THE DISTRIBUTION FUNCTIONS OF TSAO'S TRUNCATED SMIRNOV STATISTICS[1]

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1. Introduction and summary. Tsao (1954) defined two new statistics and thereby established two methods of using a truncated Smirnov test. This paper gives the distribution functions for a special case of those statistics.

Let $X_1 < X_2 < \cdots < X_n$ represent an ordered random sample from the continuous distribution function $F(x)$, with the empirical cumulative distribution function $S_n(x) = k/n$ if $X_k \leqq x < X_{k+1}$, where $X_0 = -\infty$ and $X_{n+1} = \infty$. Let $Y_1 < Y_2 < \cdots < Y_m$ represent an ordered random sample from the continuous distribution function $G(x)$, with the empirical cumulative distribution function $S_m'(x)$. As test statistics for testing $H_0:F(x) \equiv G(x)$ against $H_1:F(x) \not\equiv G(x)$, Tsao (1954) proposed

$$d_r = \max_{x \leqq X_r} |S_n(x) - S_m'(x)|, \qquad r \leqq n,$$

and

$$d_r' = \max_{x \leqq \max(X_r, Y_r)} |S_n(x) - S_m'(x)|, \qquad r \leqq \min(m, n).$$

It seems natural to consider also the test statistic

$$d_r'' = \max_{x \leqq \min(X_r, Y_r)} |S_n(x) - S_m'(x)|, \qquad r \leqq \min(m, n)$$

Tsao described a counting procedure to obtain the probabilities associated with the distribution functions of d_r and d_r', and illustrated this procedure in the relatively simple case where $m = n$. Tables were compiled using the procedure for various values of r and $m(= n)$.

In this paper the asymptotic distributions of $N^{\frac{1}{2}} d_r$, $N^{\frac{1}{2}} d_r'$, and $N^{\frac{1}{2}} d_r''$ are given, where $N = mn/(m + n)$. Also, for $m = n$, the exact closed form of the distribution functions of d_r, d_r', and d_r'' are derived under the null hypothesis. Also shown are the relationships

$$P(d_r \leqq x) = \tfrac{1}{2}P(d_r' \leqq x) + \tfrac{1}{2}P(d_r'' \leqq x);$$

$$P(d_r'' \leqq x) = P(d_{r-c}' \leqq x), \qquad \text{for } c < r, \text{ where } c = [nx],$$

$$= 1, \text{ for } c \geqq r,$$

and therefore

$$P(d_r \leqq x) = \tfrac{1}{2}P(d_r' \leqq x) + \tfrac{1}{2}P(d_{r-c}' \leqq x), \qquad \text{for } c < r,$$

$$= \tfrac{1}{2}P(d_r' \leqq x) + \tfrac{1}{2}, \text{ for } c \geqq r,$$

illustrating that tables for $P(d_r \leqq x)$ and $P(d_r'' \leqq x)$ are superfluous while tables for $P(d_r' \leqq x)$ exist.

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Epstein (1955) compared the power of Tsao's d_r' with three other nonparametric statistics on the basis of 200 pairs of random samples of size 10 drawn from tables of normal random numbers. Rao, Savage, and Sobel (1960) considered d_r' as a special case in the general scheme of censored rank order statistics.

2. Preliminaries. It is useful to establish the following identity: Let $(A)_j$ denote the falling factorial $A(A-1)(A-2)\cdots(A-j+1)$.

LEMMA 1.

$$(2.1)\quad \begin{aligned}(s+t)!(u+1)[v!(v-u-1)!]^{-1}\textstyle\sum_{j=0}^{s-v-1}(-1)^j/j!(s-v-1-j)!\\ (v+1+j)(v-u+j)(t+v+1+j)_{t+u+1-v}\\ =\textstyle\sum_{j=0}^{s-v-1}\binom{s+t-2v+u-1-j}{t+u-v}\left[\binom{2v-u+j}{v}-\binom{2v-u+j}{v-u-1}\right]\\ = g(s,t,u,v),\ say.\end{aligned}$$

