PROPERTIES OF THE STATIONARY MEASURE OF THE CRITICAL CASE SIMPLE BRANCHING PROCESS[1]

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1. Introduction. Consider a simple discrete time Galton-Watson Markov branching process $\mathcal{P} = \{Z_n, n \geqq 0\}$ whose state space is the non-negative integers. Its transition probability matrix $P = \|P_{ij}\|$ and iterates possess the representations

$$\sum_{j=0}^{\infty} P_{ij}x^j = [f(x)]^i \quad \text{and} \quad \sum_{j=0}^{\infty} P_{ij}^{(n)}x^j = [f_n(x)]^i$$

where $f_n(x) = f_{n-1}(f(x))$ is the nth functional composition of a specified probability generating function $f(x) = \sum_{k=0}^{\infty} a_k x^k$, $a_k \geqq 0$, $k = 0, 1, 2, \cdots, f(1) = 1$. Interpretations and elementary properties of simple branching processes are elaborated in [1], Chapter 1, see also [2], Chapter 11.

In this paper, we deal exclusively with the critical case branching process corresponding to $f'(1) = \sum_{k=0}^{\infty} ka_k = 1$. We assume throughout unless stated explicitly to the contrary that $f^{\text{iv}}(1) < \infty$. Furthermore, we exclude the trivial case $f(x) = x$ from all future considerations.

A set of *stationary probabilities* for $\|P_{ij}\|$ is a set of numbers p_i, $i = 0, 1, 2, \cdots$ satisfying

$$p_j = \sum_{i=0}^{\infty} p_i P_{ij}, \qquad j = 0, 1, 2, \cdots; p_i \geqq 0, \sum_{i=0}^{\infty} p_i = 1.$$

If we drop the requirement that $\sum_{i=0}^{\infty} p_i$ is finite then a non-negative solution is referred to as a *stationary measure* (or alternatively as a set of *generalized stationary probabilities*). The importance of stationary measures is familiar; discussions of their relevance and interpretations in the study of boundary theory of Markoff processes can be found in numerous texts dealing with Markoff chains.

In the case of a branching process where 0 is an absorbing state it is only meaningful to consider the existence and uniqueness of a stationary measure corresponding to the truncated system of equations

$$(1) \qquad p_j = \sum_{i=1}^{\infty} p_i P_{ij}, \qquad j = 1, 2, \cdots; \text{ and } p_i \geqq 0 \ (p_i \text{ not all zero})$$

where the 0 state has been deleted. It is easily established (see [1], page 23) that the generating function $p(x) = \sum_{i=1}^{\infty} p_i x^i$ of any such stationary measure, has radius of convergence $\rho \geqq f(0) > 0$, $p(f(0))$ is finite, and $p(x)$ satisfies the functional equation

$$(2) \qquad p(f(x)) = p(x) + p(f(0)).$$

Conversely, if $p(x) = \sum_{i=1}^{\infty} p_i x^i$ is a non-trivial solution of (2) with non-nega-

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tive coefficients, then the sequence $\{p_i\}$ determines a set of generalized stationary probabilities.

It is proved in [1], page 25, that if $f'''(1) < \infty$ then

$$A(x) = \lim_{n\to\infty} [1/(1 - f_n(x)) - 1/(1 - f_n(0))] \tag{3}$$

exists for $|x| < 1$ and solves the functional equation (2) with $A(f(0)) = f''(1)/2$. Moreover, $A(x)$ is analytic in $|x| < 1$, $A(0) = 0$ and $A(x)$ satisfies the asymptotic relation

$$A(x) \backsim 1/(1 - x), \qquad x \uparrow 1. \tag{4}$$

Examination of (3) readily reveals that $A(x)$ admits a power series expansion with non-negative coefficients and $A^{(r)}(x) > 0, 0 < x < 1, r = 1, 2, 3, \cdots$. It follows that $x = B(w) = A^{-1}(w)$ (the inverse function of $A(x)$) exists for positive w and satisfies $0 < B(w) < 1$ on $0 < w < \infty$.

It had been pointed out by Fatou that if the coefficients are not required to be non-negative then (2) has infinitely many linearly independent solutions. More recently, Kingman [5] has shown than when $f'(1) \neq 1$, the solution of (2) can be non-unique even when the coefficients are required to be non-negative. His counterexample, surprisingly is the simple case of $f(s) = (1 - p)/(1 - ps)$, $p \neq \frac{1}{2}$. It seems possible that the non-uniqueness of positive solutions of (2) always prevails when $f'(1) \neq 1$. Uniqueness of the stationary measure (up to a multiplicative constant) in the case of a critical ($f'(1) = 1$) branching process is proved in Karlin and McGregor [3] under a restriction more severe than necessary but widely satisfied in applications. By exploiting Martin boundary theory and the precise form of the asymptotic behavior of the Green's function of the process, Kesten, Ney and Spitzer [4] demonstrate the uniqueness of the stationary measure subject to the moment restriction $f''(1) < \infty$.

The principal objective of this paper is to ascertain precise information concerning the asymptotic growth behavior of the generalized stationary probabilities. In particular, Theorem 4 stated below settles and refines a conjecture of Harris ([1], page 27). The importance of this study pertaining to applications in genetics is described in Karlin and McGregor [3], see also Harris [1] and Fisher [6].

In order to formulate the principal theorem of this paper, we need the following definition: A probability generating function $f(x) = \sum_{k=0}^{\infty} a_k x^k$ is said to be *aperiodic* if gcd $\{k \mid a_k > 0\} = 1$ (gcd = greatest common divisor) or equivalently if $1 - f(x) = 1 - \sum_{k=0}^{\infty} a_k x^k$ vanishes in $|x| \leqq 1$ only at $x = 1$.

THEOREM 4. *Assume f is aperiodic, $f^{\text{iv}}(1) < \infty$ and let $A(x) = \sum_{i=1}^{\infty} A_i x^i$ denote the generating function of the stationary measure defined in* (3). *Then*

$$A_i = 1 - c/i + e_i/i, \qquad i = 1, 2, 3, \cdots, \tag{5}$$

where

$$\sum_{i=1}^{\infty} e_i^2 < \infty \quad \text{and} \quad c = [(f''(1)/2)^2 - f'''(1)/6]/a, \qquad a = f''(1)/2. \tag{6}$$

The above result provides in a sense a partial asymptotic expansion for the generalized stationary probabilities.

