ON ESTIMATION OF THE MODE

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1. Summary. Let $Y_1, \cdots, Y_n$ be an ordered sample from a density with mode θ. We propose to estimate θ by suitable points in the interval formed by the first and the last of those s consecutive Y_i's which are closest together. Choices of s which yield consistency of these estimates, the speed of convergence and asymptotic distributions are discussed in this paper.

2. Introduction. Let f be a density on the real line. The following will be assumed without further statement:

(i) For some known constants a and b with $-\infty \leqq a < b \leqq +\infty$, $f(x) > 0$ if $a < x < b$ and $f(x) = 0$ otherwise;

(ii) f is continuous over (a, b);

(iii) f achieves its maximum at the unique point θ with $a < \theta < b$.

We write F for the distribution function corresponding to f.

Let $Y_1, \cdots, Y_n$ be an ordered independent sample from the density f. Intuitively speaking θ can be estimated by that point around which the greatest "clustering" of observations occurs. Depending on how this is measured various estimates of θ can be constructed. One possibility is the midpoint of that interval of length $2a$ containing the most observations. This is discussed by Chernoff [3]. Here we study the following alternative: Let $\{r_n\}$ be a sequence of integers to be specified further below. Write

$$V_j = Y_{j+r_n} - Y_{j-r_n}, \qquad j = r_n + 1, r_n + 2, \cdots, n - r_n, \tag{2.1}$$

and define K_n by

$$V_{K_n} = \min \{V_j : r_n + 1 \leqq j \leqq n - r_n\}. \tag{2.2}$$

Then two estimates of θ will be considered, viz.

$$\theta_{1n} = \tfrac{1}{2}\{Y_{K_n+r_n} + Y_{K_n-r_n}\} \quad \text{and} \quad \theta_{2n} = Y_{K_n}. \tag{2.3}$$

Theorem 1 of Section 3 gives conditions under which these estimates are strongly consistent, Theorem 2 discusses the speed of convergence for some choices of $\{r_n\}$ and Theorem 3 studies corresponding asymptotic distributions.

We remark that these estimates of the mode are related to that of Grenander [4] in that they are also based on the sample spacings—here different functionals of these spacings are being used.

3. Consistency. $F(Y_1), \cdots, F(Y_n)$ may be thought of as the order statistics of an independent sample from the distribution uniform on [0, 1] and it is well-known that they can be represented as

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$$F(Y_i) = S_i/S_{n+1} \qquad i = 1, \cdots, n, \tag{3.1}$$

where

$$S_i = Z_1 + \cdots + Z_i, \qquad i = 1, \cdots, n+1, \tag{3.2}$$

with $Z_1, \cdots, Z_{n+1}$ independent random variables each exponentially distributed with densities $\exp(-z)$ for $z \geqq 0$ and 0 for $z < 0$. Hence, writing $G = F^{-1}$,

$$Y_i = G(S_i/S_{n+1}), \qquad i = 1, \cdots, n. \tag{3.3}$$

Let $\delta > 0$ and write

$$\alpha_1(\delta) = \min\{f(x) : \theta - \delta \leqq x \leqq \theta + \delta\}, \tag{3.4}$$

$$\alpha_2(\delta) = \max\{f(x) : a < x \leqq \theta - 2\delta, \theta + 2\delta \leqq x < b\}, \tag{3.5}$$

$$\alpha(\delta) = \alpha_1(\delta)/\alpha_2(\delta). \tag{3.6}$$

THEOREM 1. *Suppose the following conditions hold.*

$$\text{For all } \delta \text{ small enough } \alpha(\delta) > 1. \tag{3.7}$$

$$n^{-1}r_n \to 0 \text{ as } n \to \infty. \tag{3.8}$$

$$\text{For all } \lambda \text{ with } 0 < \lambda < 1, \ \sum n\lambda^{r_n} < \infty. \tag{3.9}$$

Then $\theta_{1n}, \theta_{2n} \to \theta$ *with probability one* (wp 1).

