ON THE COMBINATION OF INDEPENDENT TEST STATISTICS

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1. Introduction. Let T_i be independent one-sided test statistics for testing the hypothesis $H_{i,0}:\theta_i = \theta_{i,0}$ for the independent real-valued parameter θ_i against the one-sided alternatives $\theta_i > \theta_{i,0}$, $i = 1, 2, \cdots, k$. For the sake of definiteness we suppose that large values of T_i lead to rejection of $H_{i,0}$. It is desired to combine the results of these tests, i.e. to construct a function of $T_1, T_2, \cdots, T_k$ that may be used to test the combined hypothesis $H_0:\theta_i = \theta_{i,0}$, $i = 1, 2, \cdots, k$, against the alternative $\theta_i \geqq \theta_{i,0}$, $i = 1, 2, \cdots, k$, with strict inequality at least once.

A well-known combination method is the so-called omnibus test of R. A. Fisher [4] which is based on the probability integral transformation. If T_i has a continuous distribution function F_i under the null-hypothesis $H_{i,0}$, then $F_i(T_i)$ is uniformly distributed on (0, 1) under $H_{i,0}$. As a result, under H_0, $-\log(1 - F_i(T_i))$, $i = 1, 2, \cdots, k$, have independent exponential distributions, hence

$$-\sum_{i=1}^k \log(1 - F_i(T_i))$$

has a gamma distribution with parameter k and consequently a chi-square test is applicable. Independent of Fisher's work, K. Pearson [12] proposed $-\sum_{i=1}^k \log F_i(T_i)$ as a test statistic, small values leading to rejection of H_0. L. H. C. Tippett [13] considered $\max_{1 \leqq i \leqq k} F_i(T_i)$, whereas B. Wilkinson [15] put forward the mth largest value among the $F_i(T_i)$, which has a beta distribution under H_0. A. Birnbaum [1] has shown, however, that for the exponential class of distributions Pearson's test and Wilkinson's test for $m > 1$ are inadmissible.

Generalizing the approach of Fisher and Pearson, T. Liptak [10] studied statistics of the type $\sum_{i=1}^k \alpha_i \Psi^{-1}(F_i(T_i))$, where Ψ^{-1} is the inverse of an arbitrary distribution function Ψ and α_i are arbitrary weights. Taking for Ψ the exponential distribution one obtains a weighted version of Fisher's test which was introduced by I. J. Good [5]. However, from the point of view of distribution theory a more obvious choice is Liptak's proposal to consider $\sum_{i=1}^k \alpha_i \Phi^{-1}(F_i(T_i))$, where Φ denotes the standard normal distribution function. Under H_0 this statistic is normally distributed for any set of weights.

H. O. Lancaster [9] suggested another way to add weights to Fisher's test by transforming $1 - F_i(T_i)$ to gamma (or chi-square) distributed variates with possibly different parameter values. He also gave an approximate likelihood-ratio procedure for combining k identical tests against the same simple alternative and discussed asymptotic theory and weighting.

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pends on the continuity of F_i. H. O. Lancaster [8] and E. S. Pearson [11] have proposed methods to save the situation for discrete test statistics.

Notwithstanding these various developments, many statisticians tend to disregard the procedures outlined above as soon as the total number of observations on which the k test statistics are based is at all large. Relying on the asymptotic normality of many test statistics they prefer to use $\sum_{i=1}^{k} \alpha_i T_i$ to test H_0.

Apart from the work of Lancaster [9] and Liptak [10] the above-mentioned tests are obviously motivated by a desire to obtain a simple distribution in the null-case. The present paper constitutes an attempt to find combination methods that are optimal in some sense, regardless of possible difficulties in obtaining the distribution of the test statistic.

We complete this section by noting that the formulation of the combination problem given above restricts the parameter space to the set $\theta_i \geqq \theta_{i,0}$, $i = 1, 2, \cdots, k$. Since we shall only be concerned with the case where the T_i have distributions or asymptotic distributions of exponential type, H_0 may equally well be extended to $\theta_i \leqq \theta_{i,0}$, $i = 1, 2, \cdots, k$. However, the possibility that some of the $\theta_i - \theta_{i,0}$ should be positive and others negative is simply ruled out in advance. We believe that this is essential in the definition of the one-sided combination problem. The two-sided problem of testing $H_0:\theta_i = \theta_{i,0}$, $i = 1, 2, \cdots, k$, against $\theta_i \leqq \theta_{i,0}$, $i = 1, 2, \cdots, k$, or $\theta_i \geqq \theta_{i,0}$, $i = 1, 2, \cdots, k$, with inequality at least once, may be dealt with by applying two one-sided combination procedures. The entirely different problem of testing H_0 against $\theta_i \neq \theta_{i,0}$ at least once is not being discussed here.

2. Large sample combination. With many tests, especially distribution-free tests, the power is sufficiently intractable to defeat any attempt to find optimal combination methods for small samples. However, a number of the test statistics involved have asymptotic normal distributions. The following lemma describes the relation between the problem of finding asymptotically optimal combination procedures in this case and the small sample combination problem for normally distributed test statistics. By $N(\mu, \sigma^2)$ we denote normality with expectation μ and variance σ^2.

LEMMA 2.1. *Let $Z_{1,N}, \cdots, Z_{k,N}$ be independent and, for $N \to \infty$, let $Z_{i,N}$ be asymptotically $N(\mu_i, 1)$, $i = 1, 2, \cdots, k$. Furthermore, let $Z_1, \cdots, Z_k$ be independent, where Z_i is $N(\mu_i, 1)$, $i = 1, 2 \cdots, k$. Then, if $\psi(z_1, \cdots, z_k)$ is a measurable function that is monotonic in $z_1, \cdots, z_k$,*

$$\lim_{N\to\infty} P(\psi(Z_{1,N}, \cdots, Z_{k,N}) \leqq c) = P(\psi(Z_1, \cdots, Z_k) \leqq c),$$

uniformly for all ψ and c.

PROOF. Without loss of generality we may suppose ψ to be non-decreasing in each of its k arguments. Let $F_{i,N}$ and F_i denote the distribution functions of $Z_{i,N}$ and Z_i respectively. We define

$$s(z_2, \cdots, z_k) = \sup \{z \mid \psi(z, z_2, \cdots, z_k) \leqq c\}.$$

As ψ is measurable, so is s since

$$\{(z_2, \cdots, z_k) \mid s \leqq a\} = \bigcap_{z>a} \{(z_2, \cdots, z_k) \mid \psi(z, z_2, \cdots, z_k) > c\}$$

and the sets in the right-hand member are non-decreasing in z. Hence

$$\begin{aligned} 0 &= \lim_{N\to\infty} \int \cdots \int [F_{1,N}(s(z_2, \cdots, z_k) - 0) - F_1(s(z_2, \cdots, z_k))] \\ &\qquad \cdot dF_{2,N}(z_2) \cdots dF_{k,N}(z_k) \\ &\leqq \lim_{N\to\infty} [P(\psi(Z_{1,N}, \cdots, Z_{k,N}) \leqq c) - P(\psi(Z_1, Z_{2,N}, \cdots, Z_{k,N}) \leqq c)] \\ &\leqq \lim_{N\to\infty} \int \cdots \int [F_{1,N}(s(z_2, \cdots, z_k)) - F_1(s(z_2, \cdots, z_k))] \\ &\qquad \cdot dF_{2,N}(z_2) \cdots dF_{k,N}(z_k) \\ &= 0, \end{aligned}$$

uniformly in ψ and c, since the convergence $F_{1,N} \to F_1$ is uniform because of the continuity of F_1. Repeating this procedure we arrive in k steps at the result of the lemma.

The asymptotic combination problem we have in mind may be described as follows: For $N = 1, 2, \cdots$, let $T_{i,N}$, $i = 1, 2, \cdots, k$, denote k independent test statistics for the hypothesis $H_{i,0}: \theta_i = \theta_{i,0}$ against alternatives $\theta_i > \theta_{i,0}$. As $N \to \infty$ the sample sizes on which the $T_{i,N}$ are based increase indefinitely. We suppose that there exist positive numbers $\sigma_{i,N}$ and real-valued functions $\mu_{i,N}$ such that, if $\theta_{i,N}$ are the true parameter values of θ_i,

$$(T_{i,N} - \mu_{i,N}(\theta_{i,N}))/\sigma_{i,N}, \qquad i = 1, 2, \cdots, k,$$

tend in law to the standard normal distribution for $N \to \infty$ for every sequence $\theta_{i,N}$ having $\lim_{N\to\infty} \theta_{i,N} = \theta_{i,0}$, $i = 1, 2, \cdots, k$. On the basis of $T_{1,N}, \cdots, T_{k,N}$ we wish to test the combined hypothesis $H_0: \theta_i = \theta_{i,0}$, $i = 1, 2, \cdots, k$, against alternatives $H_1: \theta_i = \theta_{i,N}$, $i = 1, 2, \cdots, k$, satisfying

$$\lim_{N\to\infty} \theta_{i,N} = \theta_{i,0}, \qquad \lim_{N\to\infty} (\mu_{i,N}(\theta_{i,N}) - \mu_{i,N}(\theta_{i,0}))/\sigma_{i,N} = \mu_i \geqq 0,$$

$i = 1, 2, \cdots, k$, with $\mu_i > 0$ at least once.

