NOTE ON DYNKIN'S (α, ξ)-SUBPROCESS OF STANDARD MARKOV PROCESS

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Let α_t be a multiplicative functional of a standard Markov process. E. B. Dynkin [2] has defined "(α, ξ)-subprocess" under certain conditions imposed to α_t. (The conditions are stated as the existence of a suitable stochastic process ξ_t.) In this note, it is shown that (α, ξ)-subprocess exists if and only if α_t is a positive supermartingale of the class (D). For the rigorous proof of this fact, "α_t-additive functional" is introduced and the Meyer decomposition of α_t-additive functional is established.

1. Notations and definitions. Let us first recall the definition of standard Markov process. Let S be a locally compact Hausdorff space with a countable open base and $S^* = S \vee \{\Delta\}$ be the space adjoined Δ to S as an isolated point. $\mathfrak{B}_{S^*}$ is the smallest σ-algebra containing all open sets of S^*. A mapping w; $T = [0, +\infty] \to S^*$ is a *path* if it satisfies (i) $x_t(w) = w_t$ is right continuous, (ii) $x_t(w) = \Delta$ for $t \geqq \zeta(w) = \inf \{t > 0; x_t(w) = \Delta\}(= +\infty$ if $\{\quad\} = \varnothing)$ and (iii) $x_t(w)$ has left hand limits in $0 \leqq t < \zeta(w)$. The space of all paths is denoted by W. $\mathfrak{B}_t$ is the smallest σ-algebra on W for which $x_s(w)$ is measurable for $s \leqq t$, and $\mathfrak{B} = \vee_{t>0} \mathfrak{B}_t$.

Let P_x, $x \varepsilon S^*$, be a family of probability measures on $(W, \mathfrak{B})$ such that $P_x(B)$, $B \varepsilon \mathfrak{B}$ is $\mathfrak{B}_{S^*}$-measurable and $P_x(x_0(w) = x) = 1$. For a bounded measure μ on $(S^*, \mathfrak{B}_{S^*})$ we define P_μ by $\int \mu(dx)P_x$. A subset N of W which is of P_μ-outer measure 0 for every μ is called a *null set*. The set of all null sets is denoted by $\mathfrak{N}$. $\mathfrak{F}_t$ is the smallest σ-algebra containing $\mathfrak{B}_t$ and $\mathfrak{N}$. Set $\mathfrak{F} = \vee_{t>0} \mathfrak{F}_t$. A non-negative $\mathfrak{F}$-measurable function T is a *stopping time* if $\{T \leqq t\} \varepsilon \mathfrak{F}_t$ holds for every $t \geqq 0$. (If $\mathfrak{F}$ and $\mathfrak{F}_t$ are replaced by $\mathfrak{B}$ and $\mathfrak{B}_t$ in the above definition, T is called $(\mathfrak{B})$-*stopping time*.) A stopping time T is called a QHT (*quasi-hitting time*) if (i) $T(\theta_t) + t = T$ for $t \leqq T$ and (ii) $\lim_{t \downarrow 0} T(\theta_t) + t = T$ hold except for a null set, where θ_t is the shift operator defined by $x_s(\theta_t w) = x_{s+t}(w)(\forall_t$, $s \geqq 0)$. For a stopping time T, we define a σ-algebra $\mathfrak{F}_T$ by $\{B \varepsilon \mathfrak{F}$; $B \cap \{T \leqq t\} \varepsilon \mathfrak{F}_t$ for every $t \geqq 0\}$.

$(x_t, \zeta, \mathfrak{F}_t, P_x)$ is called a *standard process* if the following two conditions are satisfied.

(1) (*Strong Markov property*). For each stopping time T,

$$E_x(f(\theta_T w); B) = E_x(E_{x_T}(f); B), \qquad \forall x \varepsilon S^*, \tag{1.1}$$

holds for every bounded $\mathfrak{F}$-measurable function f and $B \varepsilon \mathfrak{F}_T$.

(2) (*Quasi-left continuous before* ζ). For each increasing sequence of stopping

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times $\{T_n\}$ with limit T,

$$(1.2) \qquad P_x(\lim_{n\to\infty} x_{T_n} = x_T,\ T < \zeta) = P_x(T < \zeta), \qquad \forall x \varepsilon S^*.$$

We shall assume, through this note, Meyer's hypothesis (L); there exists a measure γ on S^* (called a reference measure) such that every excessive function u with $\int \gamma(dx)u(x) = 0$ is identically 0.