It is worthwhile to review the recent history leading up to Theorem 4. Harris with (4) in hand and taking account of the fact that $A_i \geqq 0$, $i = 1, 2, \cdots$, deduced the average asymptotic relation $(1/n) \sum_{i=1}^{n} A_i \to 1$ by direct appeal to the classical Hardy-Littlewood Tauberian theorem for power series with non-negative coefficients. With the aid of some delicate refinements on local limit theorems for the critical case simple branching process, Kesten, Ney, and Spitzer [4] improved the average convergence to that of $\lim_{i\to\infty} A_i = 1$ merely subject to the condition $f''(1) < \infty$. Observe that our result (5) entails the assumption $f'''(1) < \infty$ indispensably since the coefficient c visibly involves the value of $f'''(1)$. Actually, the hypothesis $f'''(1) < \infty$ alone will not suffice to prove (5). Somewhat more is needed. We have imposed the requirement $f^{\text{iv}}(1) < \infty$ but probably the condition $\sum_k k^{3+\delta} a_k < \infty$ for some $\delta > 0$ is enough.

The assumption that $f(x)$ is aperiodic is essential for the validity of (5) since clearly $A(x)$ is periodic to the same extent that $f(x)$ is.

The proof of Theorem 4 depends on the following two theorems of independent interest. Theorem 1 asserts a substantial strengthening of the uniqueness criteria for solutions of the functional equation (2). Theorem 2 develops a representation formula for $A(x)$ from which we can discern the growth behavior and properties of $A(x)$ more easily.

THEOREM 1. *There exists a unique (up to an additive constant) solution $p(x)$ of the equation*

$$p(f(x)) = p(x) + 1 \tag{7}$$

analytic in $|x| < 1$ with the property that $p'(x)$ and $p''(x)$ are positive on a real interval $(1 - \epsilon, 1)$ for some positive ϵ. Subject to the additional condition $p(0) = 0$ there exists a unique analytic (in $|x| < 1$) solution of (7) for which $p'(x)$ and $p''(x)$ are positive on $(1 - \epsilon, 1)$.

THEOREM 2. *The function $A(x)$ possesses a representation of the form*

$$A(x) = 1/(1 - x) + c \log (1 - x) + \psi(x) + d \quad (d \text{ is a constant and } c \text{ is defined in (6)}) \tag{8}$$

valid in the neighborhood $D_\delta = \{x \mid |x - 1| \leqq \delta, |x| \leqq 1, x \neq 1\}$ for some $\delta > 0$. Moreover $\psi(x)$ is bounded in D_δ and admits the series expression

$$\psi(x) = \sum_{n=0}^{\infty} \theta(f_n(x)), \quad (f_0(x) = x \text{ by definition}) \tag{9}$$

where $\theta(x)/(1 - x)^2$ is bounded for x in D_δ for any δ, $0 < \delta < 1$ and the series converges uniformly in the region $|x| \leqq 1 - \epsilon$ for any positive ϵ. Furthermore $\psi(x)$ is analytic on the open unit disc.

The convergence in (9) relies on the estimate

$$1/(1 - f_n(x)) = 1/(1 - x) + na + O(\log n), \quad a = f''(1)/2, \tag{10}$$

proved in [1], page 23, where the O symbol indicates a uniform bound with respect to x in D_δ for some positive δ.

We give a brief outline of the remaining contents of the paper. Various preliminaries are set forth in Section 2, including properties of the infinitesimal generator of the semi-group $B(A(x) + t) = f_t(x)$. More specifically, Section 2 is devoted primarily to estimating the growth of the series

$$\sum_{n=1}^{\infty} [1 - f_n(x)]f_n'(x), \qquad \sum_{n=1}^{\infty} [f_n'(x)]^2 \quad \text{and} \quad \sum_{n=1}^{\infty} f_n''(x)(1 - f_n(x))$$

for $|x| \leqq 1$, $x \neq 1$ and $x \to 1-$. In Section 3 we establish the existence of a solution of (7) of the form (8). The growth behavior of $\psi(x)$ for x near 1 is determined.

The proof of Theorem 1 is in Section 4.

We frequently use symbols K, K', K'', C, C' to denote absolute positive constants which may differ from equation to equation.

In a separate publication we will give some applications of Theorem 3 to some questions of interest in evolutionary theory.

2. Preliminaries and estimates. This section is devoted to developing a variety of estimates and inequalities for certain series which occur in connection with the proofs of Theorems 1–4. Several of the individual lemmas of this section may have independent interest.

Let $A(x)$ be determined as in (3). Since $A'(x) > 0$, $0 < x < 1$ it follows that $x = B(w) = A^{-1}(w)$ (the inverse function of $A(x)$) exists for positive w and satisfies $0 < B(w) < 1$ on $0 < w < \infty$. Notice that B verifies the functional equation

$$f(B(w)) = B(w + a), \quad \text{for} \quad 0 \leqq w < \infty (a = f''(1)/2)).$$

In terms of $A(x)$ and $B(w)$ we can embed f in an analytic semigroup, viz, $f_t(x) = B(A(x) + ta)$, $0 \leqq |x| < \rho_t$, whose infinitesimal generator is

$$u(x) = \partial f_t(x)/\partial t\,|_{t=0} = a/A'(x) \tag{11}$$

and $u(x)$ is analytic in a neighborhood of $0 < x < 1$.

Taking account of the functional equation satisfied by $A(x)$ we infer that

$$A'(f_n(x))f_n'(x) = A'(x), \qquad n = 1, 2, \cdots,$$

and therefore

$$u(f_n(x)) = u(x)f_n'(x), \qquad n = 1, 2, \cdots; 0 < x < 1. \tag{12}$$

For later purposes it is essential to know the nature of $u(x)$ for x approaching $1-$. In this direction the next lemma is fundamental.