Let

$$Z_{i,N} = (T_{i,N} - \mu_{i,N}(\theta_{i,0}))/\sigma_{i,N}, \qquad i = 1, 2, \cdots, k.$$

Obviously $Z_{i,N}$ is asymptotically $N(0, 1)$ under H_0 and asymptotically $N(\mu_i, 1)$ under H_1. Consider a monotonic combination procedure of limiting size α, i.e. a procedure: reject H_0 if

$$\psi(Z_{1,N}, \cdots, Z_{k,N}) \geqq c, \tag{2.1}$$

where ψ is monotonic in each of its k arguments separately and

$$\lim_{N\to\infty} \alpha_N = \lim_{N\to\infty} P(\psi(Z_{1,N}, \cdots, Z_{k,N}) \geqq c \mid \theta_{1,0}, \cdots, \theta_{k,0}) = \alpha.$$

As before, let $Z_1, \cdots, Z_k$ be independent and let Z_i be $N(\mu_i, 1)$. Consider the hypothesis $H_0^*: \mu_i = 0$, $i = 1, 2, \cdots, k$, and $H_1^*: \mu_i \geqq 0$, $i = 1, 2, \cdots, k$, with strict inequality at least once. Then, according to Lemma 2.1, the limiting power for $N \to \infty$ of the monotonic combination procedure (2.1) is equal to the power of the monotonic size-α combination procedure: reject H_0^* if $\psi(Z_1, \cdots, Z_k) \geqq c$ for testing H_0^* against H_1^*.

Suppose that we adopt some optimality criterion based on the power and that we can find an optimal combination procedure of size α for testing H_0^* against H_1^*. If this procedure is monotonic, then the procedure obtained by replacing Z_i by $Z_{i,N}$ is asymptotically optimal for testing H_0 against H_1 among all monotonic combination procedures of limiting size α for this problem.

It will become clear in the sequel that for any reasonable optimality criterion, the optimal procedure for testing H_0^* against H_1^* is necessarily monotonic. This follows from the fact that all other procedures are inadmissible. Still, for the asymptotic problem a non-monotonic procedure (or a sequence of such procedures) may be asymptotically better. This is easily verified by examples.

In practice, however, we shall simply ignore this possibility and apply an optimal combination procedure for testing H_0^* against H_1^* without hesitation to the corresponding large sample problem for testing H_0 against H_1. This may be motivated as follows: If, for each i and all limiting sizes α, the one-sided test: reject $H_{i,0}$ if $Z_{i,N} \geqq c_i$, is asymptotically most powerful among all tests based on $T_{i,N}$ for testing $H_{i,0}:\theta_i = \theta_{i,0}$ against $H_{i,1}:\theta_i = \theta_{i,N}$, where

$$\lim_{N\to\infty} \theta_{i,N} = \theta_{i,0}, \qquad \lim_{N\to\infty} (\mu_{i,N}(\theta_{i,N}) - \mu_{i,N}(\theta_{i,0}))/\sigma_{i,N} = \mu_i > 0,$$

then we can certainly not improve on large sample monotonic combination for testing H_0 against H_1 on the basis of $T_{1,N}, \cdots, T_{k,N}$. On the other hand, if the one-sided tests based on $T_{i,N}$, that formed the starting point of our investigation, perform poorly and better one-sided test statistics that are functions of $T_{i,N}$ are available, then we should not have started out on the combination of the $T_{i,N}$ in the first place. Thus the restriction to monotonic combination merely means that poor tests will give rise to poor combination procedures.

We note that this point of view coincides with that of Liptak [10] who requires monotonicity for any combination procedure. One might argue that non-monotonic procedures hardly deserve to be called combinations of the original one-sided tests based on $T_{i,N}$.

3. Combination against a simple alternative. In the small sample set-up of Section 1, let T_i possess a density $p_i(t, \theta_i)$ with respect to a σ-finite measure ν_i on R^1 for all values of θ_i, i.e.

$$P(T_i \leqq t \mid \theta_i) = \int_{-\infty}^{t} p_i(x, \theta_i)\, d\nu_i, \qquad i = 1, 2, \cdots, k.$$

We consider testing $H_0:\theta_i = \theta_{i,0}$, $i = 1, 2, \cdots, k$, against the simple alternative $H_1:\theta_i = \theta_{i,1}$, $i = 1, 2, \cdots, k$. If L_i denotes the logarithm of the likelihood ratio

$$L_i(t) = \log p_i(t, \theta_{i,1}) - \log p_i(t, \theta_{i,0}),$$

then, according to the Neyman-Pearson fundamental lemma, the most powerful test for H_0 against H_1 rejects H_0 if $\sum_{i=1}^{k} L_i(T_i) \geqq c$. If the densities $p_i(t, \theta_i)$

constitute one parameter exponential families $p_i(t, \theta_i) = C_i(\theta_i)e^{Q_i(\theta_i)t}$, $i = 1, 2, \cdots, k$, the most powerful test rejects H_0 if

$$\sum_{i=1}^{k} \{Q_i(\theta_{i,1}) - Q_i(\theta_{i,0})\}T_i \geqq c. \tag{3.1}$$

Hence in this case any given linear combination: reject H_0 if $\sum_{i=1}^{k} \alpha_i T_i \geqq c$, is most powerful against all alternatives $\theta_i = \theta_{i,1}$, $i = 1, 2, \cdots, k$, satisfying $Q_i(\theta_{i,1}) - Q_i(\theta_{i,0}) = \lambda\alpha_i, \lambda > 0$.

EXAMPLE 3.1. Consider k (2×2)-tables, $i = 1, 2, \cdots, k$.

	Success	Failure	Total
First series	A_i	C_i	m_i
Second series	B_i	D_i	n_i
Total	R_i	S_i	$m_i + n_i$

The conditional test for testing equality of the probabilities $p_{i,1}$ and $p_{i,2}$ of success in the first and second series of experiments against the alternative $p_{i,1} > p_{i,2}$ rejects for large values of A_i. If θ_i is defined by

$$\theta_i = p_{i,1}(1 - p_{i,2})/p_{i,2}(1 - p_{i,1}),$$

the conditional distribution of A_i constitutes an exponential family with respect to θ_i

$$P(A_i = a_i \mid R_i = r_i, S_i = s_i, \theta_i) = \binom{m_i}{a_i}\binom{n_i}{r_i - a_i}\theta_i^{a_i} \Big/ \sum_a \binom{m_i}{a}\binom{n_i}{r_i - a}\theta_i^{a},$$

where $Q_i(\theta_i) = \log \theta_i$. Hence the optimal conditional combination procedure for testing $H_0: \theta_i = 1$, $i = 1, 2, \cdots, k$, against a simple alternative $H_1: \theta_i = \theta_{i,1}$, $i = 1, 2, \cdots, k$, has test statistic

$$A = \sum_{i=1}^{k} \log \theta_{i,1} \cdot A_i. \tag{3.2}$$

The procedure remains optimal if the $\theta_{i,1}$ are changed in such a manner that the ratios of $\log \theta_{i,1}$ remain fixed.

However, in terms of $p_{i,1}$ and $p_{i,2}$ such a composite alternative seems rather hard to interpret and one would often prefer to test against alternatives like e.g.: $p_{i,1} - p_{i,2} = \epsilon_i$ for fixed $\epsilon_1, \epsilon_2, \cdots, \epsilon_k$. For $\epsilon_1 \to 0$ we have under this alternative

$$\log \theta_i = \epsilon_i/p_{i,1}(1 - p_{i,1}) + O(\epsilon_i^2).$$

For large sample sizes and small $\epsilon_1, \cdots, \epsilon_k$, $p_{i,1}$ may be approximated by $r_i/(m_i + n_i)$, and one may therefore expect that the test statistic

$$\sum_{i=1}^{k} [(m_i + n_i)^2/r_i s_i]\epsilon_i A_i \tag{3.3}$$

will be approximately optimal in this case. This rather dubious argument may be made rigorous by considering the asymptotic situation where $m_1, m_2, \cdots, m_k$, $n_1, n_2, \cdots, n_k$ tend to infinity, and ϵ_i tends to zero as fast as $(m_i + n_i)^{-1}$. One easily shows in this case, that the conditional procedure based on (3.3) is asymptotically equivalent to the optimal conditional procedure (3.2), except for sets of values r_i with probability tending to zero.

Similarly, for large samples and small ϵ_i the statistic $\sum_{i=1}^k [(m_i + n_i)/s_i]\epsilon_i A_i$ is approximately optimal against $p_{i,1}/p_{i,2} = 1 + \epsilon_i$ for fixed ϵ_i.

EXAMPLE 3.2. For $i = 1, 2, \cdots, k$, let $X_{i,1}, X_{i,2}, \cdots, X_{i,m_i}$ and $Y_{i,1}, Y_{i,2}, \cdots, Y_{i,n_i}$ be independent with continuous distribution functions F_i and G_i respectively, where $G_i(x) = F_i(x - \Delta_i)$. If U_i denotes the number of pairs $(X_{i,j}, Y_{i,j'})$ with $X_{i,j} < Y_{i,j'}$, $j = 1, 2, \cdots, m_i$, $j' = 1, 2, \cdots, n_i$, then Wilcoxon's two-sample test for $H_{i,0}:\Delta_i = 0$ against $\Delta_i > 0$, rejects $H_{i,0}$ if $U_i \geqq c$.

For each i, consider a sequence of such test statistics $U_{i,N}$, $N = 1, 2, \cdots$, based on $m_{i,N}$ and $n_{i,N}$ observations, where for $N \to \infty$, $m_{i,N}$ and $n_{i,N}$ tend to infinity in such a way that their ratio tends to a positive finite limit. If $\Delta_{i,N}$ are the true parameter values of Δ_i, and $\theta_{i,N} = \int F_i(x)\, dF_i(x - \Delta_{i,N})$, then $U_{i,N}$ is asymptotically $N(m_{i,N}n_{i,N}\theta_{i,N}, \frac{1}{12}m_{i,N}n_{i,N}(m_{i,N} + n_{i,N} + 1))$, whenever $\Delta_{i,N}$ tends to zero for $N \to \infty$. Combining the results of Section 2 and (3.1) we find that the asymptotically most powerful monotonic combination procedure for testing $H_0:\Delta_i = 0$, $i = 1, 2, \cdots, k$, against $H_1:\Delta_i = \Delta_{i,N}$, $i = 1, 2, \cdots, k$, where $\lim_{N\to\infty} \Delta_{i,N} = 0$ and $(m_{i,N} + n_{i,N})^{\frac{1}{2}}(\theta_{i,N} - \frac{1}{2})$ tend to finite limits, rejects H_0 if $\sum_{i=1}^k [(\theta_{i,N} - \frac{1}{2})/(m_{i,N} + n_{i,N} + 1)]U_{i,N} \geqq c_N$.

