NOTE ON THE INFINITE DIVISIBILITY OF EXPONENTIAL MIXTURES

By F. W. Steutel

Technische Hogeschool Twente, Enschede, Netherlands

1. Introduction. In one-counter waiting-time theory the Lindley case (cf. [2]) yields infinitely divisible stationary waiting-time distributions. In connection with this in the discussion to a paper by Kingman (cf. [4]) Runnenburg conjectured that the product of two independent exponentially distributed random variables is infinitely divisible. Goldie [1] proved that the product of two independent non-negative random variables is infinitely divisible if one of the two is exponentially distributed or, equivalently, that mixtures (with positive weights) of exponential random variables are infinitely divisible. In this note a slightly more general theorem is proved by a completely different method.

2. Definitions. We consider probability density functions (pdf's), which are mixtures of exponential pdf's, i.e. functions of the form

$$f(x) = \sum_{j=1}^{n} p_j \lambda_j e^{-\lambda_j x}, \tag{1}$$

where $p_j \neq 0$, $\sum_{j=1}^{n} p_j = 1$ and $\lambda_j > 0$. Without restriction we assume that the λ's are ordered in the following way

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n . \tag{2}$$

As $f(x)$ has to be non-negative it follows, by letting $x \to 0$ and $x \to \infty$ respectively, that $\sum_1^n p_j \lambda_j \geqq 0$ and $p_1 > 0$. This however is not sufficient for $f(x)$ to be positive as may be seen from the function $e^{-x} - 8e^{-2x} + 12e^{-3x}$, which is negative for $\log 2 < x < \log 6$. Simple sufficient conditions including negative p_j do not seem to be available.

The characteristic function (c.f.) corresponding to a pdf $f(x)$ will be denoted by $\phi(t)$. For definition and properties of infinitely divisible (inf div) c.f.'s we refer to Lukacs [3]. When convenient the pdf and the random variable corresponding to an inf div c.f. will also be called inf div.

3. A theorem.

Lemma 1. *A* c.f. $\phi(t)$ *is* inf div *if* $\log \phi(t)$ *can be expressed in the form*

$$\log \phi(t) = ita + \int_0^\infty [e^{itx} - 1 - itx/(1 + x^2)] g(x)\, dx, \tag{3}$$

where a is real, $g(x) \geqq 0$ and $[x^2/(1 + x^2)]g(x)$ integrable on $(0, \infty)$.

Proof. This is a special case of the Lévy-Khinchine representation (see [3]).

Lemma 2. *If $f(x) = \lambda e^{-\lambda x}$ then $\phi(t) = \lambda/(\lambda - it)$ and in the notation of Lemma* 1

$$a = \int_0^\infty (1 + x^2)^{-1} e^{-\lambda x}\, dx, \qquad g(x) = x^{-1} e^{-\lambda x}.$$

Proof. See [3].

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THEOREM. *A* pdf *of the form* (1) *and satisfying* (2) *is* inf div *if in the sequence* $p_1, p_2, \cdots, p_n$ *there is no more than one change of sign.*

PROOF. The c.f. corresponding to (1) is

$$\phi(t) = \textstyle\sum_{j=1}^{n} p_j\lambda_j/(\lambda_j - it) = P(t)/\prod_{j=1}^{n}(\lambda_j - it), \tag{4}$$

where $P(t)$ is a polynomial of a degree not exceeding $n - 1$. It follows that $\phi(t)$ cannot have more than $n - 1$ zeros. Putting $t = -i\mu$ (μ real) we see from (4) that $\phi(-i\mu)$ has at least one zero between every pair of (simple) poles λ_j and λ_{j+1} if p_j and p_{j+1} are of the same sign. If all p_j have the same sign this accounts for all $n - 1$ zeros. If there is one change in sign then $n - 2$ zeros are accounted for. If in the latter case $\sum p_j\lambda_j = 0$ then $P(t)$ is of degree $n - 2$ and no more zeros exist. If $\sum p_j\lambda_j > 0$ then $\phi(-i\mu)$ is negative for large positive values of μ and therefore there is another (the $n - 1$st) zero on (λ_n, ∞).

