SOME REMARKS ON CONTINUOUS ADDITIVE FUNCTIONALS[1]

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1. Introduction. In recent years several authors have introduced the concept of "process on the boundary" or, equivalently, "a local time" for a closed set. See, for example, [3], [5], [6], [7], [8], and [2], V-4. If X is a standard process and D is a subset of the state space E for X, then we define a *local time* for X on D to be any nonnegative continuous additive functional (CAF) A of X whose fine support is D and such that for each $t < \infty$, $A_t < \infty$ almost surely. If D consists of a single point x_0, then it is known [2], V-3, that a local time exists if and only if x_0 is regular for D and that a local time is unique up to a multiplicative constant when it exists. For a general set D a necessary condition for a local time to exist is that D be a finely closed nearly Borel set and that each point in D be regular for D. If, in addition, $\bar{D} - D$ is polar, then a local time does indeed exist. See [5] or [2], V-4. The purpose of this note is to discuss to what extent a local time is unique in the general case. The situation is roughly as follows: Suppose A and B are two local times for X on D and let τ and σ denote the inverse functionals of A and B respectively. Subject to secondary assumptions it is easy to see that $Y_t = X(\tau_t)$ and $Z_t = X(\sigma_t)$ have the same hitting distributions. It then follows from a theorem of Blumenthal, Getoor, and McKean [1] (see also [2], V-5) that Y can be "time changed" into Z and this induces a time change between A and B. Thus a local time for X on D is unique up to a (continuous) time change. We will make this precise and give a very simple proof which avoids using the (rather deep) result of Blumenthal, Getoor, and McKean. Section 2 contains some preliminary remarks and the main result appears in Section 3.

2. Preliminaries. Let $X = (\Omega, \mathfrak{F}, \mathfrak{F}_t, X_t, \theta_t, P^x)$ be a standard process with state space $(E, \mathcal{E})$. See [2] for terminology and notation. In particular, if A is a continuous additive functional, CAF, of X, then Supp (A) denotes the fine support of A (see [2], V-3). The following result is an easy consequence of known techniques and hence we will only sketch its proof.

(2.1) Proposition. *Let A and B be* CAF's *of X with bounded α-potentials for some $\alpha \geqq 0$ and suppose that* Supp $(B) \subset$ Supp (A). *Then for each initial (probability) measure, μ, there exists a sequence $\{g_n\}$ of nonnegative bounded nearly Borel measurable functions such that if $B^n(t) = \int_0^t g_n(X_s)\, dA_s$ then almost surely P^μ, $B^n(\cdot) \to B(\cdot)$ on $[0, \infty)$, the convergence being uniform on each compact subinterval.*

Proof. For simplicity we assume that $\alpha = 1$. Let $f(x) = u_B^1(x) = E^x \int_0^\infty e^{-t}\, dB_t$ be the one potential of B. Let τ be the functional inverse to A. (We use τ_t and

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$\tau(t)$ interchangeably, similarly A_t and $A(t)$, X_t and $X(t)$.) Define

$$f_n(x) = n[f(x) - E^x\{e^{-\tau(1/n)}f(X_{\tau(1/n)})\}].$$

Then f_n is nearly Borel measurable, bounded by f, and nonnegative since f is a bounded one excessive function. Moreover

$$\begin{aligned} U_A^1 f_n(x) &= E^x \int_0^\infty e^{-t} f_n(X_t)\, dA_t \\ &= E^x \int_0^\infty e^{-\tau(t)} f_n[X_{\tau(t)}]\, dt \\ &= n \int_0^{1/n} E^x\{e^{-\tau(t)} f[X_{\tau(t)}]\}\, dt \\ &\uparrow E^x\{e^{-\tau(0)} f[X_{\tau(0)}]\} \text{ as } n \to \infty. \end{aligned}$$

But $f = u_B^1$ and Supp $(B) \subset$ Supp (A); consequently this limit is just $f(x)$. (Recall that τ_0 is the hitting time of Supp (A) almost surely). Therefore, $U_A^1 f_n \uparrow f$ as $n \to \infty$. Now fix μ and let $Y_t^n = e^{-t} U_A^1 f_n(X_t)$. Clearly $\{Y_t^n, \mathfrak{F}_t\ P^\mu\}$ is a bounded potential in the sense of Meyer [4]. If

$$A_t^n = \int_0^t e^{-s} f_n(X_s)\, dA_s$$

then

$$E^\mu(A_\infty^n \mid \mathfrak{F}_t) = A_t^n + Y_t^n,$$

and so (Y_t^n) is the potential generated by the continuous increasing process (A_t^n). On the other hand $Y_t^n \uparrow e^{-t} u_B^1(X_t)$ which is the potential generated by $B_t^* = \int_0^t e^{-s}\, dB_s$. It now follows [4], VII-T36, that

$$E^\mu\{(A_\infty^n - B_\infty^*)^2\} \to 0$$

as $n \to \infty$, and this in turn implies that there exists a subsequence $\{n_k\}$ such that almost surely P^μ

$$\lim_{k\to\infty} \sup_{0 \leq t \leq \infty} |A_t^{n_k} - B_t^*| = 0 \tag{2.2}$$