PROOF. Write $[z]$ for the greatest integer not larger than z. Let $n^{-1}(r_n + 1) \leqq p \leqq 1 - n^{-1}r_n$. Then from (2.1), (3.3) and Taylor's theorem

$$V_{[np]} = (S_{[np]+r_n} - S_{[np]-r_n})S_{n+1}^{-1}G'(\phi_n(p)) \tag{3.10}$$

where

$$S_{[np]-r_n}S_{n+1}^{-1} \leqq \phi_n(p) \leqq S_{[np]+r_n}S_{n+1}^{-1}. \tag{3.11}$$

We will show that wp 1

$$\phi_n(p) \to p \text{ uniformly in } p. \tag{3.12}$$

For this it will suffice to show that wp 1

$$S_{[np]+r_n}S_{n+1}^{-1} \to p \text{ uniformly in } p, \tag{3.13}$$

$$S_{[np]-r_n}S_{n+1}^{-1} \to p \text{ uniformly in } p. \tag{3.14}$$

Since

$$\begin{aligned} S_{[np]+r_n}S_{n+1}^{-1} - p \\ = \{S_{[np]+r_n} - [np] - r_n) + ([np] - np) + p(n - S_{n+1}^{-1}) + r_n\}S_{n+1}^{-1} \end{aligned} \tag{3.15}$$

and by the strong law of large numbers $n^{-1}S_{n+1} \to 1$ wp 1, it will, in view of (3.8), suffice for (3.13) to show that wp 1

$$n^{-1}(S_{[np]+r_n} - [np] - r_n) \to 0 \quad \text{uniformly in} \quad p. \tag{3.16}$$

We have

$$\begin{aligned} P_n &= P\{\sup_p |S_{[np]+r_n} - [np] - r_n| > n\epsilon\} \\ &= P\{\text{for some } \ j,\, r_n + 1 \leqq j \leqq n - r_n\,,\ |S_j - j| > n\epsilon\} \\ &\leqq \textstyle\sum_1^n P(S_j > j + n\epsilon) + \sum_1^n P(S_j < j - n\epsilon). \end{aligned} \tag{3.17}$$

For any random variable X and any constant a,

$$P(X > a) \leqq E \exp\{t(X - a)\}, \qquad t > 0, \tag{3.18}$$

provided that this expectation exists. Applying this inequality to S_j which has density $x^{j-1}e^{-x}/\Gamma(j)$ for $x \geqq 0$ and 0 otherwise, we get

$$P(S_j > j + n\epsilon) \leqq (1 - t)^{-j}e^{-t(j+n\epsilon)}, \qquad 0 < t < 1. \tag{3.19}$$

Summing over j between 1 and n, we find that the first sum on the right in (3.17) is bounded by $h\lambda^n$ where

$$h = \{1 - (1 - t)e^t\}^{-1}, \qquad \lambda = (1 - t)^{-1}e^{-t(1+\epsilon)}. \tag{3.20}$$

Now t can be chosen small enough so that $\lambda < 1$. A similar bound for the second sum in (3.17) can be derived by applying (3.18) to $-S_j$. It follows that $\sum P_n < \infty$ and the Borel-Cantelli lemma implies that (3.16) holds. (3.13) follows and (3.14) is proved analogously. (3.12) therefore holds.

Let $q = F(\theta)$ and choose p such that $n^{-1}(r_n + 1) \leqq p \leqq F(\theta - 3\delta)$ or $F(\theta + 3\delta) \leqq p \leqq 1 - n^{-1}r_n$. Then, from (3.10),

$$V_{[np]}V_{[nq]}^{-1} = (S_{[np]+r_n} - S_{[np]-r_n})(S_{[nq]+r_n} - S_{[nq]-r_n})^{-1} \cdot G'(\phi_n(p))/G'(\phi_n(q)). \tag{3.21}$$

From (3.12), wp 1 there exists n_0 not depending on p such that for all $n > n_0$, $\phi_n(p) \leqq F(\theta - 2\delta)$ or $\phi_n(p) \geqq F(\theta + 2\delta)$ and $F(\theta - \delta) \leqq \phi_n(q) \leqq F(\theta + \delta)$. Hence, for $n > n_0$

$$G'(\phi_n(p))/G'(\phi_n(q)) = f(G(\phi_n(q)))/f(G(\phi_n(p))) \geqq \alpha(\delta). \tag{3.22}$$

