ON OBTAINING BALANCED INCOMPLETE BLOCK DESIGNS FROM PARTIALLY BALANCED ASSOCIATION SCHEMES

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1. Shrikhande and Singh [6] pointed out that if there exists a PBIB design with two associate classes for v varieties such that $p^1_{ii} = p^2_{ii} = \lambda$ ($i = 1$ or 2), then we can construct a symmetric BIB design for v varieties with parameter $v, r = n_i, \lambda$. The design is obtained by letting the jth block consist of all those varieties which are ith associates of the jth variety.

Their procedure can be extended to PBIB designs with m associate classes, $m > 2$, with the requirement that for some i, $p^k_{ii} = \lambda$ for all k, where $k = 1, \cdots, m$. Consider the following association scheme for $(s + 2)(s + 3)(s + 4)/6$ varieties. Denote each variety by one of the triplets (x, y, z) where x, y, z are integers such that $1 \leqq x < y < z \leqq s + 4$.

Two varieties are ith associates if their representations have $(3 - i)$ integers in common. This is an association scheme of a class of PBIB designs of three associate classes, John [3], with $p^1_{22} = s^2$, $p^2_{22} = (s - 1)(s + 6)/2$, $p^3_{22} = 9(s - 2)$. For $s = 3$ we have $v = 35$, $p^1_{22} = p^2_{22} = p^3_{22} = 9$. Thus we obtain a BIBD $b = v = 35$, $r = k = 18$, $\lambda = 9$ by assigning one block to each variety and letting the jth block contain all the second associates of the jth variety. For example, the block corresponding to variety $(1, 2, 3)$ consists of the eighteen varieties represented by $(1, y, z)$, $(2, y, z)$ or $(3, y, z)$ where $y, z = 4, 5, 6$ or 7 and $y < z$. (The existence of a design for $v = 35$, $k = 18$, $\lambda = 9$ is well known. A difference set for the complementary design $(35, 17, 8)$ is given by Hall [2]).

2. Other symmetric PBIB designs are obtained by assigning to the jth block the jth variety and its ith associates. For these designs $\lambda_i = p^i_{ii} + 2$, and $\lambda_k = p^k_{ii} (k \neq i)$, $k = n_i + 1$ and BIB designs are obtained when we can find association schemes with $\lambda_i, \lambda_k = \lambda$ for all k. This procedure produces nothing new from PBIB schemes with two associate classes because the design of this type for $i = 1$ is merely the complement of the design obtained by the procedure of the previous section for $i = 2$, and vice versa. However, for $m > 2$ new designs occur.

The cubic designs of Raghavarao and Chandrasekhararao [4] have $v = s^3$, $p^1_{22} = 2(s - 1)(s - 2)$, $p^2_{22} = 2(s - 1) + (s - 2)^2$, $p^3_{22} = 6(s - 2)$ $n_2 = 3(s - 1)^2$. Each variety is represented by a set of three integers (x, y, z), $1 \leqq x, y, z \leqq s$. The integers in any representation need not be all different and any two varieties are ith associates if their representations have exactly $(3 - i)$ coordinates equal. With $s = 4$ we have $p^1_{22} = p^2_{22} + 2 = p^3_{22} = 12$, $v = 64$, $n_2 + 1 = 28$. This gives a symmetric BIBD with $v = 64$, $k = 28$, $\lambda = 12$. The jth block consists of v_j and all its second associates. The block corresponding to variety $(1, 2, 3)$ consists of

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(1, 2, 3) itself and the varieties denoted by $(x, y, 3)$, $(x, 2, z)$ or $(1, y, z)$ where $x \neq 1$, $y \neq 2$, and $z \neq 3$. The case (64, 28, 12) was one of the twelve cases for which the existence of a perfect difference set was reported as still undecided by Hall [2] after he had investigated all cases (v, k, λ) with $3 \leqq k \leqq 50$.

3. In this section we establish the non-existence of some association schemes for PBIB designs with 2 associate classes.

In the associate scheme for a PBIB design with two associate classes we may write $p_{11}^1 = g$ and $p_{11}^2 = h$ where $n_1 g + n_2 h = n_1(n_1 - 1)$. Consider schemes for which $h = g + 2$; from every such scheme we can obtain a symmetrical BIBD by including in the jth block the jth variety and all its first associates. For such BIBD's the parameters are $v = 1 + n_1 + n_2 = (n_1(n_1 + 1)/h) + 1$, $k = 1 + n$, $\lambda = h$. However, some values of n_1, h give sets of parameters for which BIB designs are known to be non-existent. It follows that the corresponding association schemes do not exist. The non-existent association schemes found in this way are listed below; the parameters v, k, λ of the non-existent BIB designs are given in parentheses:

$$
\begin{array}{lllll}
v = 53, & n_1 = 12, & g = 1, & h = 3, & (53, 13, 3); \\
v = 103, & n_1 = 17, & g = 1, & h = 3, & (103, 18, 3); \\
v = 34, & n_1 = 11, & g = 2, & h = 4, & (34, 12, 4); \\
v = 106, & n_1 = 20, & g = 2, & h = 4, & (106, 21, 4); \\
v = 77, & n_1 = 19, & g = 3, & h = 5, & (77, 20, 5); \\
v = 52, & n_1 = 17, & g = 4, & h = 6, & (52, 18, 6); \\
v = 58, & n_1 = 18, & g = 4, & h = 6, & (58, 19, 6).
\end{array}
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The non-existence of the BIB designs is established by the results of Shrikhande [5] and of Chowla and Ryser [1].

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