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LIKELIHOOD RATIO TESTS FOR RESTRICTED FAMILIES OF PROBABILITY DISTRIBUTIONS[1]

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1. Introduction. Recently, a conditional likelihood ratio test has been proposed for testing for trend in a stochastic process of Poisson type [Boswell (1966)]. This is a departure from the standard literature in that the underlying family of distributions considered is essentially nonparametric. His main result is the asymptotic distribution of the likelihood ratio under the null hypothesis of no trend.

We consider likelihood ratio tests for certain geometrically restricted families of distributions. For example, let

$$\mathfrak{F}_0 = \{F \mid F(0) = 0 \text{ and } -\log [1 - F(x)]x^{-1} \text{ nondecreasing in } x \geqq 0\}.$$

Then $\mathfrak{F}_0$ is known as the IFRA (for increasing failure rate average) family of distributions. These distributions play an important role in the mathematical theory of reliability [Birnbaum, Esary, and Marshall (1966)]. However, not only is the family nonparametric but there is no sigma-finite measure relative to which all $F \varepsilon \mathfrak{F}_0$ are absolutely continuous. Hence, the usual concept of maximum likelihood estimate does not suffice. Kiefer and Wolfowitz (1956), p. 893, propose a generalization of the maximum likelihood estimate concept which we adopt. Let F_1, $F_2 \varepsilon \mathfrak{F}$ and let $f(\cdot; F_1, F_2)$ denote the Radon-Nikodym derivative of F_1 with respect to the measure induced by $F_1 + F_2$.

Definition 1. $\hat{F}$ is called the *maximum likelihood estimate* relative to $\mathfrak{F}$ if $\hat{F}$ satisfies

$$\sup_{F \varepsilon \mathfrak{F}} \prod_{i=1}^n \{f(X_i; F, \hat{F})[1 - f(X_i; F, F)]^{-1}\} = 1,$$

where $\mathbf{X} = (X_1, X_2, \cdots, X_n)$ is a random sample.

This definition is easily seen to coincide with the usual definition when the family $\mathfrak{F}$ is dominated by a sigma-finite measure.

Now consider the problem of testing $H_0: F \varepsilon \mathfrak{F}_0$ against the alternative $H_1: F \varepsilon \mathfrak{F} - \mathfrak{F}_0$ where $\mathfrak{F}_0 \subset \mathfrak{F}$. Let $\hat{F}_0(\hat{F})$ denote the maximum likelihood estimate relative to $\mathfrak{F}_0(\mathfrak{F})$ in the sense of Definition 1. We define the likelihood ratio statistic $\Lambda_n(\mathbf{X})$ based on a random sample $\mathbf{X}$ as follows:

Definition 2. $\Lambda_n(\mathbf{X})$ is called the *likelihood ratio statistic* where

$$\Lambda_n(\mathbf{X}) = \prod_{i=1}^n \{f(X_i; \hat{F}_0, \hat{F})[1 - f(X_i; \hat{F}_0, \hat{F})]^{-1}\}.$$

We will be concerned with the properties of $\Lambda_n(\mathbf{X})$ for various restricted families

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of distributions $\mathfrak{F}_0 \subset \mathfrak{F}$. Two unbiased tests ϕ^* and ϕ^{**} for testing for constant versus increasing (or decreasing) failure rate are proposed. Both tests are based on statistics which are likelihood ratio statistics for related problems. Sampling experiments indicate that ϕ^{**} has greater power against Weibull and gamma distribution alternatives than both ϕ^* and a uniform conditional test based on the total time on test statistic (see Section 6).

It will be convenient henceforth to let $X_1 \leqq X_2 \leqq \cdots \leqq X_n$ denote the order statistics from a random sample based on a distribution F.

2. IFRA distributions. Let $\mathfrak{F}_0 = \{F \mid F(0) = 0 \text{ and } -\log [1 - F(x)]x^{-1} \uparrow \text{ in } x \geqq 0\}$. The maximum likelihood estimate (MLE) $\hat{F}_0$ relative to $\mathfrak{F}_0$ under Definition 1 puts probability at each of the sample observations as well as between observations. Letting $F\{X_k\}$ denote the F probability of observation X_k, the likelihood becomes

$$L_n(\mathbf{X} \mid F) = n! \prod_{i=1}^{n} F\{X_i\}.$$

From the definition of IFRA distributions we see that

$$L_n(\mathbf{X} \mid F) = \prod_{i=1}^{n} [\exp(-\lambda_{i-1}X_i) - \exp(-\lambda_i X_i)] \tag{2.1}$$