(see the proof of (VII-T37) in [4] or the proof of (IV-3.8) in [2]). As usual denote the sequence $\{n_k\}$ by $\{k\}$. Let $g_k = f_{n_k}$ and let

$$B^k(t) = \int_0^t e^s\, dA^k(s) = \int_0^t g_k(X_s)\, dA_s.$$

Integrating by parts we obtain

$$\begin{aligned} B^k(t) &= e^t A^k(t) - \int_0^t e^s A^k(s)\, ds, \\ B(t) &= e^t B^*(t) - \int_0^t e^s B^*(s)\, ds, \end{aligned}$$

and combining this with (2.2) yields the conclusion of (2.1).

(2.3) REMARKS. Note that for each μ the sequence $\{g_n\}$ is a subsequence of $\{f_n\}$. Consequently when X has a reference measure, [2], V-1.1, each g_n is Borel measurable. Clearly when $\alpha = 0$ the desired convergence is uniform on $[0, \infty]$. If $A_t = (t \wedge \zeta)$ and X has a reference measure, then it is known that $B = \sum C^k$ where each C^k is such that when (2.1) is applied to C^k the corresponding sequence of functions may be chosen independently of μ. See the proof of V-2.1 in [2].

Finally any CAF of X is a countable sum of CAF's with bounded one potentials [2], IV-2.21.

3. The main result. Let $X = (\Omega, \mathfrak{F}, \mathfrak{F}_t, X_t, \theta_t, P^x)$ be a standard process with state space $(E, \mathcal{E})$. We assume that X has a reference measure; consequently we may, and do, assume that any CAF of X is perfect and $\mathfrak{F}^0$ measurable. A CAF, A, of X is said to be *finite* if $A_t < \infty$ almost surely for each $t < \infty$. This is equivalent to requiring that almost surely $t \to A_t$ is finite on $[0, \infty)$. Let A be a fixed *finite* CAF of X. We assume, as we may, that $t \to A_t(\omega)$ is continuous and finite on $[0, \infty)$ for all ω and that $A_{t+s}(\omega) = A_t(\omega) + A_s(\theta_t\omega)$ for all finite t, s and all ω. Let $D = \text{Supp}(A)$ be the fine support of A and let τ be the inverse functional of A. Then D is a finely closed Borel set and each point in D is regular for D. The most important situation for applications is when D is, in addition, closed and, for convenience, *we assume that D is closed.* (This is not necessary, but it serves to simplify certain statements.) Let $\hat{X} = (\Omega, \mathfrak{F}, \mathfrak{F}_{\tau(t)}, X_{\tau(t)}, \theta_{\tau(t)}, P^x)$ denote the time changed process. We write $\hat{X}_t = X_{\tau(t)}$ and $\hat{\theta}_t = \theta_{\tau(t)}$. The symbols $\hat{\mathfrak{F}}_t^0$ and $\hat{\mathfrak{F}}_t$, have their usual meanings relative to $\hat{X}$. Note that $\tau(\infty) = \infty$ since $t \to A_t$ is finite on $[0, \infty)$. Under the present assumptions $\hat{X}$ may, and will, be regarded as a standard process with state space $(D, \mathcal{B}(D))$. Here $\mathcal{B}(D)$ is the σ-algebra of Borel subsets of D. Moreover $\hat{X}$ has a reference measure[2], V-4, and so we may, and do, assume that any CAF of $\hat{X}$ is perfect and $\hat{\mathfrak{F}}^0 = \sigma(\hat{X}_t; t \geqq 0)$ measurable. Since D is the state space of $\hat{X}$ the fine support of any CAF of $\hat{X}$ is a subset of D. Of course, when discussing $\hat{X}$, x is always assumed to be in $D \cup \{\Delta\}$. Observe that $\hat{\zeta} = A(\zeta)$ is the lifetime of $\hat{X}$. Finally it will sometimes be convenient to denote the time changed process by (X, A) as well as by $\hat{X}$ in order to display its dependence on A. If $\tau(t) < \infty$ then $A[\tau(t)] = t$. On the other hand if T is any $\{\mathfrak{F}_t\}$ stopping time, then $\tau[A(T)] = T$ almost surely on $\{X_T \varepsilon D\}$. See [2], V-3.43. More generally it is not difficult to check that

$$\tau[A(T)] = T + \tau_0 \circ \theta_T \tag{3.1}$$

provided $A(T) < \infty$. Since A is finite, $T = \infty$ if $A(T) = \infty$ and so (3.1) holds also on $\{A(T) = \infty\}$. The following lemma is the key observation of the present note.