Next we show that wp 1

$$(S_{[np]+r_n} - S_{[np]-r_n})/2r_n \to 1 \text{ uniformly in } p. \tag{3.23}$$

We have

$$\begin{aligned} &P\{\sup_p (S_{[np]+r_n} - S_{[np]-r_n}) > 2r_n(1 + \epsilon)\} \\ &\qquad \leqq \textstyle\sum P\{S_{j+r_n} - S_{j-r_n} > 2r_n(1 + \epsilon)\} \\ &\qquad \leqq nP\{S_{2r_n} > 2r_n(1 + \epsilon)\} \\ &\qquad \leqq n\lambda^{2r_n}. \end{aligned} \tag{3.24}$$

A similar inequality can be derived for $P\{\inf_p (S_{[np]+r_n} - S_{[np]-r_n}) < 2r_n(1 - \epsilon)\}$ and (3.23) follows from the Borel-Cantelli lemma in view of (3.9). Hence, wp 1 there exists $n_1 \geqq n_0$ not depending on p, such that for all $n > n_1$

$$(S_{[np]+r_n} - S_{[np]-r_n})(S_{[nq]+r_n} - S_{[nq]-r_n})^{-1} > \alpha(\delta)^{-\frac{1}{2}},$$

and from (3.21), (3.22) and (3.7) it follows that

$$V_{[np]}V_{[nq]}^{-1} > \alpha(\delta)^{\frac{1}{2}} > 1, \tag{3.25}$$

so that, by definition of K_n,

$$[nF(\theta - 3\delta)] < K_n < [nF(\theta + 3\delta)].$$

Since δ may be arbitrarily small, it follows that wp 1

$$n^{-1}K_n \to q. \tag{3.26}$$

It is now easy to show that wp 1 $Y_{K_n - r_n}$, Y_{K_n}, $Y_{K_n + r_n} \to \theta$, and the theorem follows.

We remark in passing that if $\{r_n\}$ is chosen so that $r_n \sim An^{\nu}$ with $0 < \nu < 1$ and $A > 0$, then (3.8) and (3.9) hold.

4. Speed of convergence. Refinement of the analysis of Section 3 yields results on the speed of convergence of θ_{1n} and θ_{2n} to θ. This depends on the smoothness of f near θ. Roughly speaking, the more pronounced the mode the better the speed of convergence (if $\{r_n\}$ is suitably chosen). We give a general result in the next theorem and then consider special cases.

THEOREM 2. *Suppose the following conditions hold.*

(4.1) *For all δ small enough $\alpha(\delta) \geqq 1 + \rho\delta^k$ where ρ and k are positive constants.*

$$r_n = An^{2k/(1+2k)}, \quad \text{if} \quad k \geqq \tfrac{1}{2}, \qquad = An^k, \quad \text{if} \quad k < \tfrac{1}{2}, \tag{4.2}$$

with A a positive constant.

Then, wp 1, *as $n \to \infty$,*

$$\theta_{1n}, \theta_{2n} = \theta + o(\delta_n) \tag{4.3}$$

where

$$\delta_n = n^{-1/(1+2k)}(\log n)^{1/k}, \quad \text{if} \quad k \geqq \tfrac{1}{2}, \qquad = n^{-\frac{1}{2}}(\log n)^{1/k}, \quad \text{if} \quad k < \tfrac{1}{2}. \tag{4.4}$$

PROOF. First we show that wp 1

$$n^{-1}K_n = q + o(\delta_n). \tag{4.5}$$

For this purpose we need strengthened versions of (3.13), (3.14) and (3.23). We show that wp 1

$$S_{[np]+r_n}S_{n+1}^{-1} - p = o(\delta_n) \text{ uniformly in } p, \tag{4.6}$$

(4.7) $$S_{[np]-r_n}S_{n+1}^{-1} - p = o(\delta_n) \text{ uniformly in } p,$$

(4.8) $$(S_{[np]+r_n} - S_{[np]-r_n})/2r_n = 1 + o(\delta_n^{\ k}) \text{ uniformly in } p.$$

Taking $t = \epsilon/(1 + \epsilon)$ in (3.20) one finds that for all ϵ small enough $h \leqq 4/\epsilon^2$ and $\lambda \leqq 1 - \epsilon^2/4 \leqq \exp(-\epsilon^2/4)$. Hence, with ϵ replaced by $\epsilon n^{-\frac{1}{2}} \log n$ in (3.17) we obtain as bound for the first sum in (3.17).