LEMMA 1. *The infinitesimal generator* $u(x)$ *satisfies*

$$\begin{aligned} &\text{(a)} \ \lim_{x \to 1-} u(x)/(1 - x)^2 = a, \\ &\text{(b)} \ \lim_{x \to 1-} -u'(x)/(1 - x) = 2a, \\ &\text{(c)} \ u''(x) \textit{ is uniformly bounded on } 0 < \alpha \leqq x < 1. \end{aligned} \tag{13}$$

PROOF. Let $0 \leqq \alpha < 1$ and consider the intervals $I_n = [f_{n-1}(\alpha), f_n(\alpha)]$.

The map $x \to f(x)$ sends each I_n onto I_{n+1}, and $f_n(\alpha) \to 1$, so for any function $F(x)$ defined on $[\alpha, 1]$, to show that $F(x) \to L$ as $x \to 1-$ it suffices to show that as $n \to \infty$, $F(f_n(x)) \to L$ uniformly on I_1.

The convergence of the analytic functions in (3) is uniform in any disc $|x| \leqq 1 - \epsilon$, in particular, uniform in I_1. It follows by differentiation of (3) that

$$f_n'/(1 - f_n)^2 \to A' \quad \text{and} \quad f_n''/(1 - f_n)^2 + 2(f_n')^2/(1 - f_n)^3 \to A''$$

uniformly in I_1. Since $(f_n')^2/(1 - f_n)^3 \to 0$ we also have $f_n''/f_n' \to A''/A'$ uniformly in I_1.

We first apply these results to $F(x) = u(x)/(1 - x)^2$. Using (11) and (12) we have for $n \to \infty$

$$(14) \qquad F(f_n(x)) = u(x)f_n'(x)/(1 - f_n(x))^2 \to u(x)A'(x) = a$$

uniformly in I_1. This proves (a). By differentiation of (11)

$$u'(f_n(x)) = u'(x) + u(x)f_n''(x)/f_n'(x) \to u'(x) + u(x)A''(x)/A(x) = 0$$

uniformly in I_1. It follows that $u'(x) \to 0$ as $x \to 1-$. From the relation $u'(f(x)) - u'(x) = b(x)u(x)$ where $b(x) = f''(x)/f'(x)$ we obtain

$$(15) \qquad u'(f_n(x)) - u'(x) = \textstyle\sum_{k=0}^{n-1} b(f_k(x))u(f_k(x))$$

and letting $n \to \infty$,

$$-u'(x) = \textstyle\sum_{k=0}^{\infty} b(f_k(x))u(f_k(x))$$

where the series converges uniformly in I_1. For the function $F(x) = -u'(x)/(1 - x)$ we have

$$F(f_n(x)) = (1/(1 - f_n(x)))\textstyle\sum_{k=n}^{\infty} b(f_k(x))(u(f_k(x))/(1 - f_k(x))^2)(1 - f_k(x))^2.$$

When $k \to \infty$ we have, uniformly in I_1, (see (10)),

$$b(f_k(x)) \to 2a, \quad u(f_k(x))/(1 - f_k(x))^2 \to a, \quad k^2a^2(1 - f_k(x))^2 \to 1$$

and hence

$$F(f_n(x)) = 2a(1/na(1 - f_n(x)))[n\textstyle\sum_{k=n}^{\infty} C_k(x)k^{-2}]$$

where $C_k(x) \to 1$ uniformly in I_1 as $k \to \infty$. It follows by a standard summability argument that, as $n \to \infty$, $F(f_n(x)) \to 2a$ uniformly in I_1 and this proves (b).

From (10) follows $|1 - f_k(x)| \leqq C/k$ where C is a constant independent of k and x, $|x| \leqq 1$, $k = 1, 2, \cdots$. In the identity

$$(16) \quad -u''(x) = \textstyle\sum_{k=0}^{\infty} b'(f_k(x))u(f_k(x))f_k'(x) + \sum_{k=0}^{\infty} b(f_k(x))u'(f_k(x))f_k'(x)$$

we see, from the facts that $b'(x)$ is uniformly bounded, $|f_k'(x)| \leqq 1$ and $u(f_k(x)) \leqq K/k^2$ for $0 \leqq x \leqq 1$ by the above inequality and part (a) of the lemma, that the first series on the right represents a function $G(x)$ which is bounded on $0 \leqq x \leqq 1$. Now

$$\begin{aligned} -G(f_r(x)) - u''(f_r(x)) &= \textstyle\sum_{k=0}^{\infty} b(f_{k+r}(x))u'(f_{k+r}(x))f_k'(f_r(x)) \\ &= (1/f_r'(x))\textstyle\sum_{k=r}^{\infty} b(f_k(x))u'(f_k(x))f_k'(x) \end{aligned}$$

and we show this is uniformly bounded for x in I_1 and $r \to \infty$. For x in I_1, $f_k'/(1 - f_k)^2 \to A'$ so $f_k' \leqq M/k^2$ where M is independent of k or x. By part (b) of the lemma $|u'(f_k(x))| \leqq M_1/k$ where M_1 is independent of k or x (in I_1). Since $b(x)$ is bounded and $1/f_r'(x) \leqq M_2 r^2$ it is seen that

$$-G(f_r(x)) - u''(f_r(x))$$

is bounded in I_1 as $r \to \infty$ and part (c) follows.

We now prove

LEMMA 2. *The series* $\sum_{k=1}^{\infty} (1 - f_k(x))f_k'(x)$ *is uniformly bounded with respect to* x *satisfying* $0 < \delta \leqq x < 1$ *and any* $\delta > 0$.

PROOF. The result of (13c), the inequality $u(f_k(x)) \leqq K/k^2$ for $0 \leqq x < 1$ and the fact that $b'(x)$ is bounded on $0 \leqq x \leqq 1$ imply on examination of (16) that

$$\sum_{k=0}^{\infty} b(f_k(x))u'(f_k(x))f_k'(x) \tag{17}$$

is uniformly bounded for x traversing $0 < \delta \leqq x < 1$.