REMARK. Let $\mathfrak{G}$ be a σ-subalgebra of $\mathfrak{F}$ and f, a $\mathfrak{F}$-measurable function. In this note, conditional expectation $E.(f \mid \mathfrak{G})$ is defined even for nonintegrable function in the following way; Set $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$. Then $E.(f \mid \mathfrak{G})$ is defined as $E\cdot(f^+ \mid \mathfrak{G}) - E.(f^- \mid \mathfrak{G})$ if one of them is finite and by 0 if both of them are infinite. Then the strong Markov property implies the following: Let f be a $\mathfrak{F}$-measurable function and $B \varepsilon \mathfrak{F}_T$ be a set such that $E_x(|f(\theta_T)|; B) < \infty$ for each x. Then $E_{x_{T(w)}}(f)$ is well defined and is finite for $w \varepsilon \mathfrak{W}$ a.e. P_x for each x.

A real valued process $X_t(w)$, $0 \leqq t < \infty$, defined on $(W, \mathfrak{F})$ is a *functional* if it is $\mathfrak{F}_t$-measurable for each $t \geqq 0$ and $X_t(w)$ is right continuous in $0 \leqq t < \infty$, $X_t(w) = X_\zeta(w)$ for $t \geqq \zeta(w)$ except for w of a null set. A functional X_t is a *(super)martingale* if $(X_t, \mathfrak{F}_t, P_x)$ is a (super)martingale for each $x \varepsilon S^*$. Similarly, a functional X_t is a *local (super) martingale* if there exists an increasing sequence of stopping times $\{T_n\}$ with limit $+\infty$ such that each $X_{t\wedge T_n}$ is a (super)-martingale. A functional X_t is of the *class (D)* if for any family of stopping times $\{T_\alpha\}$, $\{X_{T_\alpha}, P_x\}$ is uniformly integrable for each x. If the above is true for $\{T_\alpha\}$ dominated by a constant, X_t is of the *class (DL)*.

A nonnegative functional α_t is a MF (*multiplicative functional*) if it is a supermartingale and except for w of a null set $\alpha_t(w)\alpha_s(\theta_t w) = \alpha_{t+s}(w)$ holds for every pair $t, s \geqq 0$ and $\alpha_t(w) = 0$ for $t \geqq \zeta(w)$. To avoid a minor complication, we shall assume that $\alpha_0 > 0$ a.e. P_x for $x \varepsilon S$: Then multiplicativity implies $\alpha_0 = 1$ a.e. P_x for $x \varepsilon S$. A functional X_t is called *α_t-additive* if, except for w of a null set $X_t(w) + \alpha_t(w)X_s(\theta_t w) = X_{t+s}(w)$ holds for every pair $t, s \geqq 0$. An α_t-additive supermartingale X_t is called *regular* if $E\cdot(X_{t\wedge T} \mid \bigvee_n \mathfrak{F}_{t\wedge T_n}) = \lim_{n\to\infty} X_{t\wedge T_n}$ holds for every increasing sequence of QHT $\{T_n\}$ with limit T and every constant t.

An integrable functional A_t is an *increasing process* if $A_0(w) = 0$ and $A_t(w)$ increases with t except for w of a null set. An increasing process A_t is natural if for any bounded martingale X_t, $E_x(\int_0^t X_s\, dA_s) = E_x(\int_0^t X_s^-\, dA_s)$ holds for each t and x, where $X_s^- = \lim_{n\to\infty} X_{s-1/n}$.

2. Theorems. Our first theorem is the Meyer decomposition of α_t-additive functional.

THEOREM 1. *Let X_t be an α_t-additive supermartingale. There exists an α_t-additive supermartingale M_t which is a local martingale and α_t-additive and natural increasing process A_t such that $X_t = M_t - A_t$ holds for every $t \geqq 0$ except for a null set. Moreover, the decomposition is unique. In particular,*

(1) *X_t is regular if and only if A_t is continuous,*

(2) *X_t is of the class (DL) if and only if M_t is a martingale,*

(3) *X_t is of the class (D) if and only if M_t is a martingale of the class (D) and $E_x(A_\infty) < \infty$ for every x, where $A_\infty = \lim_{t\uparrow+\infty} A_t$.*

The above theorem is immediate from [4], Appendix if the underlying process is a Hunt process and if X_t is of the class (D). But in our case we need considerable modification of their proof. It will be given at the next section.