PROOF. Consider the identity

$$(2.2)\quad \begin{aligned}\textstyle\sum_{j=0}^{s-v-1}(-1)^j z^{v-u-1+j}y^{v+j}x^{2v-u+j}[j!(s-v-1-j)!]^{-1}\\ = z^{v-u-1}y^v x^{2v-u}(1-xyz)^{s-v-1}[(s-v-1)!]^{-1}\end{aligned}$$

which is easily obtained using the binomial expansion. Integrate both sides of (2.2) first with respect to z from 0 to 1, then with respect to y from 0 to 1, and successively with respect to x from 0 to x_1, x_1 from 0 to x_2, $\cdots$, x_{t+u-v} from 0 to 1. These $t+u-v+3$ integrations, and multiplication by $(s+t)!(u+1)/v!(v-u-1)!$, readily transform the left side of (2.2) to the left side of (2.1). The right side of (2.2) is not so easily transformed. However, if the identity

$$(2.3)\quad \int_0^a w^b(1-cw)^d\,dw = \sum_{i=0}^d a^{b+1+i}(1-ca)^{d-i}(d)_i c^i/(b+1+i)_{i+1}$$

(obtained through successive integration by parts) is used for each integration, the right side of (2.2) is transformed into an expression involving $t+u-v+3$ summations. Because the final integration is from 0 to 1, the final summation involves only one nonzero term, and the previous $t+u-v$ summations merely indicate that each nonzero term from the first two summations is repeated $\binom{s+t+u-2v-\alpha-\beta}{t+u-v+1}$ times, where α and β are the indices of the first and second summations, respectively. Thus the sucessive integrations result in the expression

$$\begin{aligned}\sum_{\alpha=0}^{s-v-1}\sum_{\beta=0}^{s-v-1-\alpha}\binom{s+t+u-2v-\alpha-\beta}{t+u-v+1}(v-u-1)!(v+\alpha)!(2v-u+\alpha+\beta)!\\ [(v-u+\alpha)!(v+1+\alpha+\beta)!(s+t)!]^{-1}\end{aligned}$$

which, by letting $\beta = j-\alpha$, rearranging the order of summation, applying the identity

$$(2.4)\quad \sum_{\alpha=0}^{j}\binom{v+\alpha}{u} = \binom{v+1+j}{u+1} - \binom{u}{u+1}$$

from Feller (1957), p. 61, and multiplying by $(s+t)!\ (u+1)/v!(v-u-1)!$, becomes the right side of (2.1), and the proof is complete.

LEMMA 2. *Let $X_1 < \cdots < X_s$ and $Y_1 < \cdots < Y_t$ be ordered random samples* *from the same continuous distribution function* $F(x)$. *Then*

$$(2.5)\quad P(Y_1 < X_{u+1},\ Y_2 < X_{u+2},\ \cdots,\ Y_{v-u} < X_v) = 1 - \left[\binom{s+t}{s-u-1} - g(s, t, u, v)\right]\binom{s+t}{s}^{-1}$$

for integers u *and* v, $0 \leqq u < v \leqq s$, *and* $v - u \leqq t$; *and where* $g(s, t, u, v)$ *is given by* (2.1).

Proof. For simplicity of expression, let $w_i = F(x_i)$, and $z_i = F(y_i)$. Then the probability expressed in (2.5) is given by the integral

$$\int_0^1 \int_0^{z_t} \cdots \int_0^{z_{v-u+2}} \int_0^{z_{v-u+1}} \int_{z_{v-u}}^1 \int_{z_{v-u}}^{w_s} \cdots \int_{z_{v-u}}^{w_{v+2}} \int_{z_{v-u}}^{w_{v+1}} \int_0^{z_{v-u}} \cdots \int_0^{z_3} \int_{z_2}^{w_{u+3}} \int_0^{z_2} \int_{z_1}^{w_{u+2}} \int_0^{w_{u+1}} \cdots \int_0^{w_3} \int_0^{w_2} s!t!\, dw_1\, dw_2 \cdots dw_u\, dw_{u+1}\, dz_1\, dw_{u+2}\, dz_2 \cdots dz_{v-u-1}\, dw_v\, dw_{v+1} \cdots dw_{s-1}\, dw_s\, dz_{v-u}\, dz_{v-u+1} \cdots dz_{t-1}\, dz_t .$$

