and (6) with the help of (1), one obtains for the distribution of $V_2$ a further density symmetric about $\frac{1}{2}$, given on $(0, \frac{1}{2}]$ by:

$$g_{V_{2,2}(u)}(v) = 4v - v^3/(1 - v) + 4v^3 \ln((1 - v)/v).$$

Consider next the asymptotic distribution of $V_{2,n}(u) \equiv V_n$, to be examined as well through that of $(A_n, B_n)$. In view of (4) and (5), the conditional (on $\bar{\pi}_{2n}$) density of $(n^{\frac{1}{2}}A_n, n^{\frac{1}{2}}B_n)$ is given by

$$\begin{aligned} (7)\quad g_n^*(a, b) &\equiv g_{n^{\frac{1}{2}}A_n, n^{\frac{1}{2}}B_n}(a, b) \\ &= \tfrac{1}{2}(1 - 1/n)(ab/(a + b))^2(1 - ab/2n)^{n-2}/\Pr[\bar{\pi}_{2n}] \end{aligned}$$

on the square $(0, n^{\frac{1}{2}}] \times (0, n^{\frac{1}{2}}]$, and by similarly scaled modifications of (5) elsewhere in the domain $a > 0$, $b > 0$, $ab/(a + b) \leqq n^{\frac{1}{2}}$. Hence, in view of (7) and (1), $g_n^*(a, b)$ converges, at every point $(a, b)$ of the positive quadrant, to the function

$$\gamma(a, b) = (\tfrac{1}{2})(ab/(a + b))^2 \exp(-ab/2), \tag{8}$$

and $\gamma$ is a density, which can be seen by changing variables to $(x = a/b, b)$ and integrating first with respect to $b$. It then follows from Scheffé's theorem

[7] that integrals of $g_n^*$ over Borel sets of form $ab/(a + b) \leqq v$ converge to the corresponding integrals of $\gamma$: i.e.,

$$\begin{aligned} &\Pr [n^{\frac{1}{2}}V_n \leqq v \mid \bar{\pi}_{2n}] \\ (9) \qquad &= \Pr [(n^{\frac{1}{2}}A_n)(n^{\frac{1}{2}}B_n)/(n^{\frac{1}{2}}A_n + n^{\frac{1}{2}}B_n) \leqq v \mid \bar{\pi}_{2n}] \\ &= \int_{ab/(a+b) \leqq v} g_n^*(a, b)\, da\, db \rightarrow_n \int_{ab/(a+b) \leqq v} \gamma(a, b)\, da\, db \equiv L_1(v). \end{aligned}$$

Finally, since

$$\Pr [n^{\frac{1}{2}}V_n \leqq v] = \Pr [\pi_{2n}] \Pr [n^{\frac{1}{2}}V_n \leqq v \mid \pi_{2n}] + \Pr [\bar{\pi}_{2n}] \Pr [n^{\frac{1}{2}}V_n \leqq v \mid \bar{\pi}_{2n}],$$

the right-hand side of (9), in view of (1), is the asymptotic cdf of $n^{\frac{1}{2}}V_n$.

Note that our results for the uniform distribution are easily extended to distributions essentially equivalent to it. In other words, let $f(t)$ be a density equal to zero to the left of some $t_0$, and continuous to the right and discontinuous to the left at $t_0$. Then $n^{\frac{1}{2}}f(t_0)(V_{2,n}(f) - t_0)$ tends in distribution to $L_1$.

**3. Games of perfect information.** Define, for $0 \leqq v \leqq 1$, $\phi(v) = (1 - (1 - v)^2)^2$. Then the cdf of the random value $V_n \equiv v(\{X(i_1, \cdots, i_{2n})\})$ introduced in Section 1 is the $n$th iterate $\phi^{(n)}(v)$ of $\phi(v)$ on [0, 1]. This section is devoted to showing that $\phi^{(n)}(a + v/(4a)^n)$ converges to a non-degenerate continuous cdf $L$ where $a$ is the unique fixed point of $\phi(v)$ in (0, 1); i.e., that $(4a)^n(V_n - a)$ tends in distribution to $L$. The proof of this, given below in a series of steps, incorporates a fairly complete qualitative description of $L$. Obtaining an elementary representation for $L$ analogous to that for $L_1$ seems tied to solving the functional equation (27) below, and has not been accomplished.