Next observe that $0 < b(x) \leqq C'$ for $0 < x < 1$. Furthermore, note that $u'(x) = -a(A''(x))/[A'(x)]^2$ is of one sign on the interval $0 \leqq x < 1$. Thus all the terms of the series $\sum_{k=0}^{\infty} b(f_k(x))u'(f_k(x))f_k'(x)$ maintain a constant sign for $0 < \delta \leqq x < 1$. Since $b(x) \geqq b_0 > 0$ for the indicated x interval and the sum is uniformly bounded per demonstration above, we deduce that

$$\sum_{k=0}^{\infty} (-u'(f_k(x))f_k'(x)$$

converges boundedly on $0 < \delta \leqq x < 1$. This convergence together with the fact

$$\lim_{x \to 1-} (-u'(x))/(1 - x) = 2a > 0 \tag{18}$$

(Lemma 2) manifestly implies the bounded convergence of the series

$$\sum_{k=0}^{\infty} [1 - f_k(x)]f_k'(x), \qquad 0 < \delta \leqq x < 1.$$

With the result of Lemma 2 in hand we have available the apparatus needed to estimate the growth of the series

$$\sum_{k=0}^{\infty} (f_k'(x))^2 \quad \text{and} \quad \sum_{k=0}^{\infty} [1 - f_k(x)]f_k''(x)$$

for x approaching $1-$.

LEMMA 3.

$$\sum_{k=0}^{\infty} [f_k'(x)]^2 \leqq K/(1 - x), \qquad \sum_{k=0}^{\infty} [1 - f_k(x)]f_k''(x) \leqq K'/(1 - x),$$
$$0 < \delta \leqq x < 1,$$

where K *and* K' *are constants independent of* x *in the indicated range.*

PROOF. The starting point is the relation $u(f_k(x)) = u(x)f_k'(x)$ which multiplied by $f_k'(x)$ and summed gives

$$\sum_{k=0}^{n} [u(f_k(x))/u(x)](f_k'(x)) = \sum_{k=0}^{n} [f_k'(x)]^2.$$

Lemma 1 tells us that $0 < \eta \leqq u(x)/(1 - x)^2 \leqq D < \infty$ on $0 < \delta \leqq x < 1$

for some η and D. Moreover, $f_k(x)$ is convex on $[0, 1)$ and therefore

$$(1 - f_k(x))/(1 - x) \leqq f_k'(1) = 1.$$

Using these facts, we estimate $\sum_{k=0}^{n} [f_k'(x)]^2$ as follows:

$$\sum_{k=0}^{n} [f_k'(x)]^2 \leqq C' \sum_{k=0}^{n} [[1 - f_k(x)]^2/(1 - x)^2] f_k'(x)$$
$$\leqq [C'/(1 - x)] \sum_{k=0}^{n} [1 - f_k(x)] f_k'(x) \leqq K/(1 - x)$$

where the last bound is assured by virtue of Lemma 2.

In order to deal with the second series we differentiate (12) to obtain

$$u'(f_k(x))f_k'(x) - u'(x)f_k'(x) = f_k''(x)u(x),$$

and thus

$$(19)\quad \sum_{k=0}^{n} [1 - f_k(x)]f_k''(x) = \sum_{k=0}^{n} [u'(f_k(x))/u(x)][1 - f_k(x)]f_k'(x) - [u'(x)/u(x)] \sum_{k=0}^{n} [1 - f_k(x)]f_k'(x).$$

Lemma 1 and Lemma 2 guarantee the validity of the inequalities $|\, u'(x)/u(x) \,| \leqq K''/(1 - x)$ and

$$u'(f_k(x))/u(x) \leqq K'''(1 - f_k(x))/(1 - x)^2 \leqq K'''/(1 - x).$$

These facts in conjunction with the assertion of Lemma 2 applied to the right side of (19) yield the desired inequalities.

For the proof of Theorem 4 we will need an estimate of the series

$$\sum_{k=0}^{\infty} f_k'(x)\, (1 - f_k(x))$$

where x varies in a complex neighborhood of the form

$$D_\delta = \{x \mid |x - 1| \leqq \delta,\ |x| \leqq 1,\ x \neq 1\}$$

where δ is a positive number not exceeding 1. If x traverses any fixed angle $\mathfrak{A}$ in $|x| \leqq 1$ with vertex at 1, i.e.,

$$(20)\quad \mathfrak{A} = \{x \mid x \text{ satisfies } |1 - x| \leqq E\cdot(1 - |x|) \text{ and } x \text{ belongs to } D_\delta \text{ for some } \delta > 0\}$$

for some positive constant E, then

$$(21)\quad \sum_{k=0}^{\infty} |f_k'(x)(1 - f_k(x)| \leqq E', \qquad \text{for } x \in D_\delta \cap \mathfrak{A}$$

where E' is a constant depending on $\mathfrak{A}$. In fact when $x \in \mathfrak{A}$ we have, since $f_k(1) = 1$,

$$(22)\quad |1 - f_k(x)| = |\sum_{l=0}^{\infty} p_{kl}(1 - x^l)| \leqq E \sum_{l=0}^{\infty} p_{kl}(1 - |x|^l) = E\cdot[1 - f_k(|x|)].$$

Also trivially

$$(23)\quad |f_k'(x)| \leqq f_k'(|x|) \qquad \text{for } |x| \leqq 1.$$

The inequality (21) is a consequence of Lemma 2 taking account of (22) and (23). The above discussion has proved the first part of the following lemma.

LEMMA 4. (i) *If x traverses $D_\delta \cap \mathfrak{A}$ then*

$$\sum_{k=0}^{\infty} |f_k'(x)(1 - f_k(x)| \leqq E'$$

where the bound E' depends on the region defined by the angle $\mathfrak{A}$ but not on the choice of $x \varepsilon \mathfrak{A}$.

(ii) *If x traverses D_δ, then*

$$\sum_{k=0}^{\infty} |f_k'(x)(1 - f_k(x)| \leqq C_\delta \log (1/|1 - x|) \tag{24}$$

where C_δ is a positive constant depending only on δ.

The proof of part (ii) relies on the following lemma.

LEMMA 5. *Let $x \varepsilon D_\epsilon$ where $0 < \epsilon < 1$. Then*

$$|n^2 f_n'(x)| \leqq M/|1 - x|^\alpha \tag{25}$$

for some positive α and constant M.