The integrals with respect to $w_1, w_2, \cdots, w_u, w_{u+1}, z_1, w_{u+2}, z_2, \cdots, z_{v-u-1}$, w_v are straightforward, and result in the integrand

$$s!\,t!\,w_{v+1}^{v} z_{v-u}^{v-u-1}[v!\,(v - u - 1)!]^{-1} - s!\,t!\,z_{v-u}^{v} w_{v+1}^{v-u-1}[v!\,(v - u - 1)!]^{-1}.$$

By induction, and with the identity

$$(2.6)\qquad \textstyle\sum_{\alpha=0}^{k} (-1)^{k-\alpha}[\alpha!\,(k - \alpha)!\,(A - \alpha)]^{-1} = (A)_{k+1}^{-1}$$

from Conover (1965), it can be shown that the integral of w_i for $i = v, v + 1, \cdots, s$, gives the integrand

$$s!\,t!\,w_{i+1}^{i} z_{v-u}^{v-u-1}[i!\,(v - u - 1)!]^{-1} - s!\,t!\,z_{v-u}^{v} w_{i+1}^{i-u-1}[v!\,(i - u - 1)!]^{-1} + \textstyle\sum_{j=0}^{i-v-1} s!\,t!\,(u + 1)(-1)^{j} z_{v-u}^{2v-u+j} w_{i+1}^{i-v-1-j} \cdot [v!\,(v - u - 1)!\,(v + 1 + j)(v - u + j)j!\,(i - v - 1 - j)!]^{-1}$$

where the convention $w_{s+1} = 1$ is used. The integrand of interest is the one where $i = s$. The remaining integrals are with respect to $z_{v-u}, z_{v-u+1}, \cdots, z_{t-1}, z_t$ and result directly in the right side of (2.5), completing the proof.

Consider a system of paths in the (x, y) plane, where each path originates at $(0, 0)$, terminates at (s, t), and is composed of unit increments in the positive x or positive y direction. Let $f(s, t, u, v)$ equal the number of paths from $(0, 0)$ to (s, t) that touch the line $x = y + u + 1$ on or before touching the line $x = v$.

Lemma 3.

$$(2.7)\qquad f(s, t, u, v) = \binom{s+t}{s-u-1} - g(s, t, u, v)$$

for integers u *and* v, $0 \leqq u < v \leqq s$, $v - u \leqq t$, *where* $g(s, t, u, v)$ *is given by* (2.1).

Proof. Let $X_1 < \cdots < X_s$ and $Y_1 < \cdots < Y_t$ be ordered random samples from the same continuous distribution function $F(x)$, and let $Z_1 < \cdots < Z_{s+t}$ be the combined ordering of the two random samples. Let each arrangement of the X's and Y's in the combined sample correspond to the path from $(0, 0)$ to

(s, t) whose ith increment is in the positive x direction or the positive y direction according to whether Z_i is an X or a Y, respectively. Then there is a one-to-one correspondence between the different ways the two samples may be combined and the paths from $(0, 0)$ to (s, t). Further, since the two random samples have the same continuous distribution function, all different arrangements of the combined sample are equally likely. Since the total number of paths from $(0, 0)$ to (s, t) is $\binom{s+t}{s}$,

$$(2.8) \quad f(s, t, u, v)\binom{s+t}{s}^{-1} = 1 - P(Y_1 < X_{u+1}, Y_2 < X_{u+2}, \cdots, Y_{v-u} < X_v).$$

The use of Lemma 2 yields (2.7).

3. The distributions of d_r', d_r'', and d_r. The method of deriving the distribution function of d_r' is similar in some respects to the method of reflected paths used by Gnedenko and Koroluk (1951), also described by Fisz (1963), for finding the distribution of the Smirnov statistic $\sup_x |S_n(x) - S_n'(x)|$. Simple probability concepts are then used to find the distribution functions of d_r'' and d_r.

