SLOWLY BRANCHING PROCESSES[1]

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A (discrete time) branching process is a sequence Z_n, $n = 1, 2, \cdots$, of random variables describing the size of successive generations of a population. Here it is assumed that given $Z_n = k$ the $n + 1$st generation consists of the offspring of the k members of the nth generation; each of these k individuals having a random number U_i $(i = 1, \cdots, k)$ of children. The U_i are mutually independent with a common distribution given by its probability generating function $f(s)$.

A continuous time Markov branching process is usually described as representing the number Z_t of particles at time t. Any one of the particles present at time t has a probability $b\tau + o(\tau)$ of splitting within the time interval $[t, t + \tau]$ into a random number of new particles whose distribution is given by a probability generating function $h(s)$.

Under somewhat less restrictive assumptions on the processes Stratton and Tucker [6] have considered a sequence $\{Z_N(t), N = 1, 2, \cdots\}$ of branching processes with $Z_N(0) = N$ and such that, as $N \to \infty$, the branching rate b_N converges to zero with Nb_N approaching a finite limit b assuming that $h(s)$ is independent of N. They found that the sequence of processes $\{X_N(t)\} = \{Z_N(t) - N\}$ converges to a process with independent increments and with characteristic function

$$\psi(u, t) = \exp \{bte^{-iu}[h(e^{iu}) - e^{iu}]\}. \tag{1}$$

The interpretation of this result is that, as N increases, the branching of the process is slowed down so much that in the limit the occurrence of "higher generation" particles can be neglected and out of the original particles only a certain number (with Poisson distribution as limit of binomials) will have split. Thus the limiting process will be a compound Poisson process (see e.g. Feller [3]).

The purpose of this note is to establish the discrete time version and the continuous state space versions of the above result and at the same time to give a proof simpler than that by Stratton and Tucker.

Let us first consider the discrete time case. Here slowing down the process is achieved by letting $P(Z_1 = 1 \mid Z_0 = 1)$ tend to one without changing the conditional distribution of Z_1 given $Z_1 \neq 1$. In other words we consider a sequence $\{Z_n(N), N = 1, 2, \cdots\}$ of branching processes with $Z_0(N) = N$ such that the distribution of the number of 'offspring' of any one individual is given by

$$f(s, p) = (1 - p)s + pf(s), \qquad |s| \leqq 1, \tag{2}$$

where $f(s)$ may be any probability function and $Np \to \lambda > 0$.

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LEMMA 1. (i) *As $p \to 0$, the iterates of $f(s, p)$ satisfy*

$$(3) \qquad f_n(s, p) = s + np(f(s) - s) + o(p), \qquad |s| \leqq 1, n = 1, 2, \cdots,$$

(ii) *If $f_{nm}(s_n, s_m, p)$ is the joint probability generating function of $Z_n(N)$ and $Z_m(N)$, $n < m$, then, as $p \to 0$,*

$$(4) \qquad f_{nm}(s_n, s_m, p) = s_n s_m + s_n(m-n)p(f(s_m) - s_m) + np(f(s_n s_m) - s_n s_m) + o(p).$$

PROOF. Using the continuity of f, i.e. the relation $f(s + o(1)) = f(s) + o(1)$ we show (i) by induction. (4) then follows by conditioning on $Z_n(N)$, again using the continuity together with (3).

COROLLARY. *If $pN \to \lambda > 0$, then the joint distribution of $X_n(N)$ and $X_m(N) - X_n(N)$, $n < m$, converges to that with characteristic function*

$$(5) \qquad \varphi_{nm}(u_n, u_m) = \varphi_n(u_n)\varphi_{m-n}(u_m) = \exp\{n\lambda((f(s_n) - s_n)/s_n) + (m-n)\lambda((f(s_m) - s_m)/s_m)\},$$

where $s_k = e^{iu_k}$ and $X_k(N) = Z_k(N) - N$, for $k = 1, 2, \cdots$.

PROOF. The joint characteristic function of $X_n(N) = Z_n(N) - N$ and $X_m(N) - X_m(N) = Z_m(N) - Z_n(N)$ is

$$\varphi_{nm}(u_n, u_m; N) = E s_n^{Z_n(N)-N} s_m^{Z_m(N)-Z_n(N)} = s_n^{-N}[f_{nm}(s_n/s_m, s_m, p)]^N$$
$$= [1 + (m-n)p(f(s_m) - s_m)/s_m + np(f(s_n) - s_n)/s_n + o(p)]^N$$
$$\to \varphi_{nm}(u_n, u_m).$$

It is clear that in the same manner a version of the corollary can be established for the joint probability generating function of the sizes of any set of distinct generations. This is our final result in the discrete time case.

THEOREM 1. *If $pN \to \lambda > 0$, then the distribution of the process $\{X_n(N), n = 1, 2, \cdots\}$ converges to that of a sequence of sums $X_n = \sum_{k=1}^{n} Y_k$ of independent variables Y_k, all having the same characteristic function*

$$(6) \qquad \varphi_1(u) = Es^{Y_1} = \exp\{\lambda((f(s) - s)/s)\}, \qquad s = e^{iu}.$$

From the above discussion it seems clear that the continuous time result (1) can be shown in a similar way provided a continuous time version of the basic Lemma 1, (i) holds.

LEMMA 2. *Let $F(s, t, b)$ be the probability generating function of the variable Z_t of a time homogeneous Markov branching process with $Z_0 = 1$, with basic probability generating function $h(s)$ and with branching rate b. Then, as $b \to 0$,*

$$(7) \qquad F(s, t, b) = s + bt(h(s) - s) + o(b), \qquad |s| \leqq 1.$$

PROOF. Using the obvious relation $F(s, b, t) = F(s, t, b)$ the relation (7) is clearly equivalent to the equation

$$(8) \quad \partial F(s, t, b)/\partial t \mid_{t=0} = b(h(s) - s) = b[h(F(s, 0, b)) - F(s, 0, b)], \quad |s| \leqq 1$$

which is well known to be true (see e.g. Harris [4], V. (9.1) for $t = 0$).

Indeed it is possible to proceed from here exactly as in the discrete time case to obtain

THEOREM 2. *If* $\{Z_t, t \geqq 0\}$ *is a Markov branching process with initial state* $Z_0 = N$, *with basic probability generating function* $h(s)$ *and with branching rate* b *such that* $Nb \to \lambda > 0$, *then the distribution of the process* $X_t = Z_t - N$ *converges, as* $N \to \infty$, *to that of a process with independent increments and with marginal distributions given by* (1), i.e. *to a compound Poisson process.*

REMARK. In the proof of Theorem 1 we may admit probability generating functions of nonnegative, not necessarily integer valued, random variables (Bühler [1], [2]) without changing the argument. Also a relation similar to (8) holds for continuous time branching processes with continuous state space (equation (2.9) in [1]). Thus, both Theorem 1 and Theorem 2 remain true for for branching processes with continuous state space as introduced by Jiřina [5].

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