This combination procedure has been proposed by Ph. van Elteren [3]. For equal values of $\theta_1, \theta_2, \cdots, \theta_k$ under the alternative it reduces to what is called in [3] the locally best W-test with test statistic $\sum_{i=1}^k U_{i,N}/(m_{i,N} + n_{i,N} + 1)$. The reasoning leading to the designfree procedure that was also put forward in [3] does not apply to our problem since we restrict ourselves to one-sided alternatives where all $\Delta_i \geqq 0$ (or $\theta_i \geqq \frac{1}{2}$).

4. Decision theory for the normal case. With the asymptotic problem of Section 2 in mind, we consider independent random variables $T_1, T_2, \cdots, T_k$, where T_i is $N(\mu_i, 1)$ with $\mu_i \geqq 0$. We wish to test the hypothesis $H_0:\mu_i = 0$, $i = 1, 2, \cdots, k$, against $H_1:\mu_i \geqq 0$, $i = 1, 2, \cdots, k$, with strict inequality at least once. Much of what follows may, however, be extended to the case where the densities of T_i constitute one-parameter exponential families.

According to (3.1) the most powerful size-α test for H_0 against a simple alternative $(\mu_1, \mu_2, \cdots, \mu_k)$ rejects H_0 if

$$\sum_{i=1}^k \mu_i T_i \geqq \xi_\alpha\left(\sum_{i=1}^k \mu_i^2\right)^{\frac{1}{2}},$$

ξ_α denoting the upper α-point of the standard normal distribution. Hence the envelope power (i.e. the supremum over all size-α tests of the power at $(\mu_1, \mu_2, \cdots, \mu_k)$) is given by

$$\beta_\alpha^+(\mu_1, \cdots, \mu_k) = 1 - \Phi\left(\xi_\alpha - \left(\sum_{i=1}^k \mu_i^2\right)^{\frac{1}{2}}\right),$$

where Φ denotes the standard normal distribution function. For a size-α test with power $\beta(\mu_1, \cdots, \mu_k)$ we define the risk $R(\mu_1, \cdots, \mu_k)$ as the amount by which the actual power of the test falls short of the envelope power at a given alternative $(\mu_1, \mu_2, \cdots, \mu_k)$:

$$R(\mu_1, \cdots, \mu_k) = \beta_\alpha^+(\mu_1, \cdots, \mu_k) - \beta(\mu_1, \cdots, \mu_k).$$

Hence for a non-randomized test with acceptance region A

$$R(\mu_1, \cdots, \mu_k) = P((T_1, \cdots, T_k) \varepsilon A \mid \mu_1, \cdots, \mu_k) - \Phi(\xi_\alpha - (\textstyle\sum_{i=1}^k \mu_i^2)^{\frac{1}{2}}).$$

In order to interpret this risk as expected loss, the appropriate loss functions are $L_1(\mu_1, \cdots, \mu_k) = \beta_\alpha^+(\mu_1, \cdots, \mu_k) - 1$ and $L_2(\mu_1, \cdots, \mu_k) = \beta_\alpha^+(\mu_1, \cdots, \mu_k)$ when rejecting or accepting H_0 respectively. The fact that these losses depend on α is irrelevant since we discuss the decision problem for a fixed value of α.

Consider the size-α Bayes-test relative to a prior distribution P on the parameter space $\mu_i \geqq 0$, i.e. the size-α test that minimizes $\int R(\mu_1, \cdots, \mu_k)\, dP$, or equivalently the one that maximizes $\int \beta(\mu_1, \cdots, \mu_k)\, dP$. If P assigns probability 1 to the point $\mu_i = 0, i = 1, 2, \cdots, k$, then every test is Bayes. Among all other prior distributions we may as well restrict consideration to those that assign probability 1 to the alternative $\mu_i \geqq 0$, $\sum \mu_i > 0$, since $R(0, \cdots, 0) = 0$. According to the fundamental lemma, the size-α Bayes-test relative to such a prior distribution P is essentially (i.e. almost surely) unique and rejects H_0 if

$$(4.1) \quad \psi(t_1, \cdots, t_k) = \int \cdots \int \exp\,(\textstyle\sum_{i=1}^k \mu_i t_i) \exp\,(-\frac{1}{2} \sum_{i=1}^k \mu_i^2)\, dP \geqq c_\alpha .$$

Since all derivatives of ψ are non-negative, it follows that this combination procedure is monotonic and its acceptance region is convex. By a limiting argument one shows that the class of wide-sense Bayes-solutions coincides with the class of all monotone procedures with convex acceptance region.

It follows from their unicity that all (non-trivial) Bayes-solutions are admissible, i.e. their risk functions cannot uniformly be improved upon, if the improvement is to be strict at at least one point. Hence the wide-sense Bayes-solutions form a minimal complete class, i.e. the class of all admissible tests (cf. [14]). In this way one arrives at a result due to A. Birnbaum [2]:

LEMMA 4.1.[1] *A combination procedure for testing H_0 against H_1 is admissible if and only if its acceptance region A is convex and the procedure is monotonic* (i.e. $(t_1, t_2, \cdots, t_k) \varepsilon A$ *implies* $(t_1', t_2', \cdots, t_k') \varepsilon A$ *whenever* $t_i' \leqq t_i$ *for all* i).

We now prove a theorem on the behavior of an admissible risk function on a half-line through the origin. By a strongly increasing (decreasing) function we mean a function with positive (negative) derivative.

THEOREM 4.1. *Consider any admissible combination procedure and any fixed* $\lambda_i \geqq 0, i = 1, 2, \cdots, k$, *having* $\sum \lambda_i^2 = 1$, *with the exception of the cases mentioned below. Then*

$$f(r) = R(\lambda_1 r, \lambda_2 r, \cdots, \lambda_k r)$$

has a unique relative maximum on $(0, \infty)$ *that is also its unique absolute maximum. In fact* $f(r)$ *decreases strongly away from this maximum on both sides, vanishes for* $r = 0$ *and for* $r \to \infty$, *and has a negative second derivative at the maximum. The exceptions occur in the following two cases*:

(1) *The combination procedure rejects* H_0 *if* $\sum \lambda_i T_i \geqq \xi_\alpha$.

[1] In the formulation of results like this we shall identify procedures that are essentially identical.

(2) *The combination procedure does not involve* T_i *for those values of* i *for which* $\lambda_i > 0$.

PROOF. Let us first consider the exceptions to the theorem. The procedure in case (1) is the essentially unique most powerful size-α test against $(\lambda_1 r, \cdots, \lambda_k r)$ for every $r > 0$. Hence $f(r) \equiv 0$ on $(0, \infty)$ in case (1); since $f(r) \geqq 0$ on $(0, \infty)$, we have $f(r) > 0$ for all $r > 0$ in all other cases.

In case (2) the power of the procedure against $(\lambda_1 r, \cdots, \lambda_k r)$ does not depend on r. Since the envelope power is strongly increasing for $r \geqq 0$, the same holds for $f(r)$. If, on the other hand, $\lambda_{i_0} > 0$ and T_{i_0} is involved in an admissible procedure, then by Lemma 4.1 its acceptance region lies below a supporting hyperplane $\sum \nu_i t_i = c$, where $\nu_i \geqq 0$ for all i, $\sum \nu_i^2 = 1$, and $\nu_{i_0} > 0$. Therefore

$$0 \leqq f(r) \leqq P(\textstyle\sum_{i=1}^k \nu_i T_i \leqq c \mid \lambda_1 r, \cdots, \lambda_k r) - \Phi(\xi_\alpha - r)$$
$$= \Phi(c - r \textstyle\sum_{i=1}^k \lambda_i \nu_i) - \Phi(\xi_\alpha - r),$$

and hence $\lim_{r\to\infty} f(r) = 0$ in all cases but (2).

Disregarding the exceptions (1) and (2) for the remainder of the proof, we have found that $f(r) > 0$ for $r > 0$ and $f(r) \to 0$ for $r \to \infty$. Of course also $f(0) = 0$.

Consider an orthogonal transformation carrying $T_1, T_2, \cdots, T_k$ into $X_1, X_2, \cdots, X_k$, where $X_1 = \sum_{i=1}^k \lambda_i T_i$. Then $X_1, X_2, \cdots, X_k$ are independent and if $ET_i = \mu_i = \lambda_i r$, $i = 1, 2, \cdots, k$, then X_1 is $N(r, 1)$ and X_i is $N(0, 1)$ for $i = 2, 3, \cdots, k$. Let A denote the acceptance region of the admissible procedure of the lemma and let B be the transformed acceptance region in x-space. Consider two points $(x_1, x_2, \cdots, x_k)$ and $(x_1', x_2, \cdots, x_k)$ with $x_1' < x_1$ corresponding to points $(t_1, t_2, \cdots, t_k)$ and $(t_1', t_2', \cdots, t_k')$ respectively. If $(x_1, x_2, \cdots, x_k) \, \varepsilon \, B$ then $(t_1, t_2, \cdots, t_k) \, \varepsilon \, A$ and inverting the transformation we find

$$t_i - t_i' = \lambda_i(x_1 - x_1') \geqq 0.$$

Hence by Lemma 4.1 $(t_1', t_2', \cdots, t_k') \, \varepsilon \, A$ or $(x_1', x_2, \cdots, x_k) \, \varepsilon \, B$. It follows that if $B_x = \{(x_2, \cdots, x_k) | (x, x_2, \cdots, x_k) \, \varepsilon \, B\}$ denotes the section of B at $x_1 = x$, then the sets B_x are non-increasing in x. Hence the function $p(x) = P((X_2, \cdots, X_k) \, \varepsilon \, B_x)$, which, for $\mu_i = \lambda_i r$, is independent of r, is also non-increasing. Finally we note that for $\mu_i = \lambda_i r$ the envelope power is equal to the power of the test that rejects H_0 if $\sum \lambda_i T_i \geqq \xi_\alpha$, i.e. if $X_1 \geqq \xi_\alpha$. Therefore

$$f(r) = \int \{p(x) - I_{(-\infty,\xi_\alpha)}(x)\}\phi(x - r)\, dx, \tag{4.2}$$

where $I_{(-\infty,c)}$ denotes the characteristic function of $(-\infty, c)$ and ϕ the standard normal density.