THEOREM 1. $P(d_r' \leqq x) = 0 \qquad$ *for* $x < 0$

$$(3.1) \qquad = 1 + 2\sum_{i=1}^{i'}(-1)^i N_i/\binom{2n}{n} \qquad \text{for } x \geqq 0$$

where $c = [nx]$, $i' = \min([(r + c)/(c + 1)], [n/(c + 1)])$, *and*

$$N_i = \binom{2n}{n-ic-i} - \sum_{j=0}^{n-r-c-1} \binom{2n-2r-c-1-j}{n-r}\left[\binom{2r+c+j}{r+ic+i-1} - \binom{2r+c+j}{r-ic-i+c}\right].$$

PROOF. The theorem is trivially true for $x < 0$ and for $x \geqq 1$, by definition of d_r'. For $0 \leqq x < 1$ consider the two samples $X_1 < \cdots < X_n$ and $Y_1 < \cdots < Y_n$ from the same continuous distribution function, and let $Z_1 < \cdots < Z_{2n}$ represent the combined sample. Consider the $\binom{2n}{n}$ paths in the (x, y) plane from $(0, 0)$ to (n, n), where each path consists of $2n$ increments of unit length, the ith increment being in the positive x or y direction according to whether Z_i is an X or Y. Then there is a one-to-one correspondence between the possible orderings of the combined sample and the paths from $(0, 0)$ to (n, n). Since all possible orderings of the combined sample are equally likely, probabilities involving the arrangement of the combined sample may be found by counting appropriate paths. In particular

$$(3.2) \quad P(d_r' \leqq x) = P(Y_1 < X_{c+1}, Y_2 < X_{c+2}, \cdots, Y_r < X_{c+r}, \\ X_1 < Y_{c+1}, \cdots, X_r < Y_{c+r}) = 1 - A/\binom{2n}{n}$$

where A represents the number of paths that touch either the line segment L_1 from $(c + 1, 0)$ to $(r + c, r - 1)$, or the line segment L_2 from $(0, c + 1)$ to $(r - 1, r + c)$.

Let A_1 denote the number of paths that touch L_1 at least once; let A_2 denote the number of paths that touch, at least once, L_1 and then L_2; let A_3 denote the number of paths that touch L_1 and L_2 in the order $L_1L_2L_1$ at least once; and in general let A_i denote the number of paths that touch L_1 and L_2 in the order L_1L_2

$\cdots$, alternating for a total of i terms, at least once. Similarly let B_i denote the number of paths that touch L_1 and L_2 in the order $L_2L_1L_2 \cdots$, alternating for a total of i terms, at least once. Then it may be shown that

$$A = \sum_{i=1}^{i'} (-1)^{i-1}(A_i + B_i) = 2\sum_{i=1}^{i'} (-1)^{i-1}A_i \tag{3.3}$$

(because $B_i = A_i$ due to symmetry, and because $A_i = 0$ for $i > i'$).

Substitute (3.3) into (3.2). It remains to show that $A_i = N_i$.

The method of reflected paths is used to show $A_i = N_i$. If p represents an arbitrary path from $(0, 0)$ to (n, n) passing through an arbitrary point (a, b), the reflection of p relative to (a, b) is the path that is identical with p from $(0, 0)$ to (a, b), but after reaching (a, b) the reflected path moves in the positive x direction whenever p moves in the positive y direction, and vice versa. Thus the reflected path goes from $(0, 0)$ to $(a + n - b, b + n - a)$.

A_1 equals the number of paths from $(0, 0)$ to (n, n) that touch L_1; that is, that touch the line $x = y + c + 1$ on or before touching the line $x = r + c$. From Lemma 3,

$$A_1 = f(n, n, c, r + c) = N_1. \tag{3.4}$$

The reflections of the paths from $(0, 0)$ to (n, n) that touch the line $x = y + c + 1$, reflected relative to the point of first contact with $x = y + c + 1$, go from $(0, 0)$ to $(n + c + 1, n - c - 1)$. Further, the paths from $(0, 0)$ to (n, n) that touch, at least once, the line $x = y + c + 1$ are in one-to-one correspondence with the paths from $(0, 0)$ to $(n + c + 1, n - c - 1)$, and the paths from $(0, 0)$ to (n, n) that touch, at least once, L_1 and then L_2 correspond to the paths from $(0, 0)$ to $(n + c + 1, n - c - 1)$ that touch $x = y + 3(c + 1)$ on or before touching $x = r + 2c - 1$, that is, that touch the reflection of the points in L_2. Again from Lemma 3,