PROOF. Clearly $f_n'(x) = f'(f_{n-1}(x))f_{n-1}'(x)$ and after iteration we have

$$f_n'(x) = [\prod_{k=0}^{n-1} g(f_k(x))] \qquad \text{where } g(x) = f'(x).$$

Multiplying by n^2 and taking logarithms we have

$$H_n(x) = \log (n^2 f_n'(x)) = \sum_{k=0}^{n-1} \log [1 - \{1 - g(f_k(x)\}] + 2 \log n$$
$$\text{for } |x| \leqq 1, |x - 1| \leqq \epsilon, x \neq 1.$$

The function $H_n(x)$ is well defined since $f_n(x)$ converges uniformly to 1 for $|x| \leqq 1$ and $f'(1) = 1$. Expanding the logarithm we have

$$H_n(x) = -\sum_{k=0}^{n-1} \sum_{r=1}^{\infty} (1/r)[1 - g(f_k(x))]^r + 2 \log n.$$

It is convenient to separate the terms with $r = 1$. With this done we obtain

$$\begin{aligned} H_n(x) = &-\sum_{k=0}^{n-1} [1 - g(f_k(x))] \\ &+ 2 \log n - \sum_{k=0}^{n-1} \sum_{r=2}^{\infty} (1/r)[1 - g(f_k(x))]^r. \end{aligned} \tag{26}$$

Now the partial Taylor expansion

$$g(x) = f'(x) = 1 + 2a(x - 1) + \gamma(x)(x - 1)^2$$

where $a = f''(1)/2$ and $\gamma(x)$ is uniformly bounded, is valid in a neighborhood of 1 and $|x| \leqq 1$. Inserting this expression in (26) and performing obvious rearrangements produces the formula

$$\begin{aligned} H_n(x) = 2a &\sum_{k=0}^{n-1} (f_k(x) - 1) + 2 \log n + \sum_{k=0}^{n-1} \gamma(f_k(x)(1 - f_k(x))^2 \\ &-\sum_{k=0}^{n-1} \sum_{r=2}^{\infty} ((-1)^r/r)[2a(f_k(x) - 1) \\ &+ \gamma(f_k(x))(f_k(x) - 1)^2]^r. \end{aligned} \tag{27}$$

Using the estimate $|1 - f_k(x)| \leqq C(x)/k$, $k = 1, 2, 3, \cdots$, where $C(x)$ is uniformly bounded for $|x| \leqq 1$ it is easy to see that the last two series of (27) converge uniformly and boundedly for $|x| \leqq 1$. In order to bound $H_n(x)$ it remains to estimate the series

$$2\sum_{k=1}^{n-1} [f_k(x) - 1 + 1/ka].$$

For this purpose we avail ourselves of the asymptotic formula

$$1 - f_k(x) = 1/[1/(1 - x) + na + O(\log n)]$$

where the $O(\cdot)$ term is uniform with respect to $|x| \leqq 1$, $x \neq 1$, $|x - 1| \leqq \epsilon$ (cf.(10)). We get

$$\begin{aligned} &\sum_{k=1}^{n-1} |[1 - f_k(x) - 1/ak]| \\ &\quad = \sum_{k=1}^{n-1} |\{1/[1/(1 - x) + ka + O(\log k)] - 1/ak\}| \\ (28) \qquad &\quad \leqq \sum_{k=1}^{n-1} [[O(\log k) + 1/|1 - x|]/ak|1/(1 - x) + ka + O(\log k)|] \\ &\quad = \sum_{k=1}^{n-1} [O(\log k)/ak|1/(1 - x) + ka + O(\log k)|] \\ &\qquad + \sum_{k=1}^{n-1} [1/|1 - x|/ak|1/(1 - x) + ka + O(\log k)|]. \end{aligned}$$

Since $\operatorname{Re} 1/(1 - x) \geqq 0$ for $|x| \leqq 1$ it follows that

$$\sum_{k=1}^{n-1} [O(\log k)/ak|1/(1 - x) + ka + O(\log k)|] \leqq \tilde{C}(x) \sum_{k=1}^{n-1} [\log k/k^2]$$

and $\tilde{C}(x)$ is uniformly bounded for $|x| \leqq 1$, $x \neq 1$, $|x - 1| \leqq \epsilon$ and so this series converges boundedly. Finally, we split the second sum of the right side of (28) as indicated

$$\begin{aligned} &\sum_{k=1}^{n} [1/|1 - x|/ak|1/(1 - x) + ka + O(\log k)|] \\ (29) \qquad &\quad \leqq (1/a) \sum_{k=1}^{1/|1-x|^2} [1/k|1 + ka(1 - x) + (1 - x)O(\log k)|] \\ &\qquad\qquad + (1/a) \sum_{k>(1/|1-x|^2)}^{n} [1/k|1 + (1 - x)ka + (1 - x)O(\log k)|]. \end{aligned}$$

Since $|1 - x|k^{\frac{1}{2}} \geqq 1$ for the k values of the second sum we infer that

$$(1/a) \sum_{k=(1/|1-x|^2)}^{n} [1/k|1 + (1 - x)ak + (1 - x)O(\log k)|]$$
$$\leqq C' \sum_{k=(1/|1-x|^2)}^{n} k^{-3/2} \leqq C''$$

where C' and C'' are positive constants independent of x.

For $|x| \leqq 1$ we have $\operatorname{Re}(1 - x) \geqq 0$ and therefore

$$|1 + ka(1 - x) + (1 - x)O(\log k)| \geqq \beta \geqq 0$$

provided k is sufficiently large $k \geqq k_0$ (a fixed integer) consequently

$$(1/a) \sum_{k=1}^{1/|1-x|^2} [1/k|1 + ka(1 - x) + (1 - x)O(\log k)|] \leqq C \log |1 - x|^{-1}.$$

The above analysis established that the series $\sum_{k=1}^{n-1} \{[1 - f_k(x)] - 1/ka\}$ converges for all $|x| \leqq 1$, $x \neq 1$ and its value is estimated above by $\alpha \log |1 - x|^{-1}$ where α is some positive constant. The conclusio f the lemma now follows by exponentiating $H_n(x)$.

REMARK. It seems likely that the precise estimate in (25) has $\alpha = 2$. Some evidence for this conjecture is indicated by the validity of the local limit law

$$\lim_{n\to\infty} n^2 e^{j/n} P_{1j}^{(n)} = 1/a$$

where j is restricted such that $0 < c_1 \leqq j/n \leqq c_2 < \infty$, see [4].