(A) *Define, for all* $n \geqq 1$ *and* $0 \leqq v \leqq 1$, $\phi^{(n+1)}(v) = \phi(\phi^{(n)}(v)) = \phi^{(n)}(\phi(v))$; *then*

(i) $0 \leqq \phi^{(n)}(v) \leqq 1$ *on* [0, 1],

(ii) $\phi^{(n)}(v)$ *is monotone increasing on* [0, 1], *and*

(iii) $\phi^{(n)}(v)$ *is continuous on* [0, 1].

(B) *The number* $a$ *in* (0, 1) *satisfying* $a^2 - 3a + 1 = 0$ *is such that, for all* $n \geqq 1$,

(i) $0 < \phi^{(n)}(v) < v$ *for* $0 < v < a$,

(ii) $v < \phi^{(n)}(v) < 1$ *for* $a < v < 1$, *and*

(iii) $0$, $a$ *and* $1$ *are the only fixed points of* $\phi^{(n)}$ *on* [0, 1].

(C) *The number* $m$ *in* $(a, 1)$ *satisfying* $3m^2 - 6m + 2 = 0$ *is such that*

(i) $\phi''(v) > 0$ *for* $0 < v < m$,

(ii) $\phi''(v) < 0$ *for* $m < v < 1$,

(iii) $\phi''(m) = 0$,

(iv) $\phi(m) > m$.

(D) *Let* $b \equiv \phi'(a) = 4a$, *which is greater than one, and consider any interval* $I: [y_0, y_1]$; *there exists* $n_0$ *such that, for all* $n > n_0$ *and all* $y$ *in* $I$,

$$\phi^{(n+1)}(a + y/b^{n+1}) \geqq \phi^{(n)}(a + y/b^n). \tag{10}$$

PROOF. Given $I$, consider $n_0$ large enough so that, for $n > n_0$ and $y$ in $I$,

$$-a \leqq y/b^n \leqq m - a. \tag{11}$$

Recall that, in view of (C) (i),

$$\phi(a + z) \geqq a + bz \quad \text{for} \quad -a \leqq z \leqq m - a. \tag{12}$$

Then (11) and (12) imply that

$$\phi(a + y/b^{n+1}) \geqq a + y/b^n. \tag{13}$$

In addition, (11) implies that

$$a + y/b^n \,\varepsilon\, [0, 1] \tag{14}$$

and also that $a + y/b^{n+1}$ is in [0, 1], the latter implying in turn, in view of (A) (i), that

$$\phi(a + y/b^{n+1}) \,\varepsilon\, [0, 1], \tag{15}$$

so that (14) and (15), together with (A) (ii), imply (10).

(E) *For any* $y$, $-\infty < y < \infty$, $L(y) \equiv \lim_{n\to\infty} \phi^{(n)}(a + y/b^n)$ *exists, and* $0 \leqq L(y) \leqq 1$.

PROOF. In (D), take $I$:$[y, y]$. Then, according to (D), there exists $n_0$ such that, for $n > n_0$, $\phi^{(n+1)}(a + y/b^{n+1}) \geqq \phi^{(n)}(a + y/b^n)$, i.e., $\phi^{(n)}(a + y/b^n)$ eventually is monotone non-decreasing. In addition, in view of (14) and (A) (i), $\phi^{(n)}(a + y/b^n)$ eventually is in [0, 1].

(F) *Define, on* [0, 1], $\mu(v) \equiv a(v/a)^b$, $\lambda(v) \equiv 1 - (1 - a)((1 - v)/(1 - a))^b$ *and the iterates* $\mu^{(n)}$ *and* $\lambda^{(n)}$ *of* $\mu$ *and* $\lambda$ *on* [0, 1] *analogously to those of* $\phi$ *in* (A) *then*

(i) $\lambda(v) \leqq \phi(v) \leqq \mu(v)$, $0 \leqq v \leqq 1$,
(ii) $\mu^{(n)}(v) = a(v/a)^{b^n}$ *and* $\lambda^{(n)}(v) = 1 - (1 - a)((1 - v)/(1 - a))^{b^n}$
(iii) $\mu$ and $\lambda$ are monotone increasing on [0, 1].