$$A_2 = f(n + c + 1, n - c - 1, 3c + 2, r + 2c + 1) = N_2. \tag{3.5}$$

A_3 is obtained by considering doubly reflected paths, reflected first when they touch the line $x = y + c + 1$, and again when they first touch the reflection of $y = x + c + 1$, namely $x = y + 3(c + 1)$. The doubly reflected paths go from $(0, 0)$ to $(n + 2(c + 1), n - 2(c + 1))$, and the paths counted by A_3 are the doubly reflected ones that touch the double reflection of L_1, the segment from $(c + 4(c + 1), 0)$ to $(r + c + 2(c + 1), r - 2(c + 1))$. That is,

$$\begin{aligned} A_3 = f(n + 2(c + 1), n - 2(c + 1), \\ c + 4(c + 1), r + c + 2(c + 1)) = N_3. \end{aligned} \tag{3.6}$$

To find A_i, consider the $(i - 1)$st reflections of the paths from $(0, 0)$ to (n, n), reflected first about $x = y + c + 1$, then about the reflection of $y = x + c + 1$, namely $x = y + 3(c + 1)$, then about the double reflection of $x = y + c + 1$, namely $x = y + 5(c + 1)$, and so on, with the $(i - 1)$st reflection being about the $(i - 2)$nd reflection of $y = x + c + 1$ if i is odd, or $x = y + c + 1$ if i is even; namely $x = y + (2i - 3)(c + 1)$. Then A_i counts those paths that go from $(0, 0)$ to $(n + (i - 1)(c + 1), n - (i - 1)(c + 1))$ that touch the $(i - 1)$st reflection of L_1 if i is odd; L_2, if i is even. Equivalently,

$$(3.7)\quad A_i = f(n + (i-1)(c+1), n - (i-1)(c+1), c + 2(i-1)(c+1), r + c + (i-1)(c+1)) = N_i$$

from Lemma 2, and the proof is complete.

COROLLARY. *Let* $d_r^+ = \max_{x \leqq \max(X_r, Y_r)}[S_n(x) - S_n'(x)],\qquad r \leqq n.$ *Then*

$$d_r^+ = \max_{x \leqq X_{r+c}}[S_n(x) - S_n'(x)],\quad \textit{and}\quad P(d_r^+ \leqq x) = 1 - N_1/\binom{2n}{n}.$$

PROOF. For either way of defining d_r^+, the event "$d_r^+ \leqq x$" corresponds to the event "a path from $(0, 0)$ to (n, n) does not touch L_1"; therefore,

$$(3.8)\qquad P(d_r^+ \leqq x) = 1 - A_1/\binom{2n}{n}$$

which, with (3.4), completes the proof.

THEOREM 2. $P(d_r'' \leqq x) = 0$ *for* $x < 0$

$= 1 + 2\sum_{i=1}^{i''}(-1)^i P_i/\binom{2n}{n}$ for $x \geqq 0$

where $c = [nx]$, $i'' = [r/(c + 1)]$, *and*

$$P_i = \binom{2n}{n-ic-i} - \sum_{j=0}^{n-r-1}\binom{2n-2r+c-1-j}{n-r+c}\left[\binom{2r-c+j}{r+(i-1)(c+1)} - \binom{2r-c+j}{r-ic-i}\right].$$

PROOF. Just as the event "$d_r' \leqq x$" corresponded to the event "a path from $(0, 0)$ to (n, n) does not touch either L_1 or L_2," the event "$d_r'' \leqq x$" corresponds to the event "a path from $(0, 0)$ to (n, n) does not touch either the line segment from $(c + 1, 0)$ to $(r, r - c - 1)$ or the line segment from $(0, c + 1)$ to $(r - c - 1, r)$," and thus corresponds to the event "$d_{r-c}' \leqq x$". Therefore

$$(3.9)\qquad \begin{aligned} P(d_r'' \leqq x) &= P(d_{r-c}' \leqq x) &&\text{for}\quad r - c \geqq 1\\ &= 1 &&\text{for}\quad r \leqq c.\end{aligned}$$

Substituting $r - c$ for r in Theorem 1 provides Theorem 2.

Tsao (1954) also defines the statistic

$$(3.10)\qquad d_r^- = \min_{x \leqq X_r}[S_n(x) - S_n'(x)]$$

if the two samples are of equal size.