As p is non-increasing and $0 \leqq p(x) \leqq 1$, it follows that for any positive constant a the function $p(x) - I_{(-\infty,\xi_\alpha)}(x) - a$ changes sign at most twice; if it does have two sign-changes, the signs occur in the order $(-, +, -)$ for increasing x. Furthermore f is certainly twice continuously differentiable and the differentiation may be carried out under the integral sign in (4.2); f cannot be identically equal to a constant since $f(0) = 0$ and $f(r) > 0$ for $r > 0$; $\phi(x - r)$ is strictly totally positive of order ∞ in x and r (cf. [6]). These conditions being satisfied, we find that for any $a > 0$

$$f(r) - a = \int \{p(x) - I_{(-\infty,\xi_\alpha)}(x) - a\}\phi(x - r)\,dx$$

has at most two zeros counting multiplicities (cf. [7]). As $f(r) > 0$ for $r > 0$ and $f(r)$ tends to zero for $r \to 0$ and for $r \to \infty$, the function has a unique relative (and absolute) maximum on $(0, \infty)$. A vanishing derivative at some point $0 < r_0 < \infty$ other than the maximum would produce at least one double and one single zero of $f(r) - f(r_0)$. Choosing for a the maximum value of the function, a vanishing second derivative at the maximum would produce a triple zero of $f(r) - a$. This completes the proof of the theorem.

From the class of all combination procedures of fixed size α we wish to select an optimal procedure on the basis of the risk function R. Lacking other reasonable criteria we shall try to determine a minimax risk procedure, i.e. a procedure that minimizes the supremum of R on the set $\mu_i \geqq 0, i = 1, 2, \cdots, k$. In the terminology of hypothesis testing such a procedure is called a most stringent (MS) size-α test. According to [14] such a MS procedure exists in our case and is wide-sense Bayes. The supremum of R of a size-α MS procedure is called the size-α minimax risk. As we have already remarked that the wide-sense Bayes-solutions constitute a minimal complete class, a MS procedure is admissible.

If P is a prior distribution on the set $\mu_i \geqq 0, i = 1, 2, \cdots, k$, then

$$R(P) = \inf \int \cdots \int R(\mu_1, \cdots, \mu_k)\,dP,$$

where the infimum is taken over all size-α procedures, denotes the Bayes-risk relative to P. Any prior distribution that maximizes $R(P)$ is called least favorable (LF) for the given size α. A prior distribution is LF for the given size α if and only if its Bayes-risk is equal to the size-α minimax risk. Equivalently, a prior distribution and its size-α Bayes-solution constitute a LF distribution and a MS procedure respectively for the given size α if and only if the prior distribution assigns probability 1 to the set of absolute maxima of the risk function of the Bayes-procedure. If a LF distribution exists, every size-α MS procedure is Bayes with respect to this prior distribution (cf. [14]).

In our case we have

LEMMA 4.2. *For any size α there exists a* LF *prior distribution and a unique* MS *combination procedure. This procedure is invariant under permutation of* $T_1, T_2, \cdots, T_k$.

PROOF. For any MS procedure we consider the randomized procedure that consists of employing each of the $k!$ procedures, that may be obtained from the MS procedure by permuting $T_1, T_2, \cdots, T_k$, with probability $(k!)^{-1}$. Since this randomized procedure is again MS, it is admissible and hence it must be essentially identical to a non-randomized procedure by Lemma 4.1. Every MS procedure must therefore be (essentially) invariant under permutation of $T_1, T_2, \cdots, T_k$.

As we know that a MS procedure exists, it follows that a procedure is MS if and only if it is MS relative to the class C of admissible and permutation invariant size-α procedures. For every procedure in C the point $t_i = a$, $i = 1, 2, \cdots, k$, where $\Phi(a) = (1 - \alpha)^{1/k}$ must lie either outside the acceptance region A or on its boundary. Otherwise, by Lemma 4.1, A would contain the set $t_i \leqq a + \epsilon, i = 1, 2, \cdots, k$, for some $\epsilon > 0$ and the size of the procedure would be smaller than α. Also the invariance under permutations together with Lemma 4.1 guarantees that A has a supporting hyperplane $\sum t_i = c$ and hence that for every procedure in C the acceptance region A is contained in the set $\sum t_i \leqq ka$. Therefore

$$R(\mu_1, \cdots, \mu_k) \leqq P(\textstyle\sum_{i=1}^k T_i \leqq ka \mid \mu_1, \cdots, \mu_k) - \Phi(\xi_\alpha - (\sum_{i=1}^k \mu_i^2)^{\frac{1}{2}})$$
$$= \Phi(k^{-\frac{1}{2}}(ka - \textstyle\sum_{i=1}^k \mu_i)) - \Phi(\xi_\alpha - (\sum_{i=1}^k \mu_i^2)^{\frac{1}{2}})$$

for every procedure in C.

Let R_0 denote the size-α minimax risk. Since $R_0 > 0$, it follows from the above inequality that there exists a number ρ such that for every procedure in C $R(\mu_1, \cdots, \mu_k) < \frac{1}{2}R_0$ whenever $\sum_{i=1}^k \mu_i^2 > \rho$, $\mu_i \geqq 0$. Hence for every procedure in C the risk function assumes its maximal value $\geqq R_0$ only on the set $\sum \mu_i^2 \leqq \rho$, $\mu_i \geqq 0$. Now consider the same decision problem for the case where the parameter space is reduced to the set $\sum \mu_i^2 \leqq \rho$, $\mu_i \geqq 0$. Obviously the size-α MS procedures for the new problem are the same as those for the original problem. Also the supremum of their risk functions remains R_0 in the new problem. However, as the parameter space is now compact, there exists a LF distribution P for the new problem. Since its Bayes-risk is equal to the minimax risk R_0 in the new problem, P must also be LF for the original problem. As every size-α MS procedure is Bayes relative to P the unicity of the MS procedure follows from the unicity of the Bayes-solutions. This completes the proof.

5. Combination of two tests. In the remainder of this report we shall specialize the setup of Section 4 to the case where $k = 2$. If S and T are independent $N(\mu, 1)$ and $N(\nu, 1)$ respectively with $\mu, \nu \geqq 0$, the problem is to test $H_0: \mu = \nu = 0$ against $H_1: \mu, \nu \geqq 0, \mu + \nu > 0$. We shall sometimes find it convenient to use polar coordinates in the parameter space and write

$$\mu = r \cos\theta, \qquad \nu = r \sin\theta.$$

By Lemma 4.1 a combination procedure is admissible if and only if its acceptance region is of the form $t \leqq a(s)$, where $a(s)$ is a non-increasing function that is concave on the interval where $a(s) > -\infty$ (as a boundary case we have $a(s) = \pm\infty$ for $s < \xi_\alpha$ and $s > \xi_\alpha$ respectively). A procedure is invariant under permutation of S and T iff its acceptance region is symmetric about the line $s = t$. Such a procedure will be called symmetric. Obviously the risk function of a symmetric procedure is symmetric about the line $\mu = \nu$. An admissible procedure with decreasing $a(s)$ is symmetric iff a is its own inverse

$$a(a(s)) \equiv s.$$

THEOREM 5.1. *For every admissible combination procedure the risk function has a finite number of absolute maxima.*

PROOF. We start by assuming that the procedure depends on both S and T and that it is not linear. Hence Theorem 4.1 holds on every half-line $r \geqq 0, \theta = \theta_0$, with $0 \leqq \theta_0 \leqq \frac{1}{2}\pi$. Let R^* denote the risk as a function of r and θ

$$(5.1)\qquad \begin{aligned} R^*(r, \theta) &= P(T \leqq a(S) \mid r, \theta) - \Phi(\xi_\alpha - r) \\ &= \int \Phi(a(s) - r \sin \theta)\phi(s - r \cos \theta)\, ds - \Phi(\xi_\alpha - r). \end{aligned}$$

Since ϕ and Φ are analytic, one easily verifies that R^* is analytic for $r \geqq 0$ and $0 \leqq \theta \leqq \frac{1}{2}\pi$. By Theorem 4.1 there exists a unique value $r(\theta) > 0$ for every $0 \leqq \theta \leqq \frac{1}{2}\pi$ such that

$$(5.2)\qquad \partial R^*(r, \theta)/\partial r \,|_{r=r(\theta)} = 0.$$

Also for every $0 \leqq \theta \leqq \frac{1}{2}\pi$

$$(5.3)\qquad \partial^2 R^*(r, \theta)/\partial r^2 \,|_{r=r(\theta)} < 0.$$

It follows from the implicit function theorem that $r(\theta)$ is analytic for $0 \leqq \theta \leqq \frac{1}{2}\pi$ and hence so is $g(\theta) = R^*(r(\theta), \theta)$.