(4.9) $$h\lambda^n \leqq 4\epsilon^{-2} n(\log n)^{-2} \exp\{-\epsilon^2(\log n)^2/4\}$$

so that $\sum h\lambda^n < \infty$. Similarly for the second sum in (3.17). Hence, from (3.15), wp 1

(4.10) $$S_{[np]+r_n}S_{n+1}^{-1} - p = n^{-1}r_n + o(n^{-\frac{1}{2}} \log n) \text{ uniformly in } p,$$

and since (4.2) and (4.4) imply that

(4.11) $$n^{-\frac{1}{2}} \log n = o(\delta_n) \quad \text{and} \quad n^{-1}r_n = o(\delta_n),$$

(4.6) follows. (4.7) is proved similarly. Further, replacing ϵ by $\epsilon r_n^{-\frac{1}{2}} \log r_n$ in (3.24) and taking t as above, we get

$$\lambda^{2r_n} \leqq \exp\{-\epsilon^2(\log r_n)^2/2\} \cong c_1 n^{-c_2 \log n}, \qquad c_1, c_2 > 0.$$

It follows that $\sum n\lambda^{2r_n} < \infty$. Hence wp 1

(4.12) $$(S_{[np]+r_n} - S_{[np]-r_n})/2r_n = 1 + o(r_n^{-\frac{1}{2}} \log r_n) \text{ uniformly in } p.$$

Since (4.2) and (4.4) imply that

(4.13) $$r_n^{-\frac{1}{2}} \log r_n \sim c_3 \delta_n^{\ k}, \qquad c_3 > 0,$$

(4.8) follows. From (4.6) and (4.7), according to (3.11), wp 1

(4.14) $$\phi_n(p) = p + o(\delta_n) \text{ uniformly in } p.$$

Now choose $\epsilon > 0$. Then by (4.8), (4.14), (3.4), (3.5), (3.6) and (4.1) applied in (3.21), wp 1 there exists n_0 not depending on p such that for all $n > n_0$

$$V_{[np]}V_{[nq]}^{-1} \geqq \{1 + o(\delta_n^{\ k})\}\{1 + \rho\epsilon^k \delta_n^{\ k}\} > 1$$

whenever $n^{-1}(r_n + 1) \leqq p \leqq F(\theta - 3\epsilon\delta_n)$ or $F(\theta + 3\epsilon\delta_n) \leqq p \leqq 1 - n^{-1}r_n$. Hence, wp 1, for all n large enough

$$[nF(\theta - 3\epsilon\delta_n)] < K_n < [nF(\theta + 3\epsilon\delta_n)].$$

Since $\delta_n^{-1}\{F(\theta \pm 3\epsilon\delta_n) - q\} \to \pm 3\epsilon/f(\theta)$ as $n \to \infty$ and ϵ may be arbitrarily small, (4.5) follows.

Next we show that wp 1

(4.15) $$Y_{K_n \pm r_n} = \theta + o(\delta_n)$$

from which the theorem will evidently follow.

Choose $\epsilon > 0$. Then, wp 1, for all n large enough

$$Y_{[n(q-\epsilon\delta_n)]} < Y_{K_n-r_n} < Y_{K_n+r_n} < Y_{[n(q+\epsilon\delta)]} .$$

It will therefore suffice for (4.15) to show that wp 1 for all n large enough

$$\theta - 2\epsilon\delta_n/f(\theta) < Y_{[n(q-\epsilon\delta_n)]} < Y_{[n(q+\epsilon\delta_n)]} < \theta + 2\epsilon\delta_n/f(\theta). \tag{4.16}$$

For the first of these inequality it suffices to show that

$$\textstyle\sum_n P\{Y_{[n(q-\epsilon\delta_n)]} \leqq \theta - 2\epsilon\delta_n/f(\theta)\} < \infty. \tag{4.17}$$

The probability appearing as the nth term of this series is also $P\{B(n, z_n) \geqq [n(q - \epsilon\delta)]\}$ where $B(n, z_n)$ is a binomial random variable with parameters n and $z_n = F(\theta - 2\epsilon\delta_n/f(\theta)) = q - 2\epsilon\delta_n(1 + o(1))$ as $n \to \infty$. Using the inequality of Bernstein as in [1], one can show that (4.17) follows. Similarly for the last inequality in (4.14).