From the foregoing considerations it follows that $P(t)$ only has purely maginary zeros $t_k = i\mu_k$ with $\mu_k > 0$. The c.f. $\phi(t)$ can now be written in the orm

$$\phi(t) = \textstyle\prod_{j=1}^{n}[\lambda_j/(\lambda_j - it)]\cdot\prod_{k=1}^{n'}(\mu_k - it)/\mu_k, \tag{5}$$

where n' is either $n - 1$ or $n - 2$ and where the μ_k satisfy the inequalities

$$\mu_k > \lambda_k \qquad (k = 1, 2, \cdots, n'). \tag{6}$$

From (5) and Lemma 2 it now follows that log $\phi(t)$ has a representation of the form (3), where

$$xg(x) = \textstyle\sum_{j=1}^{n} e^{-\lambda_j x} - \sum_{k=1}^{n'} e^{-\mu_k x} > 0$$

ibecause of (6). This completes the proof.

REMARK. Letting $\lambda_n \to \infty$ in (4) we obtain a c.f. of the form $\psi(t) = \sum_1^{n-1} p_j\lambda_j/(\lambda_j - it) + p_n$ corresponding to a distribution function of the form $\sum_1^{n-1} p_j(1 - e^{-\lambda_j x}) + p_n H(x)$, where $H(x)$ denotes the unit-step function. It follows that p_n must be positive, i.e. there is no change of sign. Following the proof of the theorem we now find the $n - 1$st zero of $\psi(-i\mu)$ for $\mu > \lambda_{n-1}$: $\lim_{\mu\downarrow\lambda_{n-1}}\psi(-i\mu) = -\infty$, $\lim_{\mu\to\infty}\psi(-i\mu) = p_n > 0$. Therefore $\psi(t)$ is also inf div.

COROLLARY 1. *If X is exponentially distributed and if Y is non-negative and independent of X then XY is* inf div.

PROOF. For the c.f. of XY we have

$$\phi(t) = \lambda\textstyle\int_0^\infty\int_0^\infty e^{itxy}e^{-\lambda x}\,dx dF(y) = \int_0^\infty \lambda/(\lambda - ity)\,dF(y),$$

where $F(y)$ denotes the distribution function of Y. Apparently $\phi(t)$ is an infinite mixture of exponential c.f.'s. This can be obtained as a limit of finite mixtures by taking the last integral as a limit of Darboux sums. As every c.f. which is a limit of inf div c.f.'s is itself inf div ([3], page 82) this completes the proof.

This corollary was obtained by Goldie [1] as a corollary to a rather different theorem, which seems to be more general in some respects and less general in others. It does not, for instance, include the infinite divisibility of exponential mixtures with negative p_j.

REMARK. Y may be zero with positive probability (compare Remark following the theorem).

COROLLARY 2. *All exponential mixtures of two components are* inf div.

PROOF. This is the particular case $n = 2$ of the theorem.

4. Counterexamples. One might be tempted to conjecture that perhaps all exponential mixtures as given by (1) are inf div. It follows from the theorem that the simplest counterexamples should be looked for in the class of three-component mixtures where $p_1 > 0$, $p_2 < 0$ and $p_3 > 0$. Such an example is provided by the pdf

$$f(x) = 2e^{-x} - 6e^{-3x} + 5e^{-5x},$$

which is seen to be positive by putting $e^{-2x} = y$. This yields $e^x f(x) = 5y^2 - 6y + 2$.

The corresponding c.f.

$$\phi(t) = 2/(1 - it) - 6/(3 - it) + 5/(5 - it)$$
$$= (15 - t^2)/(1 - it)(3 - it)(5 - it)$$

has real zeros and is therefore not inf div.

Finally we give an example of a c.f. in the same class which *is* inf div:

$$f(x) = \tfrac{1}{4}e^{-x} - e^{-2x} + \tfrac{15}{4}e^{-3x},$$
$$\phi(t) = \tfrac{1}{4}(1/(1 - it) - 4/(2 - it) + 15/(3 - it))$$
$$= \tfrac{1}{2}(4 - 3it)(3 - 2it)/(1 - it)(2 - it)(3 - it).$$

For the function $g(x)$ (see (3)) we have in this case

$$xg(x) = e^{-x} + e^{-2x} + e^{-3x} - e^{-(4/3)x} - e^{-(3/2)x} > 0$$

as is not difficult to prove.

REFERENCES

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