(3.2) Lemma. *If Y is $\hat{\mathfrak{F}}^0 = \sigma(\hat{X}_s; s \geqq 0)$ measurable, then $Y \circ \hat{\theta}_{A(t)} = Y \circ \theta_t$ or all $t \varepsilon [0, \infty)$.*

Proof. By standard considerations it suffices to show that for fixed $t, s \varepsilon [0, \infty)$ one has $\hat{X}_s \circ \hat{\theta}_{A(t)} = \hat{X}_s \circ \theta_t$. But

$$\hat{X}_s \circ \hat{\theta}_{A(t)} = X_{\tau(s)} \circ \theta_{\tau[A(t)]} = X[\tau(A_t) + \tau(s) \circ \theta_{\tau(A_t)}],$$

and using (3.1) the argument of X in this last expression becomes

$$\begin{aligned} t + \tau_0 \circ \theta_t + \tau_s \circ \theta_{t+\tau_0 \circ \theta_t} &= t + \tau_0 \circ \theta_t + \tau_s \circ \theta_{\tau_0} \circ \theta_t \\ &= t + (\tau_0 + \tau_s \circ \theta_{\tau_0}) \circ \theta_t = t + \tau_s \circ \theta_t . \end{aligned}$$

Here we have used the fact that $\tau_0 + \tau_s \circ \theta_{\tau_0} = \tau_s$ holds as an identity (in ω) in view of our assumptions on A. Hence $\hat{X}_s \circ \hat{\theta}_{A(t)} = X[t + \tau_s \circ \theta_t] = X_{\tau(s)} \circ \theta_t = \hat{X}_s \circ \theta_t$, completing the proof of (3.2).

(3.3) PROPOSITION. *There is a one-to-one correspondence between* CAF's B *of* X *whose fine support is contained in* D *and* CAF's $\hat{B}$ *of* $\hat{X}$. *This correspondence is given by*

$$(3.4) \qquad B_t = \hat{B}_{A(t)}\,; \qquad \hat{B}_t = B_{\tau(t)}\,.$$

If B *and* $\hat{B}$ *correspond, then* $B_\zeta = \hat{B}_{\hat{\zeta}}$, Supp (B) = Supp $(\hat{B})$, *and* B *and* $\hat{B}$ *have the same potential in the sense that for nonnegative* f *vanishing off* D *and* x *in* D

$$\begin{aligned} U_B f(x) &= E^x \textstyle\int_0^\infty f(X_t)\, dB_t \\ &= E^x \textstyle\int_0^\infty f(\hat{X}_t)\, d\hat{B}_t = \hat{U}_{\hat{B}} f(x). \end{aligned}$$

PROOF. Suppose first of all that $\hat{B}$ is a CAF of $\hat{X}$ — $\hat{B}$ is assumed to be perfect and $\hat{\mathfrak{F}}^0$ measurable. Define $B_t = \hat{B}_{A(t)}$. Clearly $t \to B_t(\omega)$ is continuous for almost all ω. In view of the remarks (2.3) in order to show that B_t is $\mathfrak{F}_t$ measurable it will suffice to consider the case in which $\hat{B}_t = \int_0^t f(\hat{X}_s)\, ds$ where f is a bounded nonnegative continuous function vanishing off D. But in this case using [2], II-2.20, one obtains almost surely

$$B_t = \hat{B}_{A(t)} = \textstyle\int_0^{A(t)} f[\hat{X}_s]\, ds = \int_0^t f[X_{\tau(A_s)}]\, dA_s = \int_0^t f(X_s)\, dA_s\,,$$

since D is the fine support of A, $\tau(A_s) = s$ almost surely on $\{X_s \varepsilon D\}$, and $s \to f[X_{\tau(A_s)}]$ is right continuous. Consequently B_t is $\mathfrak{F}_t$ measurable. Now $\hat{B}_u$ is $\hat{\mathfrak{F}}^0$ measurable for each u and so by (3.2), $\hat{B}_u \circ \hat{\theta}_{A(t)} = \hat{B}_u \circ \theta_t$. Letting $u = A_s \circ \theta_t$ this yields

$$\hat{B}_{A_s \circ \theta_t}(\hat{\theta}_{A(t)}) = \hat{B}_{A_s \circ \theta_t}(\theta_t) = \hat{B}_{A_s \circ \theta_t}\,.$$