(G) *There is a neighborhood $J$ of 0 in which*

$$\alpha(y) \equiv 1 - (1 - a) \exp(-y/(1 - a)) \leqq L(y) \leqq a \exp(y/a) \equiv \beta(y).$$

PROOF. Define

$$Z_k \equiv \{z{:}0 \leqq \lambda^{(k)}(z + a); \mu^{(k)}(z + a) \leqq 1\}. \tag{16}$$

It is easily verified that $a + Z_1 \,\varepsilon\, [0, 1]$, so that, in view of (F) (i),

$$0 \leqq \lambda(z + a) \leqq \phi(z + a) \leqq \mu(z + a) \leqq 1 \quad \text{on} \quad Z_1. \tag{17}$$

In addition, it follows from (F) (ii) that

$$Z_1 \supset Z_2 \supset \cdots. \tag{18}$$

Now define as well the numbers $y^- < 0$ and $y^+ > 0$ by

$$\alpha(y^-) = 0; \qquad \beta(y^+) = 1. \tag{19}$$

Defining $J \equiv (y^-, y^+)$, we have, for any $y \varepsilon J$, in view of (F) (ii) and (19), that

$$(20) \qquad \lim_{n\to\infty} \lambda^{(n)}(a + y/b^n) = \alpha(y) > \alpha(y^-) = 0,$$
$$\lim_{n\to\infty} \mu^{(n)}(a + y/b^n) = \beta(y) < \beta(y^+) = 1.$$

Hence there exists an $N$ such that $y/b^n \varepsilon Z_n$ for all $n > N$, and, in view of (18),

$$(21) \qquad y/b^n \varepsilon Z_n, Z_{n-1}, \cdots, Z_1,$$

so that, in view of (17) and (21), for $y \varepsilon J$ and $n$:1, 2, $\cdots$,

$$(22) \qquad 0 \leqq \lambda(a + y/b^n) \leqq \phi(a + y/b^n) \leqq \mu(a + y/b^n) \leqq 1.$$

It follows that

$$(23) \quad \lambda^{(2)}(a + y/b^n) \equiv \lambda(\lambda(a + y/b^n)) \leqq \lambda(\phi(a + y/b^n))$$
$$\leqq \phi(\phi(a + y/b^n)) \equiv \phi^{(2)}(a + y/b^n),$$

where (22) and (F) (iii) validate the first inequality, and (22) and (F) (i) validate the second.

Similarly,

$$(24) \qquad \phi^{(2)}(a + y/b^n) \leqq \mu^{(2)}(a + y/b^n).$$

But, in view of (21), $y/b^n$ in fact is in $Z_2$, so that (23) and (24) can be improved to

$$(25) \quad 0 \leqq \lambda^{(2)}(a + y/b^n) \leqq \phi^{(2)}(a + y/b^n) \leqq \mu^{(2)}(a + y/b^n) \leqq 1.$$

The argument leading from (22) to (25), iterated $n$ times, then yields

$$(26) \quad 0 \leqq \lambda^{(n)}(a + y/b^n) \leqq \phi^{(n)}(a + y/b^n) \leqq \mu^{(n)}(a + y/b^n) \leqq 1,$$

whereupon going to the limit with $n$ establishes what was to be shown.

(H) *$L(y)$ satisfies the functional equation*

$$(27) \qquad \phi^{(k)}(L(y/b^k)) = L(y) \quad \text{for} \quad k{:}1, 2, \cdots.$$

PROOF.

$$L(y) = \lim_{n\to\infty} \phi^{(k+n)}(a + y/b^{k+n}) = \lim_{n\to\infty} \phi^{(k)}(\phi^{(n)}(a + (y/b^k)/b^n))$$
$$= \phi^{(k)}(\lim_{n\to\infty} \phi^{(n)}(a + (y/b^k)/b^n)) = \phi^{(k)}(L(y/b^k)).$$

Here the first and fourth equalities are justified by (E), and the second and third by (A).

(I) *$\phi^{(n)}(a + y/b^n)$ is convex for all $y$ such that $a + y/b^n$ is in* [0, 1] *and* $\phi^{(n)}(a + y/b^n) \leqq m$.