PROOF OF (ii) OF LEMMA 4. Let x belong to D_δ. The terms of the series $\sum_{k=0}^{\infty} |f_k'(x)(1 - f_k(x))|$ are grouped in two parts

$$\sum_{k=0}^{1/|1-x|^\alpha} + \sum_{k=(1/|1-x|^\alpha)+1}^{\infty} = I_1 + I_2$$

where α is the exponent appearing in the estimate of Lemma 5. In the sum I_1 we use the bounds $|f_k'(x)| \leqq f_k'(1) = 1$ and $|1 - f_k(x)| \leqq \tilde{C}/k$, $k = 1, 2, \cdots$, remembering that $\tilde{C}$ is independent of x, $|x| \leqq 1$. Obviously $I_1 \leqq \tilde{C} \log |1 - x|^{-1}$. For the terms of I_2 where $k|1 - x|^\alpha \geqq 1$ we use the estimate of Lemma 5

$$|f_k'(x)| \leqq M/k^2|1 - x|^\alpha \leqq M/k$$

and again the inequality $|1 - f_k(x)| \leqq C/k$. Manifestly I_2 is uniformly bounded for $|x| \leqq 1$. The proof of Lemma 4 is complete.

3. Representation of generating function of stationary measure. This section is principally concerned with the proofs of Theorems 1 and 2 (see Section 1).

Consider the functional equation (2) (altered by a multiplicative constant) in the form

$$\pi(f(x)) = \pi(x) + 1, \qquad 0 < x < 1 \text{ (cf. (7))} \tag{30}$$

We attempt to find a solution of the form

$$\pi(x) = \alpha/(1 - x) + \gamma \log(1 - x) + \psi(x), \qquad 0 < x < 1 \tag{31}$$

where α and γ are constants to be determined and $\psi(x)$ is a bounded function for $0 \leqq x < 1$.

Substituting (31) into (30) yields the identity

$$\begin{aligned} &\alpha/(1 - f(x)) + \gamma \log[1 - f(x)] + \psi(f(x)) \\ &\qquad = \alpha/(1 - x) + \gamma \log(1 - x) + \psi(x) + 1. \end{aligned} \tag{32}$$

Our hypotheses imply that

$$1 - f(x) = 1 - x - a(1 - x)^2 + b(1 - x)^3 - d(1 - x)^4 + \rho(x), \tag{33}$$

$a = \frac{1}{2}f''(1)$, $b = f'''(1)/6$, $d = f^{\text{iv}}(1)/24$, $\rho(x) = o(1 - x)^4$ valid for $|x| \leqq 1$ and $|x - 1| \leqq \epsilon$, with ϵ positive and sufficiently small. We define

$$\begin{aligned} \theta(x) = \alpha\{1/(1 - f(x)) - 1/(1 - x)\} + \gamma \log[(1 - f(x))/(1 - x)], \\ |x| \leqq 1, \qquad x \neq 1, \end{aligned} \tag{34}$$

and set

$$\alpha = 1/a, \qquad \gamma = (a^2 - b)/a^2. \tag{35}$$

Straightforward manipulations using the expansion (33) with the special choices of α and γ given in (35) reduces (34) to an expression of the form

$$\theta(x) = e(1 - x)^2 + \delta(x) \tag{36}$$

where e is a suitable constant and the function $\delta(x)$ satisfies

$$\lim_{x\to 1-} [\delta(x)/(1 - x)^2] = 0.$$

Because of the specifications (35) and the definition of (34) it is simple to see that (32) becomes an identity provided $\psi(x)$ is determined to satisfy the functional equation

$$\theta(x) + \psi(f(x)) = \psi(x), \qquad 0 \leqq x < 1. \tag{37}$$

We exhibit a solution of (37) outright, namely

$$\psi(x) = \sum_{k=0}^{\infty} \theta(f_k(x)). \tag{38}$$

The uniform convergence of this series is validated as follows. In view of (36) we have $\theta(x) \leqq K|1 - x|^2$ for $|x| \leqq 1$, $x \neq 1$, $|x - 1| \leqq \epsilon$ and moreover the inequality $|1 - f_k(x)| \leqq C/k$, $k = 1, 2, \cdots, 0 \leqq x < 1$, where C is a positive constant, implies the estimate

$$|\theta(f_k(x))| \leqq C'/k^2, \qquad k = 1, 2, \cdots,$$

and consequently the convergence in (38).

Examination of the definition of $\theta(x)$ reveals that this function is analytic on the interior of the unit circle. Evidently each term of the series (38) is likewise analytic in any compact part of $|x| < 1$. Since $f_k(x) \to 1$, the convergence is uniform for $|x| \leqq 1 - \epsilon$ and any positive ϵ and we infer that $\psi(x)$ is analytic in the same region.

Note the following important fact: If $f(x)$ is *aperiodic* then the estimates

$$\theta(x) \leqq C(|1 - x|^2) \quad \text{and} \quad |1 - f_k(x) \leqq C/k|, \quad k = 1, 2, \cdots,$$

holds for all complex x satisfying $|x| \leqq 1$, where C is a positive constant and consequently $\psi(x)$ is uniformly bounded over the region $|x| \leqq 1$.

We sum up the conclusions of the preceding discussion as the following theorem.

THEOREM 3. *There exists a solution* $\pi(x)$ *of the functional equation*

$$\pi(f(x)) = \pi(x) + 1 \tag{39}$$

of the form

$$\begin{aligned} \pi(x) &= (1/a)/(1 - x) + (c/a) \log (1 - x) + \psi(x), \\ c &= [[f''(1)/2]^2 - f'''(1)/6]/a, \\ a &= f''(1)/2, \end{aligned} \tag{40}$$

where $\psi(x)$ *defined in* (38) *is analytic in* $|x| < 1$ *and bounded in the region* $D_\delta =$

$\{x \mid |x| \leqq 1, |x - 1| \leqq \delta, x \neq 1\}$ with δ appropriately small but fixed. If $f(x)$ is aperiodic then $\psi(x)$ is uniformly bounded for all $|x| \leqq 1, x \neq 1$.