Remarks. With the approach used in the proof above, it seems unlikely that the speed of convergence found can be improved significantly by choosing $\{r_n\}$ differently. For, one can hardly do better than (4.12) and as (4.8) is required one must have at least $\delta_n^{\,k} = O(r_n^{-\frac{1}{2}} \log r_n)$ which, for $\{r_n\}$ of the form $r_n \sim An^{\nu}$ implies $\delta_n = O(n^{-\nu/2k}(\log n)^{1/k})$. We also need (4.11) for (4.6) and (4.7) and this forces the inequalities $\nu \leqq 2k/(1 + 2k)$ and $\nu \leqq k$. Within these restrictions the best result is obtained by choosing ν as in the theorem.

Special cases. We assume here that for all δ small enough $\min \{f(x): \theta - \delta \leqq x \leqq \theta + \delta\}$ equals either $f(\theta + \delta)$ or $f(\theta - \delta)$ and that $\max \{f(x): a < x \leqq \theta - 2\delta, \theta + 2\delta \leqq x < b\}$ equals either $f(\theta + 2\delta)$ or $f(\theta - 2\delta)$.

Case (i). Suppose f satisfies

$$f(x) = \gamma_0 - \tfrac{1}{2}\gamma(x - \theta)^2 + o(|x - \theta|^2) \quad \text{as} \quad x \to \theta; \qquad \gamma_0, \gamma > 0. \tag{4.18}$$

Then (4.1) holds with $k = 2$ and (4.2) specifies $r_n = An^{4/5}$ while according to (4.3) and (4.4) $\theta_{1n}, \theta_{2n} = \theta + o(n^{-1/5}(\log n)^{\frac{1}{2}})$.

Case (ii). Suppose f satisfies

$$\begin{aligned} f(x) &= \gamma_0 - \gamma_1(x - \theta) + o(|x - \theta|) \quad \text{for} \quad x \geqq \theta, \quad x \to \theta, \\ &= \gamma_0 - \gamma_2(\theta - x) + o(|x - \theta|) \quad \text{for} \quad x < \theta, \quad x \to \theta, \end{aligned} \tag{4.19}$$

with $\gamma_0, \gamma_1, \gamma_2 > 0$. Then (4.1) holds with $k = 1$ and (4.2) requires $r_n = An^{\frac{2}{3}}$ while $\theta_{1n}, \theta_{2n} = \theta + o(n^{-\frac{1}{3}} \log n)$.

5. Asymptotic distributions. Because it is somewhat simpler to deal with we will restrict attention here to $\theta_{2n} = Y_{K_n}$. Until further conditions are specified, assume that the conditions of Theorem 1 hold. Then

$$\theta_{2n} - \theta = G(S_{K_n}/S_{n+1}) - G(q) = (S_{K_n}/S_{n+1} - q)G'(\Psi_n) \tag{5.1}$$

with Ψ_n a point closer to q than S_{K_n}/S_{n+1} so that $\Psi_n \to q$ wp 1. Further

$$(5.2)\quad S_{K_n}/S_{n+1} - q = (S_{K_n} - K_n)/S_{n+1} + (K_n - nq)/S_{n+1} + q(n - S_{n+1})/S_{n+1},$$

$$(5.3)\quad (S_{K_n} - K_n)/S_{n+1} = n^{-\frac{1}{2}}\{(S_{K_n} - K_n)/K_n^{\frac{1}{2}}\}\{S_{n+1}/n\}^{-1}\{K_n/n\}^{\frac{1}{2}}.$$