Let α_t be a MF. It is known [2, 5] that there exists a standard process $(x_t, \zeta, \mathfrak{F}_t^\alpha, P_x^\alpha)$ defined on S^* and $(W, \mathfrak{B})$ with the transition function $P^\alpha(t, x, E) = E_x(\alpha_r; x_t \varepsilon E)(E \varepsilon \mathfrak{B}_{S^*})$. Suppose that there exists a nonnegative $\mathfrak{F}$-measurable stochastic process ξ_t satisfying the following $(\xi.1)$–$(\xi.3)$. Except for a null set,

$$(\xi.1) \qquad \alpha_t(\theta_s)\xi_{t+s} \leqq \xi_s \quad \text{if} \quad t + s < \zeta \wedge T_\alpha.$$

$$(\xi.2) \qquad \psi_t \equiv \alpha_t\xi_t \text{ is right continuous in } 0 \leqq t < \zeta,$$

and

$$(\xi.3) \qquad E_x(\xi_T \mid \mathfrak{F}_T) = 1 \qquad \text{holds on } \{T < \zeta\} \text{ a.e. } P_x(x \varepsilon S),$$

for every stopping time T. Here, $T_\alpha = \inf\{t > 0; \alpha_t = 0\}$.
Dynkin [2], Chap. X, Section 4, has shown a direct method of constructing measure P_x^α from P_x using this ξ_t. (Such $(x_t, \zeta, \mathfrak{F}_t^\alpha, P_x^\alpha)$ is called (α, ξ)-subprocess.) We are interested under which condition there is ξ_t satisfying these $(\xi.1)$–$(\xi.3)$.

THEOREM 2. *The following three conditions are equivalent:*

(1) *There exists ξ_t satisfying $(\xi.1)$–$(\xi.3)$.*

(2) *α_t is of the class (D).*

(3) *For every increasing sequence of $(\mathfrak{B})$-stopping times $\{T_n\}$ with limit T, $(P_x^\alpha, \mathfrak{N}_T^-)$ is absolutely continuous with respect to $(P_x, \mathfrak{N}_T^-)$ for every x of S, where $\mathfrak{N}_T^- = \bigvee_n \mathfrak{B}_{T_n}[\bigcap_n \{T_n < \zeta\}]$ and the notation $\mathfrak{G}[\ \ ]$ means the restriction of the σ-algebra $\mathfrak{G}$ to the set $[\ \ ]$.*

PROOF. (1) $\Rightarrow$ (2). Let T be an arbitrary stopping time. Since $\alpha_T \xi_T \leqq \xi_0$ by $(\xi.1)$, $\alpha_T = E.(\alpha_T\xi_T \mid \mathfrak{F}_T) \leqq E.(\xi_0 \mid \mathfrak{F}_T)$ by $(\xi.3)$. Hence α_t is of the class (D).
(2) $\Rightarrow$ (1). Set $X_t = \alpha_t - 1$. Then X_t is an α_t-additive supermartingale as is easily shown. Let $X_t = M_t - A_t$ be the Meyer decomposition. If α_t is of the class (D), M_t is a martingale by Theorem 1, (3). α_t-additivity of M_t implies $M_0 = 0$ a.e. P_x for $x \varepsilon S$. Then we obtain $E_x(\alpha_\infty) + E_x(A_\infty) = 1$ for every x of S, where $\alpha_\infty = \lim_{t\to\infty} \alpha_t$ (exists because α_t is a supermartingale of the class (D)). Set $\xi_t(w) = \alpha_\infty(\theta_t w) + A_\infty(\theta_t w)$. Then ξ_t is a nonnegative $\mathfrak{F}$-measurable stochastic process and satisfies $\psi_t = \alpha_\infty + A_\infty - A_t$. Hence $(\xi.2)$ follows. Since

$$\alpha_t(\theta_s)\xi_{t+s} = (\alpha_\infty + A_\infty - A_{t+s})/\alpha_s \leqq (\alpha_\infty + A_\infty - A_s)/\alpha_s = \xi_s \quad \text{if} \quad \alpha_s > 0,$$

$(\xi.1)$ follows. $(\xi.3)$ follows from

$$E.(\xi_T \mid \mathfrak{F}_T) = E.(\alpha_\infty(\theta_T) + A_\infty(\theta_T) \mid \mathfrak{F}_T) = E_{x_T}(\alpha_\infty + A_\infty) = 1.$$

