A NOTE ON THE UNIMODALITY OF DISTRIBUTION FUNCTIONS OF CLASS L

By Tze-Chien Sun

Wayne State University

1. Let $X_1, X_2, \cdots, X_n, \cdots$ denote a sequence of independent random variables. Set

$$\zeta_n = B_n^{-1} \sum_{k=1}^{n} X_k - A_n, \qquad n = 1, 2, \cdots,$$

where $B_n > 0$ and A_n are some constants. Let $F_n(x)$ be the distribution functions of ζ_n for $n = 1, 2, \cdots$.

We say that a distribution function F *belongs to the class* L ([2], p. 145) if there is a sequence of independent random variables X_n such that

(i) for suitably chosen B_n and A_n, the distribution functions F_n converge weakly to F;

(ii) the random variables $\xi_{nk} = X_k/B_n$ are asymptotically constant.

We say a distribution function F is *unimodal* ([2], p. 157) if there exists at least one value $x = a$ such that $F(x)$ is convex for $x < a$ and concave for $x > a$.

A number of people have been interested in the following problem:

Are all the distribution functions belonging to the class L unimodal?

First, B. V. Gnedenko gave a proof to show that the answer to the problem was positive. Later, K. L. Chung pointed out that Gnedenko's proof, based on a wrong theorem of A. I. Lapin, was not valid (see [1] and the Appendix II in [2]). The problem thus remained open.

In 1957, I. A. Ibragimov gave an example of non-unimodal probability distributions of class L ([3]). It appeared that the problem was finally settled on the negative side. However, in this note, we shall point out that Ibragimov's proof is not valid either and moreover we shall show that the distribution functions given in [3] are in fact all unimodal. Therefore, this problem again remains open.

2. In [3], the following theorem was stated:

Among the distributions of class L, the logarithm of whose characteristic function is given by Lévy's formula by the functions

$$\begin{aligned} M(u) &\equiv 0, \\ N(u) &= \lambda \log u, \qquad u \leqq 1, \\ &= 0, \qquad\qquad u > 1, \end{aligned}$$

with $\lambda \varepsilon [\frac{1}{2}, \frac{3}{2}]$, *are non-unimodal ones.*

Let $f_\lambda(t)$ denote the characteristic function corresponding to λ. We find

$$\log f_\lambda(t) = i\gamma t + \lambda \int_0^t (e^{iu} - 1)/u \, du.$$

We shall assume that $\gamma = 0$.

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Let $F_\lambda(x)$ and $y_\lambda(x)$ denote the distribution function and the density function of $f_\lambda(t)$ respectively.

Ibragimov's proof proceeds as follows:

(1) $|f_\lambda(t)| = O(|t|^{-\lambda})$ as $t \to \pm\infty$.

(2) Suppose $F_\lambda(x)$, $\lambda \in [\frac{1}{2}, 1]$, are unimodal, then the mode of $F_\lambda(x)$ is at zero for each $\frac{1}{2} \leqq \lambda \leqq 1$.

(3) When $\lambda > 1$, $y_\lambda(x) = 0$ for $x < 0$.

(4) When $\lambda > 1$, $xy_\lambda'(x) = (\lambda - 1)y_\lambda(x) - \lambda y_\lambda(x - 1)$, $x \neq 0$.

(5) Suppose $F_\lambda(x)$, $\lambda \in [\frac{1}{2}, \frac{3}{2}]$, are unimodal. Let a_λ denote the mode of $F_\lambda(x)$. Since, as $\lambda \to 1^+$, $f_\lambda(t) \to f_1(t)$, $F_\lambda(x) \Rightarrow F_1(x)$, we have $a_\lambda \to a_1 = 0$ (from part (2)). So there is $a_\lambda < \frac{1}{2}$ for some $\lambda > 1$. By part (3) and part (4), we have $a_\lambda y_\lambda'(a_\lambda) = (\lambda - 1)y_\lambda(a_\lambda)$. Since $y_\lambda'(a_\lambda) = 0$, it follows that $y_\lambda(a_\lambda) = 0$, $y_\lambda(x) \equiv 0$ and this is impossible.

This argument is correct until part (5). The statement in part (5), "$a_\lambda \to a_1 = 0$ as $\lambda \to 1^+$" is incorrect. Although he proved $a_1 = 0$ in part (2), he did not prove that $F_1(x)$ has a unique mode. If $F_1(x)$ has more than one mode, a_λ does not necessarily converge to the mode at zero. In fact, we can see from the following that all points in [0, 1] are modes of $F_1(x)$ and $a_\lambda \to 1$ as $\lambda \to 1^+$.

3. We shall prove the following:

The distribution functions of class L, the logarithm of whose characteristic function is given by Lévy's formula by the functions

$$M(u) \equiv 0,$$

$$N(u) = \lambda \log u, \qquad u \leqq 1,$$

$$= 0, \qquad u > 1,$$

with $\lambda \in [\epsilon, \epsilon^{-1}]$, *for any* $0 < \epsilon < 1$, *are all unimodal.*

PROOF. Recall that $f_\lambda(t) = \exp\{\lambda \int_0^t (e^{iu} - 1)/u \, du\}$. The result in part (1) of [3] remains to be true here,

$$\text{(a)} \qquad |f_\lambda(t)| = O(|t|^{-\lambda}), \qquad \lambda \in [\epsilon, \epsilon^{-1}], \qquad \text{as} \quad \lambda \to \pm\infty.$$

By the inversion formula,

$$F_\lambda(x) - F_\lambda(0) = (2\pi)^{-1} \int_{-\infty}^{\infty} [(e^{-itx} - 1)/-it] f_\lambda(t) \, dt.$$