Therefore using the fact that both A and $\hat{B}$ are perfect we have almost surely

$$\begin{aligned} B_{t+s} = \hat{B}_{A_{t+s}} &= \hat{B}_{A_t + A_s \circ \theta_t} = \hat{B}_{A_t} + \hat{B}_{A_s \circ \theta_t}(\hat{\theta}_{A_t}) \\ &= \hat{B}_{A_t} + \hat{B}_{A_s} \circ \theta_t = B_t + B_s \circ \theta_t\,. \end{aligned}$$

Thus B is a CAF of X. Since Supp $(\hat{B})$ is contained in D = Supp (A), it is immediate that Supp (B) = Supp $(\hat{B})$. Also $B_\zeta = \hat{B}_{A(\zeta)} = \hat{B}_{\hat{\zeta}}$. Let $f \geqq 0$ be a continuous function vanishing off D and let x be in D. Then, as above,

$$\begin{aligned} U_B f(x) &= E^x \textstyle\int_0^\infty f(X_t)\, dB_t = E^x \int_0^\infty f[X_{\tau(A_t)}]\, d\hat{B}_{A_t} = E^x \int_0^{A(\infty)} f[X_{\tau(t)}]\, d\hat{B}_t \\ &= E^x \textstyle\int_0^\infty f(\hat{X}_t)\, d\hat{B}_t = \hat{U}_{\hat{B}} f(x), \end{aligned}$$

and so B and $\hat{B}$ have the same potential.

Suppose next that B is a CAF of X with Supp $(B) \subset D$. Define $\hat{B}_t = B_{\tau(t)}$. Clearly $t \to \hat{B}_t$ is right continuous and

$$\hat{B}_{t+s} = B_{\tau(t+s)} = B_{\tau(t)+\tau(s)\circ\theta_{\tau(t)}} = B_{\tau(t)} + B_{\tau(s)} \circ \theta_{\tau(t)} = \hat{B}_t + \hat{B}_s \circ \hat{\theta}_t\,.$$

Let φ be a strictly positive bounded function such that $A_t^* = \int_0^t \varphi(X_s)\, dA_s$ has a bounded one potential [2], IV-2.21. Since $\{\hat{\zeta} > t\} = \{\tau(t) < \infty\}$, $\hat{B}_t = B_\infty = B_\zeta$ on $\{\hat{\zeta} \leqq t\}$. Assume for the moment that $\hat{B}$ is continuous. Then in order to show that $\hat{B}_t$ is $\hat{\mathfrak{F}}_t$ measurable it suffices to show that $\hat{B}_t I_{[0,\hat{\zeta})}(t) = B_{\tau(t)} I_{[0,\infty)}(\tau_t)$ is $\hat{\mathfrak{F}}_t$ measurable. In view of (2.1) and (2.3) it suffices to consider the case in which $B_t = \int_0^t f(X_s)\, dA_s^* = \int_0^t \varphi(X_s) f(X_s)\, dA_s$ and hence the case $B_t = \int_0^t g(X_s)\, dA_s$ where g is a bounded nonnegative continuous function vanishing off D. But then if $\tau(t) < \infty$

$$B_{\tau(t)} = \int_0^{\tau(t)} g(X_s)\, dA_s = \int_0^t g(\hat{X}_s)\, ds$$

and hence $B_{\tau(t)} I_{[0,\infty)}(\tau_t)$ is $\hat{\mathfrak{F}}_t$ measurable. We must still show $t \to \hat{B}_t$ is actually continuous. If $\hat{B}$ is not continuous, then there exists an x in D and an $\epsilon > 0$ such that $P^x(T < \infty) > 0$ where $T = \inf\{t: \hat{B}_t - \hat{B}_{t-} > \epsilon\}$. But almost surely X_u is not in $D \supset \text{Supp}\,(B)$ for any $u \,\varepsilon\, (\tau(T-), \tau(T))$. Therefore almost surely $u \to B_u$ is constant on $[\tau(T-), \tau(T)]$. See [2], V-3.8. Now B is continuous and so $\hat{B}_{T-} = B_{\tau(T-)} = B_{\tau(T)} = \hat{B}_T$ almost surely on $\{T < \infty\}$ and this contradicts the fact that $P^x(T < \infty) > 0$. Thus $\hat{B}$ is a CAF of $\hat{X}$.