(2) $\Rightarrow$ (3). Let $\{T_n\}$ be an increasing sequence of stopping times with limit T. Then

$$P_x^\alpha(B \cap \{T_n < \zeta\}) = E_x(\alpha_{T_n}; B \cap \{T_n < \zeta\}), \; B \varepsilon \mathfrak{B}_{T_k}, \quad k \leqq n.$$

(See [5].) Letting $n \to \infty$, we obtain

$$E_x^{\alpha}(B \cap [\bigcap_n \{T_n < \zeta\}]) = E_x(\lim_{n\to\infty} \alpha_{T_n}; B \cap [\bigcap_n \{T_n < \zeta\}]).$$

The above holds for every $B \in \bigvee_n B_{T_n}$. Hence $\lim_{n\to\infty} \alpha_{T_n}$ is the Radon-Nikodym derivative of $(P_x^{\alpha}, \mathfrak{N}_T^-)$ relative to $(P_x, \mathfrak{N}_T^-)$ for every x of S.

(3) $\Rightarrow$ (2). Let $\alpha_{T-}^{(x)}$ be the Radon-Nikodym derivative of $(P_x^{\alpha}, \mathfrak{N}_T^-)$ relative to $(P_x, \mathfrak{N}_T^-)$. We extend $\alpha_{T-}^{(x)}$ to W by setting 0 on the complement of $\bigcap_n \{T_n < \zeta\}$. Then we have

$$E_x(\alpha_{T_n}; B \cap \{T_n < \zeta\}) = P_x^{\alpha}(B \cap \{T_n < \zeta\})$$
$$\geqq E_x(\alpha_{T-}^{(x)}; B \cap \{T_n < \zeta\})$$

if $B \in \mathfrak{B}_{T_n}$. Therefore $(\alpha_{T_1}, \cdots, \alpha_{T_n}, \cdots, \alpha_{T-}^{(x)})$ is a supermartingale and we obtain $\lim_{n\to\infty} \alpha_{T_n} \geqq E_x(\alpha_{T-}^{(x)} \mid \bigvee_n \mathfrak{F}_{T_n})$. While we have

$$\lim_{n\to\infty} E_x(\alpha_{T_n}) = \lim_{n\to\infty} P_x^{\alpha}(T_n < \zeta) = P_x^{\alpha}(\bigcap_n \{T_n < \zeta\}) = E_x(\alpha_{T-}^{(x)}).$$

Therefore $\lim_{n\to\infty} E_x(\alpha_{T_n}) = E_x(\lim_{n\to\infty} \alpha_{T_n})$. Then $\{\alpha_{T_n}\}$ is uniformly integrable relative to P_x by [7], Chapter II, T21.

3. Proof of Theorem 1. If we assume the existence of the Meyer decomposition, uniqueness is immediate from [7], Chapter VII, T21. "If" part of (1) is clear. It is not difficult to see (2) and (3). We shall prove here the existence of the Meyer decomposition and the "only if" part of (1).

LEMMA 1. *Let X_t be an α_t-additive and regular supermartingale. Then the Meyer decomposition exists and the corresponding increasing process is continuous.*

PROOF. Set

$$(3.1) \quad A_t^n = n \int_0^t \alpha_s E_{x_s}(X_{1/n})\, ds (= n \int_0^t \{X_s - E.(X_{s+1/n} \mid \mathfrak{F}_s)\}\, ds)$$

and

$$(3.2) \quad X_t^n = X_t - n \int_0^{1/n} \alpha_t E_{x_t}(X_s)\, ds (= n \int_t^{t+1/n} E.(X_s \mid \mathfrak{F}_t)\, ds).$$

Then A_t^n is an α_t-additive and continuous increasing process and X_t^n is an α_t-additive supermartingale increasing to X_t. Further X_t^n and A_t^n are related as

$$(3.3) \quad E.(A_T^n \mid \mathfrak{F}_{t\wedge T}) - A_{t\wedge T}^n = X_{t\wedge T}^n - E.(X_T^n \mid \mathfrak{F}_{t\wedge T}),$$