COROLLARY $\quad d_r^- = \max_{x \leqq \min(X_r, Y_r)}[S_n'(x) - S_n(x)]$,

and

$$P(d_r^- \leqq x) = 1 - P_1/\binom{2n}{n} = P(d_{r-c}^+ \leqq x).$$

Proof. For either way of defining d_r^-, the event "$d_r^- \leqq x$" corresponds to the event "a path from $(0, 0)$ to (n, n) does not touch the line segment from $(0, c + 1)$ to $(r - c - 1, r)$," which by symmetry corresponds to the event "$d_{r-c}^+ \leqq x$."

Theorem 3.

$$P(d_r \leqq x) = 0 \quad \textit{for} \quad x < 0$$
$$= 1 + \sum_{i=1}^{i'} (-1)^i N_i / \binom{2n}{n} + \sum_{i=1}^{i''} (-1)^i P_i / \binom{2n}{n} \quad \textit{for} \quad x \geqq 0,$$

where i', i'', c, N_i, and P_i are given by Theorems 1 and 2.

Proof. Theorem 3 follows from the relationship

$$\begin{aligned} P(d_r \leqq x) &= P(d_r \leqq x \mid X_r > Y_r) \cdot P(X_r > Y_r) \\ &\quad + P(d_r \leqq x \mid X_r < Y_r) \cdot P(X_r < Y_r) \\ &= P(d_r' \leqq x) \cdot P(X_r > Y_r) + P(d_r'' \leqq x) \cdot P(X_r < Y_r) \\ &= \tfrac{1}{2} P(d_r' \leqq x) + \tfrac{1}{2} P(d_r'' \leqq x). \end{aligned} \tag{3.11}$$

Because Tsao (1954) presents tables for both d_r and d_r', the following corollary, relating the two statistics, is of interest.

Corollary.

$$P(d_r \leqq x) = \tfrac{1}{2} P(d_r' \leqq x) + \tfrac{1}{2} P(d_{r-c}' \leqq x) \quad \textit{for} \quad r > c$$
$$= \tfrac{1}{2} P(d_r' \leqq x) + \tfrac{1}{2} \quad \textit{for} \quad r \leqq c.$$

Proof. The corollary follows directly from (3.9) and (3.11).

4. The asymptotic distribution functions. A referee mentioned that the asymptotic distribution functions may be obtained from known results as follows.

Theorem 4. *Let Z be the standardized normal random variable, and let $N = mn/(m + n)$. If $m, n, r \to \infty$ in such a way that $r/n \to b$, and $r/m \to b'$, then*

(i) $\lim P(N^{\frac{1}{2}} d_r \leqq z) = P(0, b, z)$,
(ii) $\lim P(N^{\frac{1}{2}} d_r' \leqq z) = P(0, \max(b, b'), z)$
(iii) $\lim P(N^{\frac{1}{2}} d_r'' \leqq z) = P(0, \min(b, b'), z)$
where $P(0, b, z) = \sum_{j=-\infty}^{\infty} (-1)^j e^{-2z^2 j^2} P(|Z - 2jz((1 - b)/b)^{\frac{1}{2}}| < z(b - b^2)^{-\frac{1}{2}})$.

Outline of Proof. The distribution functions of the random variables $N^{\frac{1}{2}} d_r$ and $N^{\frac{1}{2}} \sup_{F(x) \leqq b} |S_n(x) - S_m'(x)|$ converge to the same limiting distribution function because $F(X_r) \to b$ and $P(0, F(X_r), z) \to P(0, b, z)$. Also, in the same way that Doob (1949) showed the distribution functions of $N^{\frac{1}{2}} \sup_x |S_n(x) - S_m'(x)|$ and $n^{\frac{1}{2}} \sup_x |S_n(x) - F(x)|$ converge to the same limiting distribution function, the random variables

$$N^{\frac{1}{2}} \sup_{F(x) \leqq b} |S_n(x) - S_m'(x)| \quad \text{and} \quad n^{\frac{1}{2}} \sup_{F(x) \leqq b} |S_n(x) - F(x)|$$

may be shown to have the same limiting distribution. However, the latter distribution was given by Anderson and Darling (1952), pp. 209–210, as $P(0, b, z)$. The proofs of (ii) and (iii) are similar.

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