From Theorem 4.1 we know that the absolute maxima of R^* lie on the curve $r = r(\theta)$. If R^* and hence g would have an infinite number of absolute maxima, $g(\theta)$ would be identically equal to a constant on $[0, \frac{1}{2}\pi]$ because of its analyticity. However, this is impossible since R^* has a local maximum at the boundary-point $\theta = 0, r = r(0)$, because of (5.2), (5.3) and

$$(5.4)\qquad \partial R^*(r, \theta)/\partial \theta \,|_{\theta=0} = -r \int \phi(a(s))\phi(s - r)\, ds < 0.$$

It remains to consider the exceptions to Theorem 4.1. If the procedure depends on both S and T but is linear, e.g. rejects H_0 if $\lambda_1 S + \lambda_2 T \geqq c, \lambda_1, \lambda_2 > 0$, then the conclusion of Theorem 4.1 continues to hold for every half-line $r \geqq 0$, $\theta = \theta_0$, with $0 \leqq \theta_0 \leqq \frac{1}{2}\pi, \theta_0 \neq \theta_1$, where $\text{tn}\, \theta_1 = \lambda_2/\lambda_1, 0 < \lambda_2/\lambda_1 < \infty$. Hence in this case we have analyticity of $r(\theta)$ and g on $[0, \theta_1)$ as well as on $(\theta_1, \frac{1}{2}\pi]$. The conclusion of the theorem then follows from (5.2), (5.3), (5.4) and

$$\begin{aligned} \partial R^*(r, \theta)/\partial \theta \,|_{\theta=\frac{1}{2}\pi} &= -r \int \Phi(a(s) - r)\phi(s)s\, ds \\ &= -r \int \phi(a(s) - r)\phi(s)a'(s)\, ds > 0, \end{aligned}$$

since $a'(s) \equiv -\lambda_1/\lambda_2 < 0$.

Finally, if the procedure does not depend on both S and T, e.g. rejects H_0 if $S \geqq \xi_\alpha$, then $R(\mu, \nu)$ is a strongly increasing function of ν for every value of $\mu \geqq 0$ and R does not possess any absolute maxima at all. This completes the proof of the theorem.

As a LF prior distribution assigns probability 1 to the set of absolute maxima of the risk function of the MS procedure, we have

COROLLARY 5.1. *For each α, every* LF *prior distribution assigns probability* 1 *to a finite pointset.*

Now let us, for a moment, restrict the parameter space to the half-lines $\mu = 0$, $\nu \geqq 0$, and $\nu = 0$, $\mu \geqq 0$. By the same reasoning as that of Lemma 4.2, there exists a LF prior distribution and a unique and symmetric MS procedure for every size α for the new problem. Since this MS procedure is admissible for the original problem and depends on both S and T because of its symmetry, its risk function has exactly one maximum on each of the half-lines $\mu = 0$ and $\nu = 0$ by Theorem 4.1. Also because of the symmetry of the procedure, this risk function is symmetric about the line $\mu = \nu$ and hence it assumes the same maxmum value on both half-lines $\mu = 0$ and $\nu = 0$ at points $\mu = 0, \nu = r$ and $\nu = 0, \mu = r$ respectively. It follows that for the new problem the LF distribution concentrates on the two points $(0, r)$ and $(r, 0)$ and hence by (4.1) the MS procedure for the new problem rejects H_0 if

$$pe^{rS} + (1-p)e^{rT} \geqq c', \qquad 0 \leqq p \leqq 1.$$

From the symmetry of the acceptance region we find that $p = \frac{1}{2}$, i.e. the LF distribution assigns probabilities $\frac{1}{2}$ to each of the points $(0, r)$ and $(r, 0)$. Because of the unicity of the MS procedure, the constants r and c', that depend on α, are uniquely determined by the requirements that the size of the procedure be equal to α and that its risk function assumes its maximum for $\mu = 0$ at $\nu = r$.

Returning to our original problem we consider the behavior of the risk function of the above procedure on the entire parameter space $\mu, \nu \geqq 0$. If this risk function assumes its absolute maximum anywhere on the boundary $\mu = 0$ (or $\nu = 0$) of the parameter space, then the above procedure is not only MS on the restricted parameter space $\mu = 0$ and $\nu = 0$, but also on the entire parameter space $\mu, \nu \geqq 0$. Hence we have proved (we find it convenient to replace c' by e^{rc})

THEOREM 5.2. *For each α there exists a unique size-α combination procedure that rejects H_0 if*

$$e^{r(\alpha)S} + e^{r(\alpha)T} \geqq e^{r(\alpha)c(\alpha)}$$

and for which $R(0, \nu)$ assumes its maximum at $\nu = r(\alpha)$. If, for a certain α, $R(0, r(\alpha))$ is also the maximum value of R on the entire parameter space $\mu, \nu \geqq 0$, then the procedure is MS *for this value of α.*

The usefulness of this theorem depends heavily on our ability to check whether the condition of the theorem is fulfilled for a given value of α. In this respect the following lemma will prove helpful.

LEMMA 5.1. *Consider an admissible and symmetric combination procedure for which $a(s)$ is continuously differentiable on the interval where $a(s) > -\infty$, and let s_0 denote the point where $a(s_0) = s_0$. If $g(s) = s + a(s)a'(s) \leqq 0$ on $(-\infty, s_0)$, then the risk function R of the procedure assumes its absolute maximum only on the boundary of the parameter space ($\mu = 0$ or $\nu = 0$). If $g(s)$ changes sign exactly once in the order $(-, +)$ for increasing s on $(-\infty, s_0)$, then R can attain its absolute maximum only on the boundary of the parameter space and on the half-line $\mu = \nu$.*

PROOF. Let

$$\lim_{s\to-\infty} a(s) = a \qquad \text{(finite or infinite)},$$

then by the symmetry of the procedure

$$\lim_{s\uparrow a} a(s) = -\infty.$$

As before, let $R^*(r, \theta)$ denote the risk as a function of the polar coordinates r and θ. We shall prove the lemma by studying the behavior of R^* for fixed $r > 0$ as a function of θ. Since the risk is symmetric about $\theta = \pi/4$ we restrict attention to the interval $0 \leqq \theta \leqq \pi/4$. According to (5.1) we have

$$\begin{aligned} R_\theta^*(r, \theta) &= (\partial/\partial\theta)R^*(r, \theta) \\ &= r \int_{-\infty}^{a} \{-\cos\theta\, \phi(a(s) - r\sin\theta)\phi(s - r\cos\theta) \\ &\quad +\sin\theta\, \Phi(a(s) - r\sin\theta)\phi'(s - r\cos\theta)\}\, ds \\ &= -r\int_{-\infty}^{a} \{\cos\theta + a'(s)\sin\theta\}\phi(a(s) - r\sin\theta)\phi(s - r\cos\theta)\, ds \\ &= -(re^{-\frac{1}{2}r^2}/2\pi)\int_{-\infty}^{a} \{\cos\theta + a'(s)\sin\theta\} \\ &\quad \cdot[\exp(r(s\cos\theta + a(s)\sin\theta))]\exp[-\tfrac{1}{2}(s^2 + a^2(s))]\, ds \\ &= -(e^{-\frac{1}{2}r^2}/2\pi)\int_{-\infty}^{a} \{s + a(s)a'(s)\} \\ &\quad \cdot[\exp(r(s\cos\theta + a(s)\sin\theta))]\exp[-\tfrac{1}{2}(s^2 + a^2(s))]\, ds \end{aligned}$$

by repeated partial integration. By substitution of $s = a(s')$ or $s' = a(s)$ we may change the integral from s_0 to a into an integral from $-\infty$ to s_0 and obtain

$$R_\theta^*(r, \theta) = \int_{-\infty}^{s_0} g(s)f_r(\theta, s)\, d\lambda_r(s), \tag{5.5}$$

where

$$g(s) = s + a'(s)a(s),$$

$$f_r(\theta, s) = [\exp(r(a(s)\cos\theta + s\sin\theta))] - [\exp(r(s\cos\theta + a(s)\sin\theta))],$$

and the measure λ_r is defined by

$$d\lambda_r(s) = (e^{-\frac{1}{2}r^2}/2\pi)\exp[-\tfrac{1}{2}(s^2 + a^2(s))]\, ds.$$

We proceed to study the function f_r for $0 < \theta < \pi/4$ and $s < s_0$. Since $a(s) > s$ for $s < s_0$ and $\cos\theta > \sin\theta$ for $0 < \theta < \pi/4$, we have

$$a(s)\cos\theta + s\sin\theta - s\cos\theta - a(s)\sin\theta = (a(s) - s)(\cos\theta - \sin\theta) > 0,$$

and hence $f_r > 0$. Furthermore consider the determinant

$$\begin{aligned} D &= \begin{vmatrix} f_r(\theta, s) & (\partial/\partial\theta)f_r(\theta, s) \\ (\partial/\partial s)f_r(\theta, s) & (\partial^2/\partial\theta\partial s)f_r(\theta, s) \end{vmatrix} \\ &= [\exp(r(a(s) + s)(\cos\theta + \sin\theta))][r^2(\cos^2\theta - \sin^2\theta)(a(s) - s)(a'(s) - 1) \end{aligned}$$

$$- r(\cos\theta - \sin\theta)(a'(s) + 1)]$$
$$+ [\exp(2r(a(s)\cos\theta + s\sin\theta))][-ra'(s)\sin\theta + r\cos\theta]$$
$$+ [\exp(2r(a(s)\sin\theta + s\cos\theta))][ra'(s)\cos\theta - r\sin\theta].$$

Let us denote the sum of the last two terms in this expression by D^* and consider the inequality $\alpha e^x - \beta e^{-x} > (\alpha - \beta) + (\alpha + \beta)x$, whenever $\alpha \geqq \beta$, $\alpha + \beta > 0$ and $x > 0$. We have

$$\begin{aligned} D^* = {} & [\exp(r(a(s) + s)(\cos\theta + \sin\theta))][r(-a'(s)\sin\theta + \cos\theta) \\ & \cdot[\exp(r(\cos\theta - \sin\theta)(a(s) - s))] \\ & - r(-a'(s)\cos\theta + \sin\theta)[\exp(-r(\cos\theta - \sin\theta)(a(s) - s))] \\ > {} & [\exp(r(a(s) + s)(\cos\theta + \sin\theta))] \\ & \cdot[r^2(\cos^2\theta - \sin^2\theta)(a(s) - s)(1 - a'(s)) + r(\cos\theta - \sin\theta)(a'(s) + 1)], \end{aligned}$$

since $a'(s_0) = -1$ because of the symmetry and hence $-1 \leqq a'(s) \leqq 0$ for $s < s_0$. It follows that $D > 0$ and hence that the function f_r is strictly totally positive of order 2 for $0 < \theta < \pi/4$ and $s < s_0$ (cf. [7]).