By (3.26) and the strong law of large numbers the last two factors in (5.3) tend to $q^{\frac{1}{2}}$ wp1 and by the central limit theorem for a random number of summands, [2], the third last factor tends in law to a $N(0, 1)$ distribution. Hence

$$(5.4)\quad (S_{K_n} - K_n)/S_{n+1} = O_P(n^{-\frac{1}{2}}).$$

A similar result holds for the last term in (5.2). Hence

$$(5.5)\quad S_{K_n}/S_{n+1} - q = O_P(n^{-\frac{1}{2}}) + (K_n - nq)/S_{n+1}.$$

Suppose that $\{a_n\}$ is a sequence of positive numbers satisfying

$$(5.6)\quad n^{-\frac{1}{2}}a_n \to 0 \quad \text{and} \quad a_n \to \infty \quad \text{as} \quad n \to \infty.$$

Then multiplication of (5.5) by a_n and substitution into (5.1) readily shows that if $a_n(n^{-1}K_n - q)$ has a limiting distribution then $a_n f(\theta)(\theta_{2n} - \theta)$ has the same limiting distribution, which we will characterize below for some situations. The analysis to follow is largely motivated by that of Chernoff, [3], to whom we will refer for some finer details.

To begin with, suppose f satisfies (4.18). Then, for $p \to q$ we have

$$(5.7)\quad G(p) = G(q) + \gamma_0^{-1}(p - q) + \tfrac{1}{6}(p - q)^3\{\gamma\gamma_0^{-3} + o(1)\}.$$

Select $\{r_n\}$ so that

$$(5.8)\quad r_n \sim An^{4/5}$$

and write

$$(5.9)\quad U_n = 2^{\frac{1}{3}}n^{\frac{1}{3}}r_n^{-\frac{2}{3}}\gamma^{-\frac{2}{3}}\gamma_0^{2}, \qquad z_n = U_n^{-1}(n^{-1}K_n - q).$$

Then, by definition of K_n, z_n is a choice of the real variable z which minimizes $V_{[n(q+U_nz)]}$ and hence also

$$(5.10)\quad (2U_nn)^{-\frac{1}{2}}S_{n+1}\gamma_0\{V_{[n(q+U_nz)]} - V_{[nq]}\}.$$

From (4.10) we get

$$(5.11)\quad S_{[n(q+U_nz)]+r_n}/S_{n+1} = q + U_n z + n^{-1}r_n + o(n^{-\frac{1}{2}}\log n)$$

and similar expressions hold for $S_{[n(q+U_nz)]-r_n}/S_{n+1}$, $S_{[nq]+r_n}/S_{n+1}$ and $S_{[nq]-r_n}/S_{n+1}$. Substituting for the V's in terms of the Y's, for these in terms of the S's and expanding according to (5.7), (5.10) becomes, after some simplification,

$$(5.12)\quad Z_n(z) + R_n(z)$$

where

$$(5.13)\quad Z_n(z) = (2U_nn)^{-\frac{1}{2}}\{(S_{[n(q+U_nz)]+r_n} - S_{[n(q+U_nz)]-r_n}) - (S_{[nq]+r_n} - S_{[nq]-r_n})\}$$

and $R_n(z)$ is the term which results from the third order terms in this expansion. Using (5.11) and its equivalents, we find

$$
\begin{aligned}
R_n(z) \sim \tfrac{1}{6}\gamma\gamma_0^{-3}(2U_n)^{-\frac{1}{2}}n^{\frac{1}{2}}\{&(n^{-1}r_n + U_n z + o(n^{-\frac{1}{2}} \log n))^3(1 + o(1)) \\
&- (-n^{-1}r_n + U_n z + o(n^{-\frac{1}{2}} \log n))^3(1 + o(1)) \\
&- (n^{-1}r_n + o(n^{-\frac{1}{2}} \log n))^3(1 + o(1)) \\
&+ (-n^{-1}r_n + o(n^{-\frac{1}{2}} \log n))^3(1 + o(1))\}.
\end{aligned} \tag{5.14}
$$