where T is a bounded stopping time. Suppose for a moment that there exists an increasing sequence of stopping times $\{S_p\}$ with limit $+\infty$ such that for each p, $\{A_{S_p}^n\}$ is a $L_2(P_x)$-Cauchy sequence for every x. Then by a well known martingale inequality, $\sup_{t<S_p} |E.(A_{S_p}^n \mid \mathfrak{F}_{t\wedge S_p}) - E.(A_{S_p}^m \mid \mathfrak{F}_{t\wedge S_p})|$ tends to 0 as $n \to +\infty$ in $P.$-probability. Also, $\sup_{t<S_p} |E.(X_{S_p}^n \mid \mathfrak{F}_{t\wedge S_p}) - E.(X_{S_p}^m \mid \mathfrak{F}_{t\wedge S_p})|$ tends to 0 in $P.$-probability as $n, m \to \infty$. Then $\sup_{t<S_p} |A_t^n - A_t^m|$ does by (3.3). We can choose an α_t-additive and continuous increasing process A_t such that $\sup_{t<S_p} |A_t - A_t^n| \to 0$ in P_x-probability as well as $E_x((A_{S_p} - A_{S_p}^n)^2) \to 0$ for every x, by the same method as [4], Appendix. This A_t satisfies

$$E.(A_{S_p} \mid \mathfrak{F}_{t\wedge S_p}) - A_{t\wedge S_p} = X_{t\wedge S_p} - E.(X_{S_p} \mid \mathfrak{F}_{t\wedge S_p})$$

from (3.3). Hence $X_t + A_t$ is a local martingale.

To prove the existence of such sequence $\{S_p\}$, set

$$T_n = \inf\{t > 0, X_t - X_t^n > \epsilon\alpha_t\}.$$

Then $\{T_n\}$ is an increasing sequence of QHT. The regularity of X_t and X_t^n concludes that $\{T_n\}$ tends to T_α, where $T_\alpha = \inf\{t > 0; \alpha_t = 0\}$. (The proof is obtained from a trivial modification of [7], Chapter VII, T36). We define Y_t by $X_t - E.(X_N \mid \mathfrak{F}_t)$ if $t < N$ and by 0 if $t \geqq N$, where N is a positive constant. Set $S = S_{c,N} = \inf\{t > 0; Y_t > c\} \wedge (N-1)$. Since $S_{c,N}$ increases to $+\infty$ as $c, N \to +\infty$, it suffices to show $E.[(A_S^n - A_S^m)^2] \to 0$ as $n, m \to +\infty$. Set $S_k = T_k \wedge S$. Then

$$\begin{aligned} & E.[(A_{S_k}^n - A_{S_k}^m)^2] \\ (3.4) \qquad & = 2E.[\textstyle\int_0^{S_k} \{(A_{S_k}^n - A_{S_k}^m) - (A_t^n - A_t^m)\}\, d(A_t^n - A_t^m)] \\ & = 2E.[\textstyle\int_0^{S_k} (Y_t^n - Y_t^m - Y_{S_k}^n + Y_{S_k}^m)\, d(A_t^n - A_t^m)] \\ & \leqq 4E.[\sup_{t<S_p} |Y_t^n - Y_t^m - Y_{S_k}^n + Y_{S_k}^m|^2]^{\frac{1}{2}} E\cdot[(A_{S_k}^n)^2 + (A_{S_k}^m)^2]^{\frac{1}{2}}. \end{aligned}$$

Here $Y_t^n = X_t^n - E.(X_N \mid \mathfrak{F}_t)$ and we have used the relation (3.3). Notice that $0 \leqq Y_t^n \leqq Y_t \leqq c$ on $t < S_k$ and $Y_{S_k}^n \geqq 0$, we have

$$(3.5) \quad \sup_{t<S_k} |Y_t^n - Y_t^m - Y_{S_k}^n + Y_{S_k}^m| \leqq \{\epsilon \sup_{t<S_k} \alpha_t + |Y_{S_k}^n - Y_{S_k}^m|\} \wedge 2c.$$

By a similar estimation as (3.4),

$$(3.6) \qquad E.[(A_{S_k}^n)^2] \leqq 2cE.(A_{S_k}^n) \leqq -2cE.(X_{S_k}^n) \leqq -2cE.(X_N)$$

and

$$(3.7) \qquad E.[(A_S^n - A_{S_k}^n)^2] \leqq 2cE.(A_S^n - A_{S_k}^n) \leqq 2cE.(X_{S_k}^n - X_S^n).$$