Returning to expression (5.5) we note that $g(s)$ cannot be identically zero for $s < s_0$ almost everywhere $[\lambda_r]$, since $2g(s)$ is the derivative of $s^2 + a^2(s)$ which tends to infinity for $s \to -\infty$. Therefore, if $g(s) \leqq 0$ on $(-\infty, s_0)$, we find that $R_\theta^*(r, \theta) < 0$ for all $r > 0$ and $0 < \theta < \pi/4$ because $f_r > 0$. Since R^* is symmetric about $\theta = \pi/4$ it can only have absolute maxima for $\theta = 0$ and $\theta = \frac{1}{2}\pi$.

Similarly, if $g(s)$ changes sign exactly once in the order $(-, +)$ for increasing s on $(-\infty, s_0)$, then expression (5.5) together with the strict total positivity of f_r ensures that for any $r > 0$, $R_\theta^*(r, \theta)$ has at most one zero for $0 < \theta < \pi/4$; if it does have one zero it changes sign at this zero in the order $(-, +)$ for increasing θ (cf. [7]). Hence for every $r > 0$, $R^*(r, \theta)$ has at most one minimum and no maximum for $0 < \theta < \pi/4$. Because of the symmetry of R^* about $\theta = \pi/4$ its absolute maxima can only occur for $\theta = 0$, $\theta = \pi/4$ and $\theta = \pi/2$, which completes the proof of the lemma.

A procedure that rejects H_0 if

$$e^{rS} + e^{rT} \geqq e^{rc}, \qquad r > 0,$$

will be called an exponential combination procedure with parameters r and c. We prove

THEOREM 5.3. *For any exponential combination procedure the risk function can assume its absolute maxima only on the half-lines $\mu = 0$, $\nu = 0$ and $\mu = \nu$. Moreover, if $rc \leqq 1 + \log 2$, this absolute maximum can only be attained on the half-lines $\mu = 0$ and $\nu = 0$.*

PROOF. For an exponential procedure we have for $-\infty < s < c$

$$a(s) = r^{-1}\log(e^{rc} - e^{rs})$$
$$g(s) = s + a(s)a'(s) = s - (e^{rs}/r(e^{rc} - e^{rs}))\log(e^{rc} - e^{rs}).$$

The point s_0 where $a(s_0) = s_0$ is given by $s_0 = c - r^{-1}\log 2$.

To study the sign-changes of g on $(-\infty, s_0)$ we set $x = e^{rs}$, $b = e^{ra}$, and consider the function

$$h(x) = r(e^{rc} - e^{rs})g(s) = (b - x) \log x - x \log (b - x)$$

for $0 < x < e^{rs_0} = \frac{1}{2}b$. We have

$$\lim_{x\to 0} h(x) = -\infty, \qquad h(\tfrac{1}{2}b) = 0,$$

$$h'(x) = -\log x - \log (b - x) + (b - x)/x + x/(b - x),$$

$$\lim_{x\to 0} h'(x) = +\infty, \qquad h'(\tfrac{1}{2}b) = 2(1 - \log b/2),$$

$$h''(x) = (1/(b - x) - 1/x) + b(1/(b - x)^2 - 1/x^2) < 0$$

for $0 < x < \frac{1}{2}b$.

If $rc \leqq 1 + \log 2$, i.e. $b \leqq 2e$, then $h'(\frac{1}{2}b) \geqq 0$ and since h' is decreasing, it is positive on $(0, \frac{1}{2}b)$. Hence h is negative on $(0, \frac{1}{2}b)$ and so is g on $(-\infty, s_0)$.

If $rc > 1 + \log 2$, i.e. $b > 2e$, then $h'(\frac{1}{2}b) < 0$ and since h' is decreasing, it changes sign exactly once on $(0, \frac{1}{2}b)$ in the order $(+, -)$ for increasing x. Hence h has one maximum and no minimum on $(0, \frac{1}{2}b)$. It follows that h changes sign exactly once on $(0, \frac{1}{2}b)$ in the order $(-, +)$ for increasing x, and so does g on $(-\infty, s_0)$ for increasing s.

Application of Lemma 5.1 completes the proof.

Combining Theorems 5.2 and 5.3 we have

COROLLARY 5.2. *For a given size α the exponential combination procedure of Theorem* 5.2 *is* MS *if and only if one of the following conditions is satisfied:*

(1) $r(\alpha)c(\alpha) \leqq 1 + \log 2$,

(2) *the maximum risk of the procedure on the half-line $\mu = \nu$ does not exceed that on the half-line $\mu = 0$.*

Corollary 5.2 admits at least a partial solution to the problem of finding the size-α MS procedure. By varying r and c for a given size α it is fairly simple to determine numerically the exponential procedure of Theorem 5.2 for which the risk assumes its (unique) maximum for $\mu = 0$ at $\nu = r$. Once $r(\alpha)$ and $c(\alpha)$ have been determined, the validity of conditions (1) or (2) is easily checked. At most the computations involve the determination of the (unique) maximum of the risk function for $\mu = \nu$.

It turns out that condition (1) is of little practical interest since it covers only large values of α. For $\alpha \geqq 0.75$ the acceptance region of any exponential procedure can not include the origin as an interior point, since it would then strictly contain the set $s, t \leqq 0$ that has probability 0.25 under H_0. Therefore, for $\alpha \geqq 0.75$, $e^{rc} \leqq 2$ for any size-α exponential procedure and hence in particular $r(\alpha)c(\alpha) \leqq \log 2$ and the procedure of Theorem 5.2 is MS. Of course the estimate involved is rather rough and it turns out that the procedure of Theorem 5.2 has $r(\alpha)c(\alpha) = \log 2$ for $\alpha \approx 0.60$ and reaches the point where $r(\alpha)c(\alpha) = 1 + \log 2$ only for $\alpha \approx 0.24$.

Below this point the validity of condition (1) for $r(\alpha)$ and $c(\alpha)$ seems to end and we have to rely on condition (2). For $\alpha = 0.10$ and $\alpha = 0.05$ the procedures of Theorem 5.2 still satisfy condition (2) and we find that the MS combination procedures reject H_0 if

$$e^{1.635S} + e^{1.635T} \geqq 16.52 \quad \text{for} \quad \alpha = 0.10, \tag{5.6}$$

$$e^{1.900S} + e^{1.900T} \geqq 44.47 \quad \text{for} \quad \alpha = 0.05. \tag{5.7}$$

The point where the risk function of the procedure of Theorem 5.2 assumes equal maxima on the half-lines $\mu = \nu$ and $\mu = 0$ is reached for $\alpha = \alpha_0 \approx 0.043$. Although we have proved no such result, numerical evidence strongly suggests that the procedure of Theorem 5.2 is MS for all $\alpha \geqq \alpha_0 \approx 0.043$.

For $\alpha < \alpha_0$ the situation becomes more complicated. We conjecture that a LF prior distribution that is symmetric about the half-line $\mu = \nu$ will continue to exist and that for values of α slightly below α_0 it will assign positive probability to three points $(\mu(\alpha), 0)$, $(0, \mu(\alpha))$, $(\mu^*(\alpha), \mu^*(\alpha))$ in the (μ, ν)-plane. By (4.1) the MS procedure would then reject H_0 if

$$e^{\mu(\alpha)S} + e^{\mu(\alpha)T} + \lambda(\alpha)e^{\mu^*(\alpha)(S+T)} \geqq c^*(\alpha),$$

where $\lambda(\alpha)$, $\mu(\alpha)$, $\mu^*(\alpha) > 0$. As α decreases further towards zero the LF distribution will supposedly concentrate on an indefinitely increasing (but finite) number of points. As a result, the number of terms involved in the test statistic of the MS procedure would also increase indefinitely for $\alpha \to 0$, and the task of determining the MS procedure would rapidly become hopeless.

Obviously, what remains to be done is to find an asymptotically good procedure for $\alpha \to 0$. To this end we consider the likelihood ratio (LR) test for the hypothesis $H_0: \mu = \nu = 0$ against the composite alternative $H_1: \mu, \nu \geqq 0, \mu + \nu > 0$. One easily verifies that the size-α LR procedure rejects H_0 if

$$S^2 I_{(0,\infty)}(S) + T^2 I_{(0,\infty)}(T) \geqq \rho_\alpha^2, \tag{5.8}$$

where $I_{(0,\infty)}$ denotes the characteristic function of the set $(0, \infty)$ and $\rho_\alpha > 0$. We note that these LR procedures have size $\alpha < \frac{3}{4}$ since the set $s, t \leqq 0$ is always strictly contained in the acceptance region A_α of the procedure. The region A_α is bounded by the quarter-circle $s^2 + t^2 = \rho_\alpha^2$ in the first quadrant and by the half-lines $t = \rho_\alpha$ and $s = \rho_\alpha$ in the second and fourth quadrants respectively. It follows from Lemma 4.1 that the LR procedures are admissible; however, these procedures are not (strict-sense) Bayes, since one easily shows from (4.1) that the acceptance region of a Bayes procedure is either $s < \xi_\alpha$, or $t < \xi_\alpha$, or $t < \alpha(s)$, where $a(s)$ is strongly decreasing.