PROOF. In view of (C) (iv), $\phi(m) > m$, so that, by (A) (ii),

$$(28) \qquad \phi^{-1}(m) < m.$$

Hence

$$(29) \qquad [v \varepsilon [0, 1]; \phi^{(n)}(v) \leqq m] \Rightarrow [\phi^{(n-1)}(v) \leqq m],$$

since then, in view of (A) (ii) and (28), $\phi^{(n-1)}(v) \leqq \phi^{-1}(m) < m$. Now suppose that $\phi^{(n-1)}(v)$ is convex for all $v$ such that $v$ is in [0, 1] and $\phi^{(n-1)}(v) \leqq m$, and consider any $v_1$, $v_2$ in [0, 1] such that $\phi^{(n)}(v_i) \leqq m$. Then, in view of (29),

$$\phi^{(n-1)}(v_i) \leqq m$$

and, in view of the convexity assumption concerning $\phi^{(n-1)}(v)$,

$$(30) \qquad \phi^{(n-1)}((v_1 + v_2)/2) \leqq \tfrac{1}{2}(\phi^{(n-1)}(v_1) + \phi^{(n-1)}(v_2)),$$

which leads to

$$\phi^{(n)}((v_1 + v_2)/2) = \phi(\phi^{(n-1)}((v_1 + v_2)/2))$$
$$\leqq \phi(\tfrac{1}{2}(\phi^{(n-1)}(v_1) + \phi^{(n-1)}(v_2))) \leqq \tfrac{1}{2}(\phi^{(n)}(v_1) + \phi^{(n)}(v_2)),$$

where the first inequality follows from (30) and (A) (ii), and the second follows from (30) and (C) (i).

The induction is now complete since (C) (i) shows $\phi(v)$ to be convex on $[0, m]$, and our original assertion follows since a linear transformation preserves convexity.

(J) *There is a neighborhood of zero in which L is convex.*

Proof. In view of (G), there is a neighborhood $[\underline{y}, \bar{y}]$ of zero in which $L(y) < m$, and, in view of (D), there is an $n_0$ such that, for $n > n_0$, $\phi^{(n)}(a + y/b^n)$ is monotone non-decreasing in $n$ for every $y$ in $J$. Moreover, for all such $y$, $\phi^{(n)}(a + y/b^n)$ tends to $L(y)$, in view of (E). Hence, for $n > n_0$ and $y$ in $[\underline{y}, \bar{y}]$,

$$(31) \qquad \phi^{(n)}(a + y/b^n) \leqq L(y) < m.$$

Hence, in view of (I) and (31), $\phi^{(n)}(a + y/b^n)$ is convex for $n > n_0$ and $y$ in $[\underline{y}, \bar{y}]$, so that $L(y)$ is convex in $[\underline{y}, \bar{y}]$, since the limit preserves convexity.

(K) *There is a neighborhood of zero in which L is continuous.*

That $L$ is continuous in $(\underline{y}, \bar{y})$ follows from (J).

(L) *L is continuous everywhere.*

Proof. Given any $y_0$, there is a $k$ large enough so that $y_0/b^k$ is in $(\underline{y}, \bar{y})$, hence, in view of (K), so that $L(y/b^k)$ is continuous at $y_0$. But then, in view of (A) (iii) and (E), $\phi^{(k)}(L(y/b^k))$ is continuous at $y_0$, and hence also $L(y)$, in view of (H).

(M) *L is monotone increasing.*

Proof. To begin with, $L(y)$ is monotone non-decreasing since $\phi^{(n)}(a + y/b^n)$ is monotone increasing. It therefore remains, only, to show that there cannot be $y_1$ and $y_2$ with $y_1 \neq y_2$ and $L(y_1) = L(y_2)$. Suppose, then, that there is such a pair $(y_1, y_2)$; then, in view of (G), $y_1$ and $y_2$ must be on the same side of $a$, say $a < y_1 < y_2$. (H) and (A) (ii) then imply that $L(y_1/b^n) = L(y_2/b^n)$ for any $n$, which, since $L$ is monotone non-decreasing and in view of (J), implies in turn that $L$ is constant in a right neighborhood of 0. But this is a contradiction of (G).

(N) $L(-\infty) = 1 - L(+\infty) = 0$.