Therefore from (3.4)–(3.7) we have the following inequality

$$\begin{aligned} & \lim_{n,m\to\infty} E.[(A_S^n - A_S^m)^2] \\ (3.8) \qquad & \leqq 2 \lim_{n,m\to\infty} \{E.[(A_{S_k}^n - A_{S_k}^m)^2] + E.[(A_S^n - A_{S_k}^m)^2] \\ & \qquad + E.[(A_S^m - A_{S_k}^m)^2]\} \\ & \leqq -32cE.(X_N)E.[(\epsilon \sup_{t<S_k} \alpha_t) \wedge 2c] + 8cE.(X_{S_k} - X_S). \end{aligned}$$

Note that $\lim_{k\to\infty} X_{S_k} - X_{S\wedge T\alpha}$ coincides with $\lim_{k\to\infty} X_{N\wedge T_k} - X_{N\wedge T\alpha}$ on the set $\bigcap_k \{S_k < S\}$ which belongs to $\bigvee_k \mathfrak{F}_{N\wedge T_k}$. Then the regularity of X_t together with uniform integrability of $\{X_{S_k}\}$ implies

$$\lim_{k\to\infty} E.(X_{S_k} - X_{S\wedge T\alpha}) = E.(\lim_{k\to\infty} X_{N\wedge T_k} - X_{N\wedge T\alpha}\, ; \textstyle\bigcap_k \{S_k < S\}) = 0.$$

But since $X_S = X_{S\wedge T\alpha}$ except for a null set, $E.(X_{S_k} - X_S) \to 0$ as $k \to \infty$. Now since ϵ is arbitrary, we obtain $\lim_{n,m\to\infty} E.[(A_S^n - A_S^m)^2] = 0$ by the inequality (3.8).

To prove that $M_t = X_t + A_t$ is a supermartingale, it suffices to show that a positive local martingale is a supermartingale because we can reduce the proof to this case. Let Z_t be a positive local martingale and $\{S_n\}$ be an increasing sequence of stopping times with limit $+\infty$ such that each $Z_{t\wedge S_n}$ is a martingale. Then $E.(Z_{t\wedge S_n})$ is finite and does not depend on n. Hence by Fatou's lemma $E.(Z_t)$ is finite and Z_t is integrable. If we notice that $Z_{t\wedge S_n}$ is a positive martingale, we obtain the inequality $E.(Z_t\,;\, t < S_n,\, \Lambda) \leqq \leqq E.(Z_s\,;\, s < S_n,\, \Lambda)$ where $t \geqq s$ and $\Lambda \,\varepsilon\, \mathfrak{F}_{s\wedge S_m}$ $(m \leqq n)$. Hence we obtain $E.(Z_t\,;\,\Lambda) \leqq E\cdot(Z_s\,;\,\Lambda)$ by tending $n \to \infty$, which implies that Z_t is a supermartingale.

LEMMA 2. *Let X_t be an α_t-additive supermartingale. If X_t is not regular, there exists an α_t-additive and natural increasing process B_t which is purely discontinuous (and not identically* 0) *such that $X_t + B_t$ is an α_t-additive supermartingale.*

PROOF. Let us define $T_n = \inf\{t > 0; X_t - X_t^n > \epsilon\alpha_t\}$, where X_t^n is a regular supermartingale defined by (3.2) and ϵ is a positive constant. Then since X_t^n increases to X_t as $n \to \infty$, $X_{T_n} - X_{T_n}^m \geqq X_{T_n} - X_{T_n}^n \geqq \epsilon\alpha_{T_n}$ holds for $m \leqq n$. (We put $X_\infty = 0$ conventionally.) Letting first $n \to \infty$ and next $m \to \infty$, we have, by putting $T = \lim_{n\to\infty} T_n$

$$\lim_{n\to\infty} X_{T_n} - E.(X_T \mid \vee_n \mathfrak{F}_{T_n}) \geqq \epsilon \lim_{n\to\infty} \alpha_{T_n}. \tag{3.9}$$

Proof of Lemma 1 shows actually that if X_t is not regular, there exists $\epsilon > 0$ sufficiently small such that the left hand of (3.9) is not identically 0. We shall show that the left hand of (3.9) coincides with $-\lim_{n\to\infty} \alpha_{T_n} E_{x_{T_n}}(X_T)$ for w with $T(\mathrm{w}) < \infty$. α_t-additive property $X_{T_n} - X_T = -\alpha_{T_n} X_{T(\theta_{T_n})}(\theta_{T_n})$ for $T < \infty$ together with the strong Markov property implies

$$E\cdot((X_{T_n} - X_T)I(T < \infty) \mid \mathfrak{F}_{T_n}) = -\alpha_{T_n} E_{x_{T_n}}(X_T),$$

where I is the indicator function. On the other hand, since $E.(X_T \mid \vee_n \mathfrak{F}_{T_n}) = \lim_{n\to\infty} E.(X_T \mid \mathfrak{F}_{T_n})$, $\lim_{n\to\infty} \alpha_{T_n}\cdot E_{x_{T_n}}(X_T)$ exists and equals to the left hand of (3.9).