The risk function of the size-α LR procedure is given by

$$R_\alpha(\mu, \nu) = \Phi(\rho_\alpha - \nu)\Phi(-\mu) + \int_0^{\rho_\alpha} \Phi((\rho_\alpha^2 - s^2)^{\frac{1}{2}} - \nu)\phi(s - \mu)\, ds - \Phi(\xi_\alpha - (\mu^2 + \nu^2)^{\frac{1}{2}}). \tag{5.9}$$

Substituting $\mu = \nu = 0$ we find that ρ_α is determined by the relation

$$\Phi(\rho_\alpha) - \tfrac{1}{4}e^{-\frac{1}{2}\rho_\alpha^2} = 1 - \alpha = \Phi(\xi_\alpha). \tag{5.10}$$

If the acceptance region A_α is written in the form $t < a_\alpha(s)$, then for $s < 2^{-\frac{1}{2}}\rho_\alpha$,

$s^2 + a_\alpha^2(s)$ is obviously non-increasing and $s + a_\alpha(s)a_\alpha'(s) \leqq 0$. Hence by Lemma 5.1, R_α assumes its absolute maximum only on the half-lines $\mu = 0$ and $\nu = 0$. Let $\mu_{0,\alpha}$ denote the unique value of μ for which $R_\alpha(\mu, 0)$ assumes its maximum (cf. Theorem 4.1). Then $R_\alpha(\mu, \nu) \leqq R_\alpha(\mu_{0,\alpha}, 0)$ for all $\mu, \nu \geqq 0$, and since the second term in the right-hand member of (5.9) is smaller than $\Phi(\rho_\alpha - \mu) - \Phi(-\mu)$

$$(5.11) \quad R_\alpha(\mu, \nu) \leqq \{\Phi(\rho_\alpha) - 1\}\Phi(-\mu_{0,\alpha}) + \Phi(\rho_\alpha - \mu_{0,\alpha}) - \Phi(\xi_\alpha - \mu_{0,\alpha})$$

for all $\mu, \nu \geqq 0$.

Now $\rho_\alpha > \xi_\alpha$, and as α tends to zero, both ρ_α and ξ_α tend to infinity. Moreover, as

$$\Phi(x) = 1 - x^{-1}\phi(x) + O(x^{-3}\phi(x)) \quad \text{for} \quad x \to \infty,$$

we have from (5.10)

$$\tfrac{1}{4}e^{-\frac{1}{2}\rho_\alpha^2} + O(\rho_\alpha^{-1}\phi(\rho_\alpha)) = (2\pi)^{-\frac{1}{2}}\xi_\alpha^{-1}e^{-\frac{1}{2}\xi_\alpha^2} + O(\xi_\alpha^{-3}\phi(\xi_\alpha)),$$

or, taking logarithms,

$$\tfrac{1}{2}\rho_\alpha^2 = \tfrac{1}{2}\xi_\alpha^2 + \log \xi_\alpha + O(1).$$

It follows that

$$\rho_\alpha = \xi_\alpha + \xi_\alpha^{-1}\log \xi_\alpha + O(\xi_\alpha^{-1}) \quad \text{for} \quad \alpha \to 0,$$

and hence in particular $\lim_{\alpha\to 0}(\rho_\alpha - \xi_\alpha) = 0$. Combining this with (5.11) we obtain

$$(5.12) \qquad \lim_{\alpha\to 0} R_\alpha(\mu, \nu) = 0 \quad \text{uniformly for all} \quad \mu, \nu \geqq 0.$$

Though property (5.12) is obviously a desirable one, it remains to be seen what other families of combination procedures besides the LR procedures possess this property. We proceed to show that, in a sense to be made precise below, any family of admissible procedures that satisfies (5.12) approaches to the LR procedures for $\alpha \to 0$.

Consider an arbitrary family of admissible procedures with acceptance regions $\tilde{A}_\alpha$ $(0 < \alpha < 1)$, where the procedure characterized by $\tilde{A}_\alpha$ has size α and risk function $\tilde{R}_\alpha$. If ρ and η denote polar coordinates in the (s, t)-plane,

$$s = \rho \cos \eta, \qquad t = \rho \sin \eta,$$

the acceptance region $\tilde{A}_\alpha$ may be written as $\rho < \tilde{b}_\alpha(\eta)$. We note that in the special case where $\tilde{A}_\alpha = A_\alpha$ we have $\tilde{b}_\alpha(\eta) = \rho_\alpha$ for $0 \leqq \eta \leqq \pi/2$, where $\rho_\alpha \sim \xi_\alpha$ for $\alpha \to 0$.

THEOREM 5.4.

$$\lim_{\alpha\to 0} \sup_{\mu\geqq 0,\nu\geqq 0} \tilde{R}_\alpha(\mu, \nu) = 0 \text{ iff } \lim_{\alpha\to 0} \sup_{0\leqq\eta\leqq\pi/2} |\tilde{b}_\alpha(\eta) - \xi_\alpha| = 0.$$

PROOF. We start by remarking that $\tilde{b}_\alpha(\eta) \geqq \xi_\alpha$ for all α and η, since otherwise there would exist a line of support of $\tilde{A}_\alpha$ at a distance from the origin smaller than ξ_α, and as a result the procedure corresponding to $\tilde{A}_\alpha$ would have a size $> \alpha$. We proceed to prove the "only if" assertion of the theorem.

Let δ be an arbitrary positive number,

$$\epsilon = \tfrac{1}{8}P(S^2 + T^2 \leqq \tfrac{1}{4}\delta^2 \mid \mu = \nu = 0),$$

and let $p > \frac{1}{2}\delta$ be so large that

$$P(S^2 + T^2 \geqq p^2 \mid \mu = \nu = 0) < \epsilon.$$

Furthermore let α_0 be so small that for all $\alpha < \alpha_0$ we have

(1) $\rho_\alpha - \xi_\alpha < \frac{1}{2}\delta$;

(2) $\rho_\alpha > \{\frac{1}{2}\delta + (p^2 - \frac{1}{4}\delta^2)^{\frac{1}{2}}\}\{1 - p^{-1}(p^2 - \frac{1}{4}\delta^2)^{\frac{1}{2}}\}^{-1}$.

Suppose that for some $\alpha_1 < \alpha_0$ and $0 \leqq \eta_1 \leqq \frac{1}{2}\pi$, $\tilde{b}_{\alpha_1}(\eta_1) - \xi_{\alpha_1} = d > \delta$, and hence, because of (1), $\tilde{b}_{\alpha_1}(\eta_1) - \rho_{\alpha_1} = d_1 > \frac{1}{2}\delta$. Returning to the cartesian coordinate system in the (s, t)-plane, let L_1 be the line through the origin at an angle η_1 to the positive s-axis, and let P_1 and $P_2 = (s_2, t_2)$ denote the points of intersection of L_1 with the boundaries of A_{α_1} and $\tilde{A}_{\alpha_1}$ respectively. Define the region G_p by

$$G_p = \{(s, t) \mid (s - s_2)^2 + (t - t_2)^2 < p^2\}.$$

We shall show that the boundaries of A_{α_1} and $\tilde{A}_{\alpha_1}$ have no common points in the set G_p. Suppose to the contrary that such a point would exist, say P_3. We note that this would imply that $P_1 \,\varepsilon\, G_p$ or that $d_1 < p$. Denote the line through P_2 and P_3 by L_2 and let ζ be the positive acute angle between L_2 and the line of support of A_{α_1} at P_1. Let L_3 be the line through the origin orthogonal to L_2 and let P_4 be the point of intersection of L_2 and L_3. Then $\overline{OP_4} = (\rho_{\alpha_1} + d_1) \cdot \cos\zeta < (\rho_{\alpha_1} + p)\cos\zeta$, where O denotes the origin. Since

$$\sin\zeta = d_1/\overline{P_2P_3} > \delta/2p,$$

we have by (2) and (1)

$$\overline{OP_4} < (\rho_{\alpha_1} + p)(1 - (\delta/2p)^2)^{\frac{1}{2}} < \rho_{\alpha_1} - \tfrac{1}{2}\delta < \xi_{\alpha_1}.$$

Since P_2, P_3 and P_4 are collinear and P_3 is situated between P_2 and P_4, P_4 lies outside $\tilde{A}_{\alpha_1}$ or on its boundary. This follows from the fact that $\tilde{A}_{\alpha_1}$ is convex and that P_2 and P_3 are boundary points. But this contradicts $\overline{OP_4} \leqq \xi_{\alpha_1}$ (cf. the remark at the beginning of the proof) and hence the assertion that A_{α_1} and $\tilde{A}_{\alpha_1}$ have no common boundary points in G_p is proved.

Hence $(G_p \cap A_{\alpha_1}) \subset (G_p \cap \tilde{A}_{\alpha_1})$, and $(G_p \cap \tilde{A}_{\alpha_1}) - (G_p \cap A_{\alpha_1})$ contains a circle sector C_{d_1} of a circle with centre P_2, radius d_1 and extending over an angle $\frac{1}{2}\pi$. Taking $\mu_0 = s_2$, $\nu_0 = t_2$, it follows from the definitions of ϵ, p and G_p that

$$\tilde{R}_{\alpha_1}(\mu_0, \nu_0) - R_{\alpha_1}(\mu_0, \nu_0) > P((S, T) \,\varepsilon\, C_{d_1} \mid \mu_0, \nu_0) - \epsilon > \epsilon.$$

Hence $\tilde{R}_{\alpha_1}(\mu_0, \nu_0) > \epsilon$, which proves the "only if" assertion of the theorem.