The leading term in $\{\cdot\}$ on the right here is found to be $6n^{-1}r_nU_n^2z^2(1 + o(1))$ and it follows from (5.9) and (5.8) that

$$R_n(z) = z^2(1 + o(1)) \text{ wp } 1 \text{ as } n \to \infty. \tag{5.15}$$

Now $Z_n(z)$ may be thought of as a stochastic process with parameter z. We have $EZ_n(z) = 0$ and a straight forward but tedious calculation from (5.13) shows that with

$$\lim_{n\to\infty} n^{-1}r_nU_n^{-1} = B = A^{5/3}2^{-\frac{1}{3}}\gamma^{\frac{2}{3}}\gamma_0^{-2}, \tag{5.16}$$

we have

$$
\begin{aligned}
&\operatorname{Cov}\{Z_n(z), Z_n(z^*)\} \\
&\qquad \to \tfrac{1}{2}\{\min(|z|, 2B) + \min(|z^*|, 2B) - \min(|z - z^*|, 2B)\}.
\end{aligned} \tag{5.17}
$$

In particular

$$\operatorname{Var}\{Z_n(z)\} \to \min(|z|, 2B). \tag{5.18}$$

Since $\{Z_n(z)\}$ is a suitable linear combination of independent random variables, it can be shown that $\{Z_n(z)\}$ tends in law to a Gaussian process with expectation 0 and covariance function given by the limit (5.17). This limiting process will be denoted by $\{Z(z)\}$.

We note that with $a_n = U_n^{-1}$ the conditions in (5.6) are satisfied. The analysis above leads one to expect that for large n $z_n = U_n^{-1}(n^{-1}K_n - q)$, and hence $U_n^{-1}\gamma_o(\theta_{2n} - \theta)$, is distributed as the variable $\mathbf{z}$ which minimizes the process $Z(z) + z^2$. The details of this limiting argument are similar to that outlined by Chernoff, [3], and need not be given here. We have therefore proved

THEOREM 3a. *If f satisfies* (4.18) *and* $\{r_n\}$ *is chosen according* (5.8) *then* $2^{-\frac{1}{3}}A^{\frac{2}{3}}\gamma^{\frac{2}{3}}\gamma_o^{-1}n^{1/5}(\theta_{2n} - \theta)$ *is asymptotically distributed as the variable* $\mathbf{z}$ *which minimizes the process* $\{Z(z) + z^2\}$ *where* $\{Z(z)\}$ *is a Gaussian process with expectation* 0 *and covariance function given by the limit in* (5.17).

Next we consider the effect of a slightly stronger assumption on f. Suppose that

$$f(x) = \gamma_o - \tfrac{1}{2}\gamma(x - \theta)^2 + \tfrac{1}{6}\gamma_3(x - \theta)^3 + o(|x - \theta|^3) \quad \text{as} \quad x \to \theta. \tag{5.19}$$

Then (5.7) can be strengthened accordingly. Select $\{r_n\}$ so that

$$r_n \sim An^{\nu} \quad \text{with} \quad \tfrac{4}{5} \leqq \nu < \tfrac{7}{8}. \tag{5.20}$$

With U_n and z_n as in (5.9) one again finds that z_n minimizes (5.12) with Z_n as in (5.13) and $R_n(z)$ consisting of third and fourth order terms in the expansion. Instead of (5.14) we now get

$$R_n(z) = z^2(1 + o(1)) + R_n^*(z) \tag{5.21}$$

with $R_n^*(z)$ an expression similar to the right hand side of (5.14) but involving fourth powers. Under (5.20) we find

$$R_n^*(z) = o(1) \text{ wp } 1. \tag{5.22}$$

While if, instead of (5.20) we have

$$r_n \sim An^{7/8} \tag{5.23}$$

then we find

$$R_n^*(z) = -3^{-1}2^{-\frac{1}{3}}A^{\frac{8}{3}}\gamma^{-\frac{1}{3}}\gamma_o^{-4}\gamma_3\, z(1 + o(1)) = Cz(1 + o(1)), \text{ say.} \tag{5.24}$$

Also, under (5.20) and (5.23) $B = \infty$ in (5.16) and the limits in (5.17) and (5.18) become the covariance and variance functions of a two-sided Wiener-Levy process. Hence