We now define sequences of stopping times $\{T^p\}$ and $\{T_n^p\}$ by induction; $T^p = T^{p-1} + T(\theta_{T^{p-1}})(T^0 = 0, T^1 = T)$ and $T_n^p = T^{p-1} + T_n(\theta_{t_n^{p-1}})(T^0 = 0, T_n^1 = T_n)$. Then by the same reasoning as that of the preceding paragraph, $-\lim_{n\to\infty} \alpha_{T_n^p} E_{x_{T_n^p}}(X_T)$ exists and coincides with $\lim_{n\to\infty} X_{T_n^p} - E.(X_T \mid \vee_n \mathfrak{F}_{T_n^p})$ for w with $T^p(w) < \infty$. Put $B_t^p = -\lim_{n\to\infty} \alpha_{T_n^p} E_{x_{T_n^p}}(X_T) I(0 < T^p \leqq t)$ and $B_t = \sum_{p=1}^\infty B_t^p$. Then B_t is clearly a purely discontinuous increasing process not identically 0. Furthermore, each B_t^p is natural by the proof of [7], Chapter VII, T49. Hence B_t is natural if it is integrable.

We shall next prove that B_t is integrable and $X_t + B_t$ is a supermartingale. But we shall only prove $X_t + B_t^p$ is a supermartingale; then this fact can be obtained repeating the same argument inductively. For an arbitrary pair of constants $0 < s < t$, we define T' by T if $s < T^p \leqq t$, by s if $T^p \leqq s$ and by t if $T^p > t$. T_n' is defined similarly from T_n^p. Then we obtain

$$E.(Y_t' - Y_s' \mid \mathfrak{F}_s) = E.(X_t - X_{T'} \mid \mathfrak{F}_s) + E.(\lim_{n\to\infty} X_{T_n'} - X_s \mid \mathfrak{F}_s),$$

where $Y_t' = X_t + B_t^p$. Since Y_t' is integrable and since each term of the right hand is negative, Y' is a supermartingale.

It remains to prove the α_t-additivety of β_t. Let us notice the relation

$$(3.10) \qquad T^{p+q} = t + T^q(\theta_t) \quad \text{if} \quad T^p \leqq t < T^{p+1} \quad \text{and} \quad q \geqq 1.$$

(Similarly, $T_n^{p+q} = t + T_n^q(\theta_t)$ holds if $T^p \leqq t < T_n^{p+1}$ and $q \geqq 1$). In fact, since $T(\theta_{T^p}) > t - T^p$ holds in the case $T^p \leqq t < T^{p+1}$, $T(\theta_{t-T^p} \circ \theta_{T^p}) + t - T^p = T(\theta_{T^p})$ holds because T is a QHT. But since $\theta_{t-T^p} \circ \theta_{T^p} = \theta_t$ holds, we have $T(\theta_t) + t = T^p + T(\theta_{T^p}) = T^{p+1}$. The general case can be proved by induction. In fact, suppose the relation (3.10) holds for $q - 1$; then

$$T^{p+q} = T^{p+q-1} + T(\theta_{T^{p+q-1}}) = t + T^{q-1}(\theta_t) + T(\theta_{T^{p+q-1}}),$$

while we have $\theta_{T^{p+q-1}} = \theta_{T^{q-1}(\theta_t)} \circ \theta_t$, so that the last expression above coincides with $t + T^q(\theta_t)$.