Finally to complete the proof we must show that the above correspondence is one-to-one. If $\hat{B}$ is a CAF of $\hat{X}$ and $B_t = \hat{B}_{A(t)}$, then $B_{\tau_t} = \hat{B}_{A(\tau_t)} = \hat{B}_t$ since $A(\tau_t) = t$ if $\tau_t < \infty$, while if $\tau_t = \infty$ then $t \geqq A(\zeta) = \hat{\zeta}$ and so $B_{\tau_t} = \hat{B}_{A(\infty)} = \hat{B}_t$. Conversely if B is a CAF of X with Supp $(B) \subset D$ and if $\hat{B}_t = B_{\tau(t)}$, then $\hat{B}_{A(t)} = B_{\tau(A_t)} = B_{t+\tau_0\circ\theta_t} = B_t + B_{\tau_0} \circ \theta_t$. But $B_{\tau_0} = 0$ almost surely since Supp $(B) \subset D$ and τ_0 is the hitting time of D. Thus, by continuity, almost surely $\hat{B}_{A(t)} = B_t$ for all t. This completes the proof of (3.2).

Remark. It is clear in view of our assumed normalizations of CAF's and the above proof that (3.4) holds only almost surely, that is, there exists Ω_0 with $P^x(\Omega_0) = 0$ for all x such that (3.4) holds for all t and $\omega \not\varepsilon \Omega_0$. The justification for writing (3.4) without the qualifying phrase is the usual identification of equivalent additive functionals.

The following corollary is the precise formulation of the statement in the introduction.

(3.5) Corollary. *Let A and B be finite* CAF's *of X with the same (closed) fine support D, that is, local times for X on D. Then (X, A) may be transformed into (X, B) by a continuous time change.*

Proof. Let τ and σ denote the functionals inverse to A and B respectively. By (3.3), $\hat{B}_t = B_{\tau(t)}$ defines a CAF of (X, A) whose fine support is D. As a result $\hat{B}$ is strictly increasing and so the inverse of $\hat{B}$, $\hat{\sigma}$, is continuous on $[0, \hat{B}_\infty)$. The following statements hold for all t and all $\omega \not\varepsilon \Omega_0$ where $P^x(\Omega_0) = 0$ for all x. Since $B_t = \hat{B}_{A(t)}$ it follows that $\sigma(t) = \tau(\hat{\sigma}_t)$ if $\hat{\sigma}_t < \infty$. Let $\hat{\zeta}$ be lifetime of (X, A); then $\hat{B}_{\hat{\zeta}} = B_\zeta$ and B_ζ is the lifetime of (X, B) while $\hat{B}_{\hat{\zeta}}$ is the lifetime of $((X, A), B)$—the transform of (X, A) by the time change $\hat{\sigma}$. Finally $\{\hat{\sigma}_t < \infty\} = \{t < \hat{B}_{\hat{\zeta}}\}$. Combining these observations we see that $X[\sigma(t)] = X[\tau(\hat{\sigma}_t)]$ for all t, completing the proof of (3.5).

Note added in proof. The following extension of the Blumenthal, Getoor, and McKean result is another consequence of Proposition 3.3. Let X be as above and let D be a closed subset of E such that each point of D is regular for D. Let $\tilde{X}$ be a standard process with state space $(D, \mathfrak{B}(D))$ whose hitting distributions are the same as those of X when restricted to D; that is, if x is in D and F is a

Borel subset of D then $\tilde{P}_F(x, \cdot) = P_F(x, \cdot)$. Then there exists a CAF, B of X whose fine support is D and such that $X^* = (X, B)$ is equivalent to $\tilde{X}$. To see this let A be any finite CAF of X whose fine support is D and let $\hat{X} = (X, A)$. See Section V-4 of [2] for the existence of such an A. Now $\hat{X}$ is a standard process with state space $(D, \mathcal{B}(D))$ and one easily checks that $\hat{X}$ and $\tilde{X}$ have the same hitting distributions. Thus by the result of Blumenthal, Getoor, and McKean there exists a CAF, $\hat{B}$ of $\hat{X}$ with Supp $(\hat{B}) = D$ such that $(\hat{X}, \hat{B})$ and $\tilde{X}$ are equivalent. Finally define $B_t = \hat{B}_{A(t)}$. It now follows from (3.3) as in the proof of (3.5) that $X^* = (X, B)$ is equivalent to $\tilde{X}$.

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