Note that the above integral is absolutely integrable, so $F_\lambda(x)$ is continuous. For $x > 0$, set $tx = s$. Then

$$\begin{aligned} F_\lambda(x) - F_\lambda(0) &= (2\pi)^{-1} \int_{-\infty}^{\infty} [(e^{-is} - 1)/-is] f_\lambda(s/x) \, ds \\ &= (2\pi)^{-1} \int_{-\infty}^{\infty} [(e^{-is} - 1)/-is] \exp\{\lambda \int_0^{s/x} (e^{iu} - 1)/u \, du\} \, ds. \end{aligned}$$

Differentiating both sides with respect to x, we have

$$\begin{aligned} y_\lambda(x) &= (2\pi)^{-1} \int_{-\infty}^{\infty} [(e^{-is} - 1)/-is] f_\lambda(s/x) \lambda (1 - e^{is/x})/x \, ds \\ &= \lambda (2\pi)^{-1} \int_{-\infty}^{\infty} [(e^{-itx} - 1)/-itx] f_\lambda(t) (1 - e^{it}) \, dt \\ &= (\lambda/x)[F_\lambda(x) - F_\lambda(0) - (F_\lambda(x - 1) - F_\lambda(-1))] \\ &= (\lambda/x) \{[F_\lambda(x) - F_\lambda(x - 1)] - [F_\lambda(0) - F_\lambda(-1)]\}, \quad x > 0. \end{aligned}$$

Differentiation under the integral sign is legitimate here because the last integral converges uniformly in $x \varepsilon [\delta, \infty)$ for any $\delta > 0$. The same formula can be obtained for $x < 0$. Thus

(b) $y_\lambda(x) = (\lambda/x)\{[F_\lambda(x) - F_\lambda(x-1)] - [F_\lambda(0) - F_\lambda(-1)]\}, \quad x \neq 0.$

First we note that $y_\lambda(x)$ is continuous at all $x \neq 0$. Also it is easily seen from (b) that $F_\lambda(0) - F_\lambda(-1) = 0$, otherwise we would have negative values for $y_\lambda(x)$ for sufficiently large x. Therefore, (b) becomes

$$y_\lambda(x) = (\lambda/x)[F_\lambda(x) - F_\lambda(x-1)], \qquad x \neq 0.$$

Since $F_\lambda(x) - F_\lambda(x-1) \geqq 0$, and $y_\lambda(x) \geqq 0$, we have $y_\lambda(x) \equiv 0$ for $x < 0$. Consider

(c) $$y_\lambda(x) = (\lambda/x)[F_\lambda(x) - F_\lambda(x-1)] \quad \text{for} \quad x > 0.$$

Differentiating (c), we have

(d) $$xy_\lambda'(x) = (\lambda - 1)y_\lambda(x) - \lambda y_\lambda(x-1), \qquad x > 0.$$

We see from (d) that $y_\lambda'(x)$ is continuous at all x except at $x = 0$ and $x = 1$.

CASE 1. $\lambda \leqq 1$.

From (d), we have $y_\lambda'(x) \leqq 0$ for all $x > 0$. So $y_\lambda(x)$ is monotonically decreasing for $x > 0$. It follows $F_\lambda(x)$ is unimodal. Actually, by integrating (d) for $0 < x < 1$, we have $y_\lambda(x) = cx^{\lambda-1}$ where c is a non-negative constant. $c > 0$ otherwise $y_\lambda(x)$ would be zero for all x. Thus, for $\lambda < 1$, $F_\lambda(x)$ has a unique mode at zero and for $\lambda = 1$, all points in $[0, 1]$ are modes of $F_1(x)$.

CASE 2. $\lambda > 1$.

For $0 < x < 1$, again, we have $y_\lambda(x) = cx^{\lambda-1}$, $c > 0$. Since $y_\lambda(x)$ now is continuous at $x = 0$, we see that $y_\lambda'(x)$ is continuous for all $x \neq 0$. Note that $y_\lambda'(x) > 0$ for $0 < x < 1$, also $y_\lambda(x) \to 0$ as $x \to \infty$. It follows $y_\lambda(x)$ has at least one local maximum in $1 < x < \infty$.

Let x_1 be the smallest point where $y_\lambda(x)$ achieves maximum. Then

(i) $x_1 > 1$;

(ii) $y_\lambda(x)$ is strictly increasing in $0 \leqq x \leqq x_1$,

(iii) $y_\lambda(x_1)/y_\lambda(x_1 - 1) = \lambda/(\lambda - 1)$ by (d).

Suppose $y_\lambda(x)$ has at least a local minimum in $x_1 < x < \infty$. Let x_2 be the smallest such point. Then

(iv) $x_2 > x_1$;

(v) $y_\lambda(x)$ is strictly decreasing in $x_1 \leqq x \leqq x_2$;

(vi) $y_\lambda(x_2)/y_\lambda(x_2 - 1) = \lambda/(\lambda - 1) > 1$.

(v) and (vi) imply $0 < x_2 - 1 < x_1$. But $x_2 - 1 > x_1 - 1 > 0$, by (ii), $y_\lambda(x_2 - 1) > y_\lambda(x_1 - 1)$. By (iii) and (vi), we then have

$$y_\lambda(x_2) > y_\lambda(x_1).$$

This is impossible. Hence $y_\lambda(x)$ is strictly decreasing in $x_1 \leqq x < \infty$. Therefore $y_\lambda(x)$ has a unique mode at x_1. Q.E.D.

REFERENCES

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