Coming back to the proof of α_t-additivity of B_t, let us rewrite

$$\{t < T^p \leqq t + s\} = \textstyle\bigcup_{q=0}^{p-1} [\{T^q \leqq t < T^{q+1}\} \cup \{T^{p-q}(\theta_t) \leqq s\}], \quad p \geqq 1.$$

Then multiplicativity of α_t and the property (3.10) deduce

$$B_{t+s}^p - B_t^p = \alpha_t \textstyle\sum_{q=0}^{p-1} B_s^{p-q}(\theta_t) I(T^q \leqq t < T^{q+1}).$$

Summing up the above for $p = 1, 2, 3, \cdots$ and changing the order of summations relative to p and q, we have

$$\begin{aligned} B_{t+s} - B_t &= \alpha_t \textstyle\sum_{q=0}^{p-1} [\sum_{p \geqq q+1}^{\infty} B_s^{p-q}(\theta_t)] T(T^q \leqq t < T^{q+1}) \\ &= \alpha_t B_s(\theta_t) I(T^\infty > t), \end{aligned}$$

where $T^\infty = \lim_{p\to\infty} T^p$. Thus B_t is α_t-additive if the inequality $T^\infty \geqq T_\alpha \equiv \inf\{t > 0; \alpha_t = 0\}$ is satisfied, which we shall prove henceforth.

By inequality (3.9), we have $B_t \geqq \epsilon \sum_{p=1}^{\infty} \lim_{n\to\infty} \alpha_{T_n^p} I(T^p \leqq t)$ and we get $\limsup_{p\to\infty} \lim_{n\to\infty} \alpha_{T_n^p} = 0$ because B_t is finite. Therefore we have

$$E.(\alpha_{T^\infty} \mid \vee_p \vee_n \mathfrak{F}_{T_n^p}) \leqq \lim_{p\to\infty} E.(\alpha_{T^p} \mid \vee_n \mathfrak{F}_{T_n^p}) \leqq \limsup_{p\to\infty} \lim_{n\to\infty} \alpha_{T_n^p} = 0,$$

which implies $T^\infty \geqq T_\alpha$.

Proof of the Existence of the Meyer Decomposition. Let X_t be an α_t-additive supermartingale. For every countable ordinal η, we define an α_t-additive supermartingale X_t^η by the transfinite induction. Suppose X_t^ξ, $\xi < \eta$, are well defined. If η is a limit ordinal, set $X_t^\eta = \sup_{\xi<\eta} X_t^\xi$. Suppose η is an isolated ordinal. If $X_t^{\eta-1}$ is regular, define X_t^η by $X_t^{\eta-1}$. If $X_t^{\eta-1}$ is not regular, define X_t^η by $X_t^{\eta-1} + B_t^{\eta-1}$, where $B_t^{\eta-1}$ is a natural increasing process constructed from $X_t^{\eta-1}$ by the method of Lemma 2.

Let γ be a reference measure of the standard process. There is then a countable ordinal η_0 such that $X_t^\xi = X_t^{\eta_0}$ holds a.e. P_γ for every $\xi \geqq \eta_0$. Then $X_t^\xi \geqq X_t^{\eta_0}$ is satisfied a.e. P_x for every $\xi \geqq \eta_0$. Indeed, suppose on the contrary that there exists $\xi > \eta_0$ such that $X_t^\xi > X_t^{\eta_0}$ holds on a set with P_x-positive probability for some x. Set $f_t(x) = E_x(X_t^\xi - X_t^{\eta_0})$. Then $f_{t+s}(x) = f_t(x) + E_x(\alpha_t f_s(x_t))$ holds and f_t increases to f_∞ as $t \uparrow \infty$ and decreases to 0 as $t \downarrow 0$. Hence $f_\infty(x) \geqq E_x(\alpha_t f_\infty(x_t))$ and the right hand increases to $f_\infty(x)$ as $t \downarrow 0$. Then $f_\infty(x)$ is excessive relative to the transformed process $(x_t, \zeta, \mathfrak{F}_t^\alpha, P_x^\alpha)$. Since γ is also a reference measure of the transformed process and since $\int \gamma(dx) f_\infty(x) = 0$, u must be identically 0, which is a contradiction. Hence $X_t^\xi = X_t^{\eta_0}$ holds for every $\xi \geqq \eta_0$ except for a null set. Then $X_t^{\eta_0}$ is regular.

By Lemma 1, $X_t^{\eta_0}$ has the Meyer decomposition $M_t - A_t^c$, where M_t is a local martingale and A_t^c is a continuous increasing process both of which are α_t-additive. Since $X_t = X_t^{\eta_0} + \sum_{\xi \leqq \eta_0} B_t^\xi$, we obtain $X_t = M_t - A_t$ by setting $A_t = A_t^c + \sum_{\xi \leqq \eta_0} B_t^\xi$ and this is the desired result.

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