The growth properties of the function represented by (40) as $x \uparrow 1-$ are easily determined. The following lemma contains the desired information

LEMMA 6. *Let $\psi(x)$ be determined as in (38) where $\theta(x)$ is defined in (34) and α and γ in (35). Then*

$$\psi'(x) = \sum_{k=0}^{\infty} \theta'(f_k(x))f'_k(x) \tag{41}$$

is uniformly bounded on $0 \leqq x < 1$ and

$$\psi''(x) = \sum_{k=0}^{\infty} \theta''(f_k(x))[f_k'(x)]^2 + \sum_{k=0}^{\infty} \theta'(f_k(x))f''_k(x) \tag{42}$$

obeys the growth condition

$$|\psi''(x)| \leqq C/(1 - x), \qquad 0 \leqq x < 1. \tag{43}$$

PROOF. The term by term differentiations in (41) and (42) for $|x| < 1$ are justified by appealing to the standard theorem on convergence of series of analytic functions. From the definition of $\theta(x)$ it is easy to show that $|\theta'(x)| \leqq K|1 - x|$ and $|\theta''(x)| \leqq K, 0 \leqq x < 1$, where K denotes, as usual, an absolute constant. The series in (41) is clearly bounded by

$$K\sum_{k=0}^{\infty} [1 - f_k(x)]f_k'(x), \qquad 0 < \delta \leqq x < 1,$$

which according to Lemma 2 is uniformly bounded.

The first series of (42) can be estimated above by $K\sum_{k=0}^{\infty} [f_k'(x)]^2$ and the second by $K\sum_{k=0}^{\infty} [1 - f_k(x)]f_k''(x)$. The growth property expressed in (43) is verified by citing the results of Lemma 3.

A direct corollary of Theorem 3 and Lemma 6 is

LEMMA 7. *Let $\pi(x)$ be the solution of (39) constructed in the proof of Theorem 3 (see (40)). Then*

$$\lim_{x\to 1-} (1 - x)\pi(x) = \alpha = 1/a, \qquad \lim_{x\to 1-} (1 - x)^2\pi'(x) = \alpha, \tag{44}$$
$$\lim_{x\to 1-} (1 - x)^3\pi''(x) = 2\alpha.$$

PROOF. The assertions of (44) are checked by direct differentiation of the expression (40) taking account of the conclusions of Lemma 6.

We are now in a position to prove Theorem 1.

PROOF OF THEOREM 1. We denote by $\pi(x)$ the solution of (39) represented in (40). Notice that $\pi^{-1}(w) = I(w)$ exists ($\pi^{-1}(w)$ is the inverse function of $\pi(x)$) for w sufficiently large. In fact, by virtue of (44) we infer that $\pi(x)$, $\pi'(x)$ and $\pi''(x)$ are strictly positive in an appropriate interval $(1 - \epsilon, 1)$ with $\epsilon > 0$.

Let $p(x)$ denote another analytic solution of (39) for which $p'(x)$ and $p''(x)$ are both positive on $(1 - \epsilon, 1)$. Consider

$$p(I(w)) - w = g(w), \qquad w^* < w < \infty \ (w^* = \pi(1 - \epsilon)). \tag{45}$$

It is straightforward to verify, as a consequence of (39), that $g(w)$ is periodic of period 1 for $w \varepsilon [w^*, \infty)$. Observe that

$$p(x) = \pi(x) + g(\pi(x)), \qquad 1 - \epsilon \leqq x < 1. \tag{46}$$

On differentiating (46) and multiplying by $[\pi(x)]^3/\pi'(x)$ we obtain

$$p'(x)[\pi^3(x)/\pi'(x)] = \pi^3(x)[1 + g'(\pi(x))], \quad 1 - \epsilon \leqq x < 1. \tag{47}$$

We claim that the left hand side of (47) is non-decreasing on $(1 - \epsilon_1, 1)$ where ϵ_1 is an appropriate positive number satisfying $\epsilon_1 < \epsilon$. Indeed, note that

$$(d/dx)([\pi(x)]^3/\pi'(x)) = \pi^2(x)[3 - \pi(x)\pi''(x)/[\pi'(x)]^2]$$

which is certainly positive for $1 - \epsilon_1 \leqq x < 1$ because of (44). Since $p'(x)$ and $p''(x)$ are positive by hypothesis and certainly $\pi^3(x)/\pi'(x)$ is positive on $(1 - \epsilon_1, 1)$ we may conclude that $p'(x)\pi^3(x)/\pi'(x)$ is monotone increasing on $(1 - \epsilon_1, 1)$ as claimed.

The derivative of the right hand side of (47) is

$$\pi^2(x)\pi'(x)[3 + 3g'(w) + wg''(w)], \qquad w = \pi(x). \tag{48}$$

Since g is periodic on $[w^*, \infty)$ it follows that $g'(w)$ is uniformly bounded. Now, if $g(w)$ is not constant then there exists infinitely many values of the form $w_k = w_0 + k > w^*$ with $k = k_0, k_0 + 1, \cdots, k_0$ an integer (obviously $w_k \to \infty$) such that $g''(w_k) = g''(w_0) < 0$. As x approaches $1-$, $w = \pi(x) \to \infty$ and (48) is manifestly negative at w_k when k is sufficiently large. This fact contradicts the statement following (47). The only tenable inference is that $g(w) \equiv d_0 =$ constant $w^* < w < \infty$. Since $\pi(x)$ and $p(x)$ are analytic on $0 \leqq x < 1$ it follows from (45) that

$$\pi(x) \equiv p(x) + d_0, \qquad 0 \leqq x < 1.$$

The validation of the last statement of Theorem 1 is clear.

Proof of Theorem 2. This requires simply putting together the results of Theorems 1 and 3.

4. Proof of Theorem 4. We have available most of the apparatus to complete the proof of Theorem 4. To ease the exposition we divide the analysis into stages by stating four further lemmas. It is convenient at this point to introduce some additional notation. Let D_δ be a small region about $x = 1$ of the form

$$D_\delta = \{x \mid |x - 1| \leqq \delta, |x| \leqq 1, x \neq 1\}.$$

Let $\mathcal{A}_\epsilon = \{x \mid \arg(x - 1) \leqq \pi/2 + \epsilon\}$ and $D_{\delta,\epsilon} = D_\delta - \mathcal{A}_\epsilon$, the complement of $\mathcal{A}_\epsilon$ in D_δ. Finally S shall denote the unit circle $S = \{x \mid |x| \leqq 1\}$.