To prove the converse, suppose to the contrary that $\tilde{b}_\alpha(\eta) - \xi_\alpha$, and hence $\tilde{b}_\alpha(\eta) - \rho_\alpha$, converges uniformly to zero on $[0, \frac{1}{2}\pi]$, and that sequences $\{\alpha_i\}$

and $\{\mu_i, \nu_i\}$ exist such that $\lim_{i\to\infty} \alpha_i = 0$, $\mu_i, \nu_i \geqq 0$ and

$$\tilde{R}_{\alpha_i}(\mu_i, \nu_i) > \epsilon \quad \text{for} \quad i = 1, 2, \cdots, \tag{5.13}$$

where ϵ is a positive number. Define $d > 0$ by

$$P(S^2 + T^2 > d \mid \mu = \nu = 0) = \tfrac{1}{4}\epsilon,$$

and let

$$C_i = \{(s, t) \mid (s - \mu_i)^2 + (t - \nu_i)^2 \leqq d\}, \qquad i = 1, 2, \cdots.$$

Furthermore, let D_i be the intersection of C_i with the symmetric difference of A_{α_i} and $\tilde{A}_{\alpha_i}$.

The uniform convergence of $\tilde{b}_\alpha(\eta) - \rho_\alpha$ on $[0, \frac{1}{2}\pi]$ also ensures the uniform convergence of the boundary of $\tilde{A}_\alpha$ to the boundary of A_α in strips with width d along the s- and t-axes outside the first quadrant. This may be shown by the same line of argument that we used in the first part of the proof to show that G_p contained no common boundary points of A_{α_1} and $\tilde{A}_{\alpha_1}$. Hence

$$\lim_{i\to\infty} \lambda(D_i) = 0,$$

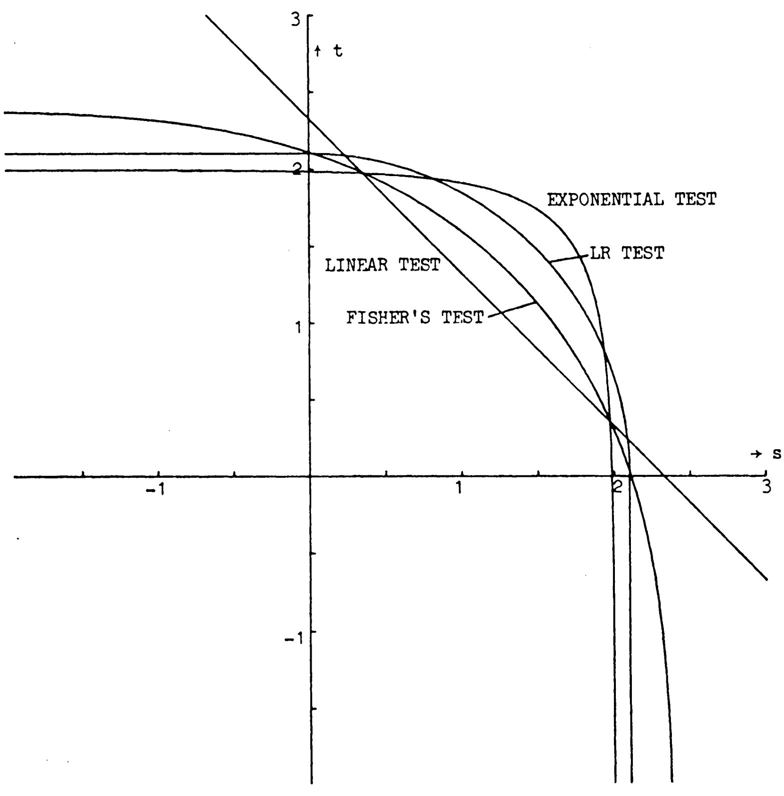

FIG. 5.1. Boundaries of the acceptance regions of 4 symmetric tests; size $\alpha = .05$.

where λ denotes Lebesgue-measure, and consequently for all sufficiently large i

$$|R_{\alpha_i}(\mu_i, \nu_i) - \tilde{R}_{\alpha_i}(\mu_i, \nu_i)| < \tfrac{1}{2}\epsilon .$$

Since by (5.12) $\lim_{i\to\infty} R_{\alpha_i}(\mu_i, \nu_i) = 0$, this contradicts (5.13), which completes the proof of the theorem.

It may be of interest to remark that Fisher's omnibus combination procedure, that rejects H_0 for large values of

$$-\log(1 - \Phi(S)) - \log(1 - \Phi(T)), \tag{5.14}$$

satisfies the convergence criterion for $\tilde{b}_\alpha$ in Theorem 5.4. As a result, for $\alpha \to 0$, it shares the property of uniformly vanishing risk of the LR procedure. The exponential combination procedure of Theorem 5.2, however, does not enjoy this property. For $\alpha \to 0$ its maximum risk tends to 1 as it approaches Tippett's procedure that rejects H_0 for large values of max (S, T). The additional fact that this limiting risk 1 is reached on every half-line through the origin except $\mu = 0$ and $\nu = 0$ makes exponential combination most unsatisfactory for very small values of α.

This unsatisfactory behavior for $\alpha \to 0$ is of course due to the fact that the exponential procedure of Theorem 5.2 is Bayes relative to prior distributions that remain concentrated on a bounded number of points as α tends to zero.

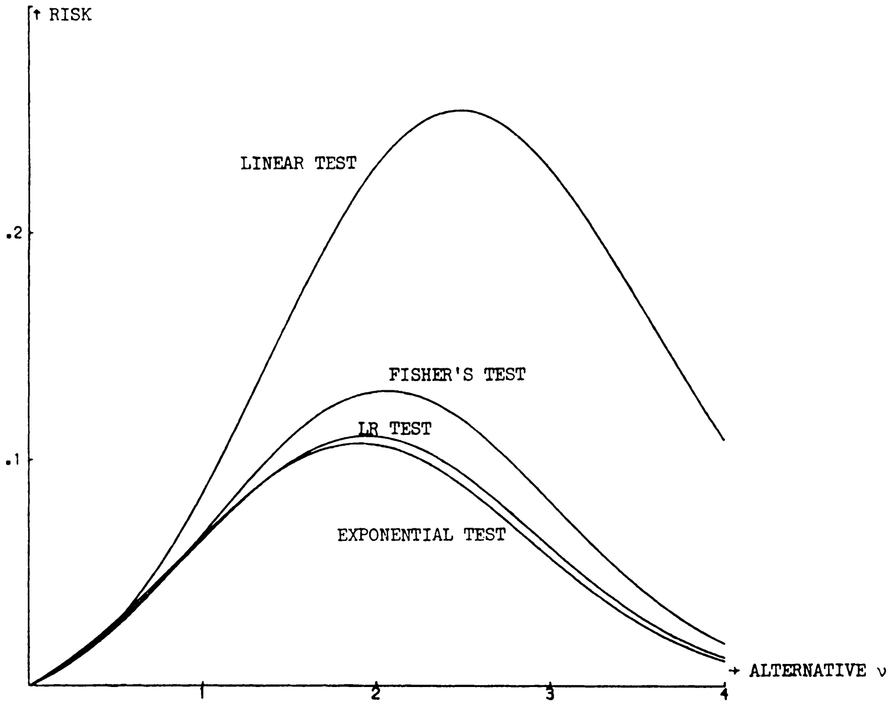

FIG. 5.2. Risk functions of 4 symmetric tests on the half-line $\mu = 0$, $\nu \geqq 0$; size $\alpha = .05$.

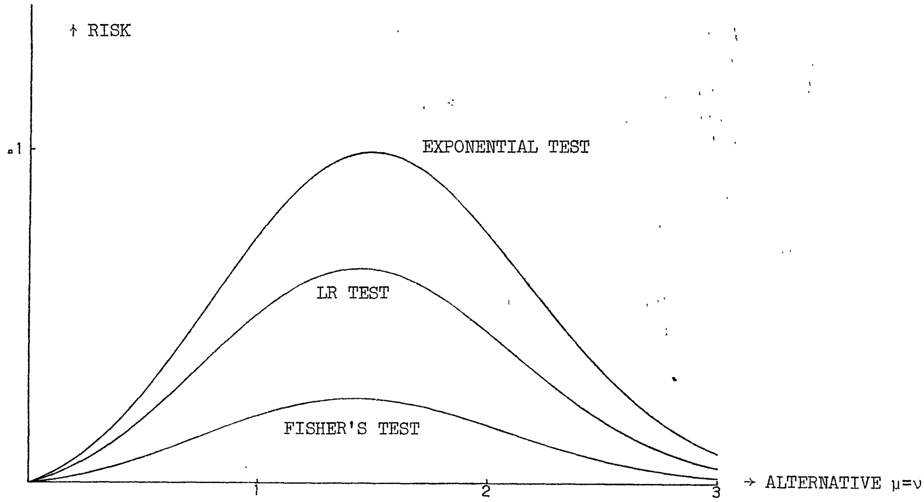

FIG. 5.3. Risk functions of 3 symmetric tests on the half-line $\mu = \nu \geqq 0$; size $\alpha = .05$.

A similar case is therefore afforded by linear combination. For $\alpha \to 0$ the procedure that rejects H_0 for large values of

$$(5.15) \qquad \lambda_1 S + \lambda_2 T, \qquad \lambda_1, \lambda_2 \geqq 0,$$

has limiting maximum risk 1, that is reached on every half-line through the origin but $\lambda_2\mu - \lambda_1\nu = 0$. The proofs of the above remarks will be omitted here.

To conclude this paper we give some numerical results that provide some indication of the performance of several procedures discussed in this paper for the time-honoured value of $\alpha = 0.05$. The following procedures have been included:

(1) Exponential combination (5.7), which is the MS procedure for $\alpha = 0.05$;
(2) Fisher's combination procedure (5.14);
(3) Likelihood-ratio (LR) procedure (5.8);
(4) Linear combination (5.15) with $\lambda_1 = \lambda_2$, which is MS among all linear procedures because of its symmetry.

For these four symmetric procedures and $\alpha = 0.05$ Figure 5.1 shows the boundary of the acceptance region. Figures 5.2 and 5.3 show the risk of these procedures on the half-lines $\mu = 0$ and $\mu = \nu$ respectively. For linear combination (4) the risk for $\mu = \nu$ is not shown since it is identically equal to zero.

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