THEOREM 3b. *If f satisfies* (5.19) *and $r_n \sim An^{\nu}$ with $\frac{4}{5} \leqq \nu \leqq \frac{7}{8}$ then $2^{-\frac{1}{3}}A^{\frac{2}{3}}\gamma^{\frac{2}{3}}\cdot \gamma_o^{-1}n^{(2\nu-1)/3}(\theta_{2n} - \theta)$ is asymptotically distributed according to the distribution of* $\mathbf{z}$ *which minimizes $\{Z(z) + z^2\}$ if $\nu < \frac{7}{8}$ and $\{Z(z) + z^2 + Cz\}$ if $\nu = \frac{7}{8}$, where $\{Z(z)\}$ is a two-sided Wiener-Levy process with expectation* 0 *and unit variance per unit z.*

The distribution of $\mathbf{z}$ in the first of the two cases of this theorem has been studied by Chernoff, ([3], p. 37).

Finally, suppose that f satisfies (4.19). Then, for $p \to q$, we have

$$\begin{aligned} G(p) &= G(q) + \gamma_o^{-1}(p - q) + \tfrac{1}{2}\gamma_o^{-3}(p - q)^2(\gamma_1 + o(1)) \text{ if } p \geqq q \\ &= G(q) + \gamma_o^{-1}(p - q) + \tfrac{1}{2}\gamma_o^{-3}(p - q)^2(-\gamma_2 + o(1)) \text{ if } p < q. \end{aligned} \tag{5.25}$$

Let

$$r_n \sim An^{\frac{2}{3}}, \qquad U_n = 2\gamma_o^4 n r_n^{-2}, \qquad \lim U_n^{-1}r_n n^{-1} = D = \tfrac{1}{2}A^3\gamma_o^{-4} \tag{5.26}$$

and let z_n be as before. By (5.11), wp 1 for all n large enough,

$$S_{[n(q+U_n z)]+r_n}/S_{n+1} \gtreqless q \text{ according as } z + D \gtreqless 0. \tag{5.27}$$

Hence, using (5.25) and (5.11)

$$\begin{aligned} Y_{[n(q+U_n z)]+r_n} &= G(q) + \gamma_o^{-1}(S_{[n(q+U_n z)]+r_n}/S_{n+1} - q) \\ &\quad + \tfrac{1}{2}\gamma_o^{-3}U_n^2(z + D(1 + o(1)))^2(\gamma_1 + o(1)) \quad \text{if } z + D \geqq 0 \\ &= G(q) + \gamma_o^{-1}(S_{[n(q+U_n z)]+r_n}/S_{n+1} - q) \\ &\quad + \tfrac{1}{2}\gamma_o^{-3}U_n^2(z + D(1 + o(1)))^2(-\gamma_2 + o(1)) \quad \text{if } z + D < 0 \end{aligned}$$

and similar expressions hold for other relevant quantities. Expanding (5.10) as before, we find that z_n minimizes $\{Z_n(z) + R_n^{**}(z)\}$ with Z_n as in (5.13) and $R_n^{**}(z)$ an expression consisting of the second order terms in this expansion. We find that

$$
\begin{aligned}
R_n^{**}(z) \to R(z) &= -2\gamma_2 z - \tfrac{1}{2}D(\gamma_1 + \gamma_2) && \text{if } z < -D \\
&= (\gamma_1 + \gamma_2)z^2/2D - z(\gamma_1 - \gamma_2) && \text{if } -D \leqq z < D \qquad (5.28)\\
&= 2\gamma_1 z - \tfrac{1}{2}D(\gamma_1 + \gamma_2) && \text{if } D \leqq z.
\end{aligned}
$$

Hence we have

THEOREM 3c. *If f satisfies* (4.19) *and $r_n \sim An^{\frac{2}{3}}$ then $\frac{1}{2}\gamma_o^{-4}A^2 n^{\frac{1}{3}}(\theta_{2n} - \theta)$ has asymptotic distribution equal to that of* $\mathbf{z}$ *which minimizes $\{Z(z) + R(z)\}$ with $\{Z(z)\}$ a Gaussian process with expectation* 0 *and covariance function given by* (5.17) *with $B = D$ and $R(z)$ given by* (5.28).

It is evident that similar results can be obtained for other situations.

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