Lemma 8. *If $f(x)$ is* aperiodic *then for each δ ($\delta > 0$) there exists $\epsilon(\delta)$ such that f maps $S - D_\delta$ into $S - \mathcal{A}_\epsilon$.*

Proof. Suppose to the contrary that for each $\epsilon > 0$ there exists x_ϵ in $S - D_\delta$ such that $f(x_\epsilon)$ lies in $S \cap \mathcal{A}_\epsilon$. Let x_0 be a limit point of x_ϵ as $\epsilon \to 0$. Obviously $f(x_0) = 1$ and $x_0 \neq 1$ in violation of the hypothesis that $f(x)$ is *aperiodic*. The proof is complete.

We need to study the growth behavior of $\psi'(x) = \sum_{k=0}^{\infty} \theta'(f_k(x))f_k'(x)$ for x in a neighborhood of 1. (For the definitions of ψ and θ see (34) and (38) respectively.) Recall the property $|\theta'(x)| \leqq K|1 - x|$ valid for x in D_δ provided δ is small enough. Thus,

$$|\psi'(x)| \leqq K \sum_{k=0}^{\infty} |[1 - f_k(x)]f'_k(x)|, \qquad x \in D_\delta, \delta < 1. \tag{49}$$

Furthermore for any $\epsilon > 0$, Lemma 4 part (i) informs us that the series in (49) is bounded by a constant E_ϵ'' for $x \in D_{\epsilon,\epsilon}$ where the bound depends on ϵ. Moreover, part (ii) of the same lemma provides the estimate $C \log |1 - x|^{-1}$ valid for all $x \in D_\delta$, $0 < \delta < 1$.

Lemma 9. *Let $f(x)$ be* aperiodic. *There exists δ positive and small such that $|\psi'(x)|$ is uniformly bounded for $x \in S - D_\delta$ and*

$$|\psi'(x)| \leqq C \log |1 - x|^{-1} \qquad \text{for } x \in D_\delta. \tag{50}$$

Proof. Let δ be chosen sufficiently small such that (49) is valid for $x \in D_\delta$ and consequently (24) holds. Corresponding to δ there exists ϵ (by Lemma 8) such that $f(x) \in S - \mathfrak{A}_\epsilon$ for all $x \in S - D_\delta$. Moreover, we can suppose that ϵ is so small that $f(x) \notin D_\epsilon$ for $x \in S - D_\delta$.

We know by Lemma 4 that

$$|\psi'(x)| \leqq E_\epsilon' \qquad \text{for all } x \in D_\delta - D_\epsilon. \tag{51}$$

Now $\psi(x)$ satisfies (37) and therefore

$$f'(x)\psi'(f(x)) + \theta'(x) = \psi'(x) \qquad \text{for all } |x| < 1. \tag{52}$$

Since $f(x)$ is *aperiodic* it is clear from the definition that $\theta'(x)$ is bounded for all $x \in S - D_\epsilon$. Let $D_\delta^{(1)}$ denote the set of points in $S - D_\delta$ which are mapped by $f(x)$ into D_δ. By virtue of the specification of ϵ, we have $f(D_\delta^{(1)}) \subset D_\delta - D_\epsilon$. Both terms on the left in (52) are absolutely uniformly bounded on $D_\delta^{(1)}$ since for these x values $f(x) \in D_\delta - D_\epsilon$ and (51) applies. Therefore $|\psi'(x)|$ is uniformly bounded on $D_\delta^{(1)}$. Next consider the set $D_\delta^{(2)}$ in S which f maps into $D_\delta^{(1)}$. A parallel argument as above demonstrates that $|\psi'(x)|$ is uniformly bounded on $D_\delta^{(2)}$. Obviously, any $x \in S - D_\delta$ is mapped by some $f_k(x)$, $k = 1, 2, \cdots, N$, (N is a suitable fixed integer) for a first time into $D_\delta - D_\epsilon$. It follows from suitable repetition of the above analysis that $|\psi'(x)|$ is uniformly bounded on $S - D_\delta$ as asserted in the lemma. The proof is complete.

Lemma 10. *If $f(x)$ is* aperiodic *then the function $\psi'(x)$ is of class H^2 in the unit circle,* i.e.,

$$\int_0^{2\pi} |\psi'(re^{i\theta})|^2 \, d\theta \leqq C^*, \qquad 0 \leqq r < 1, \tag{53}$$

where C^* is a bound independent of r.

Proof. The estimates described in Lemma 9 clearly suffice to establish (53).

Lemma 11. *Let $\psi(x) = \sum_{k=0}^{\infty} \psi_k x^k$ denote the power series expansion of $\psi(x)$ about the origin. Then $\psi_k = (1/k)e_k$, $k = 1, 2, \cdots$, where $\sum_{k=0}^{\infty} |e_k|^2 < \infty$.*

PROOF. For $0 < r < 1$ we have by Lemma 9

$$2\pi \sum_{k=1}^{\infty} |k\psi_k|^2 r^{2k} = \int_0^{2\pi} |\psi'(re^{i\theta})|^2 \, d\theta \leqq C^*.$$

Since C^* does not depend on r we obtain, by letting $r \to 1-$,

$$2\pi \sum_{k=1}^{\infty} |k\psi_k|^2 \leqq C^*$$

and this proves this result.

PROOF OF THEOREM 4. By Theorem 2 we have

$$A(x)/a = \pi(x) = (1/a) \cdot 1/(1-x) + (c/a) \log (1-x) + (1/a)\psi(x) + d_0$$

and therefore

$$\pi'(x) = (1/a)/(1-x)^2 - (c/a)/(1-x) + (1/a)\psi'(x).$$

Comparing coefficients on both sides yields

$$\pi_j = 1/a - (c/a)/j + e_j/j, \qquad j = 1, 2, \cdots.$$

The proof of Theorem 4 is complete.

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