PROOF. Consider any $y < 0$. Then, in view of (G) and (M), $0 \leqq L(y) < a$, and $\lim_{k\to\infty} \phi^{(k)}(L(y)) = 0$; hence, in view of (H), $\lim_{k\to\infty} L(b^k y) = 0$, which implies that $L(-\infty) = 0$ because of (M). That $L(+\infty) = 1$ is shown in similar fashion.

This completes the characterization of the limit law $L$. Analogously to the case of $L_1$, the limit law $L$ applies as well to a considerably larger set of payoff distributions $F$; indeed, as will be shown in a subsequent paper, to all distributions with non-zero derivative at the point $x_a$ where $F(x_a) = a$.

A final remark concerns the convergence of the moments and absolute moments of $b^n(V_n - a)$ to those of $L$. Let both $X$ and $Y$ be $R_1$, and consider any probability measure $\mathfrak{F}$, with cdf $F$, on the Borel sets $\mathfrak{b}_X$ of $X$; let $\mathfrak{L}$ be Lebesgue measure on the Borel sets $\mathfrak{b}_Y$ of $Y$. Then Fubini's theorem, applied to the function $|y|^{k-1}$ integrated with respect to $\mathfrak{F} \times \mathfrak{L}$ on the Borel set $[y \leqq 0; x \leqq y \leqq 0]$ of $\{\mathfrak{b}_X\} \times \{\mathfrak{b}_Y\}$, yields the identity

$$k \int_{-\infty}^{0} |y|^{k-1} F(y)\, dy = \int_{-\infty}^{0} |x|^k\, dF(x). \tag{32}$$

But, for $y \leqq 0$ and $n:1, 2, \cdots$,

$$\phi^{(n)}(a + y/b^n) \leqq \mu^{(n)}(a + y/b^n) < a \exp(y/a), \tag{33}$$

where the first inequality follows from (26), and the second follows from the known monotonicity of the approach of $(1 + x/n)^n$ to $e^x$. It follows from (33) and (F) that $L(y) \leqq a \exp(y/a)$ for $y \leqq 0$, so that

$$\int_{-\infty}^{0} |y|^k\, dL(y) < +\infty. \tag{34}$$

It follows as well that

$$\begin{aligned} \lim_{n\to\infty} \int_{-\infty}^{0} |x|^k\, d\phi^{(n)}(a + x/b^n) &= \lim_{n\to\infty} \left[k \int_{-\infty}^{0} |y|^{k-1} \phi^{(n)}(a + y/b^n)\, dy\right] \\ &= k \int_{-\infty}^{0} |y|^{k-1} \left[\lim_{n\to\infty} \phi^{(n)}(a + y/b^n)\right] dy \\ &= k \int_{-\infty}^{0} |y|^{k-1} L(y)\, dy \\ &= \int_{-\infty}^{0} |x|^k\, dL(x), \end{aligned} \tag{35}$$

where the first and fourth equalities follow from (32), and the second and third from (E), (32), (34) and Lebesgue's theorem.

Similarly,

$$\lim_{n\to\infty} \int_{0}^{\infty} x^k\, d\phi^{(n)}(a + x/b^n) = \int_{0}^{\infty} x^k\, dL(x) < +\infty. \tag{36}$$

Relations (35) and (36) establish the desired convergence. The latter may be of some interest from the following elementary game-theoretic point of view: Consider a composite game $G$ consisting of the successive playing of $N$ zero-sum games $G_1, G_2, \cdots, G_N$ of the type under consideration here. Then, as often happens also in the case of less trivial composite games (see [6], Appendix 8), $G$ is itself a zero-sum game for which the minimax strategies simply call for minimax strategies in the component games $G_i$. If now $N$ is large and the payoffs in the component games can be thought of as randomly selected from a single distribution, the average per-component-game gain of Player I, in a single play of $G$, will be approximated by the expectation $E(V_n)$; $E(V_n)$ thus approximates the per-component-game payment of Player I to Player II that makes $G$ fair. If $\mu$ is the first moment of $L$, the convergence of the first moment of $\phi^{(n)}(a + y/b^n)$ to $\mu$ then allows us the further approximation $E(V_n) \doteq a + \mu/b^n$ for large $n$.

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