where $0 \leqq \lambda_0 \leqq \lambda_1 \leqq \cdots \leqq \lambda_n$. We maximize likelihood subject to these restrictions by letting $\lambda_0 = 0$ and $\lambda_n = +\infty$. Letting $\Delta\lambda_j = \lambda_j - \lambda_{j-1}$ where $\lambda_0 = 0$ and $\lambda_n = +\infty$, we see that (2.1) becomes

$$L_n(\mathbf{X} \mid F) = \prod_{i=1}^{n-1} [\exp(-\Delta\lambda_i \textstyle\sum_{i+1}^{n} X_j)[1 - \exp(-\Delta\lambda_i X_i)]]. \tag{2.2}$$

Maximizing (2.2) subject to $\Delta\lambda_i \geqq 0$ $(1 \leqq i \leqq n)$ we see that

$$\Delta\hat{\lambda}_i = X_i^{-1}[\log \textstyle\sum_i^n X_j - \log \sum_{i+1}^n X_j] \geqq 0 \tag{2.3}$$

are the maximum likelihood estimates of $\Delta\lambda_i (1 \leqq i \leqq n)$. Substituting (2.3) in (2.2) we see that the maximum likelihood according to definition 1 becomes

$$L_n(\mathbf{X} \mid \hat{F}_1) = \prod_{i=1}^{n-1} [1 - X_i/\textstyle\sum_i^n X_j]^{\sum_i^n X_j/X_i - 1}[X_i/\sum_i^n X_j]. \tag{2.4}$$

If we let $\mathfrak{F}$ denote the class of all distributions on the positive axis, then $\hat{F}$ is the usual empirical distribution function and the likelihood ratio statistic for testing H_0: $F \varepsilon \{\text{IFRA}\}$ versus H_1: $F \not\varepsilon \{\text{IFRA}\}$ becomes

$$\Lambda_n(\mathbf{X}) = n^n \prod_{i=1}^{n-1} [1 - X_i/\textstyle\sum_i^n X_j]^{\sum_i^n X_j/X_i - 1}[X_i/\sum_i^n X_j]. \tag{2.5}$$

We consider the test, ϕ, which rejects H_0 when

$$\Lambda_n(\mathbf{X}) \leqq c_\alpha$$

where c_α is determined by

$$P_G\{\Lambda_n(\mathbf{Y}) \leqq c_\alpha\} = \alpha.$$

$G(x) = 1 - e^{-x}$ for $x \geqq 0$, and $\mathbf{Y}$ denotes an ordered sample from G.

Percentage points for $-\log \Lambda_n(\mathbf{Y})$ are given in Table 1. This choice of a test based on the likelihood ratio statistic is motivated by the following theorem.

TABLE 1*

Percentage points for $-\log \Lambda_n(\mathbf{Y})$

Sample Size n	Percentiles					
	.01	.05	.10	.90	.95	.99
2	0.015	0.073	0.14	2.63	3.34	4.91
3	0.21	0.48	0.71	4.33	5.28	7.25
4	0.58	1.07	1.42	5.91	6.89	9.00
5	1.10	1.75	2.22	7.50	8.60	11.09
6	1.65	2.49	3.07	8.96	10.11	12.66
7	2.37	3.35	3.93	10.41	11.63	14.13
8	3.05	4.15	4.83	11.83	13.18	15.65
9	3.72	4.97	5.73	13.20	14.66	17.54
10	4.50	5.84	6.66	14.62	16.12	19.16

* Note that we use lower percentiles for testing exponentiality versus IFRA and upper percentiles for testing IFRA versus DFRA.

THEOREM 2.1. *If* $F_2^{-1}F_1(x)/x$ *is nondecreasing for* $x \geqq 0$, $F_1(0) = F_2(0) = 0$ *and* $\mathbf{X}(\mathbf{Y})$ *denotes an ordered sample from* $F_1(F_2)$, *then*

$$\Lambda_n(\mathbf{X}) \geqq_{\text{st}} \Lambda_n(\mathbf{Y}) \tag{2.6}$$

where $\leqq_{\text{st}}$ *denotes stochastic ordering.*

PROOF. Let $\Psi(x) = F_2^{-1}F_1(x)$, $Y_i^* = \Psi(X_i)$ and note $\Psi(x)/x \uparrow$ in $x \geqq 0$. Also, Y_i^* is distributed as the ith order statistic from F_2. Now

$$\Psi(X_i)/X_i \leqq \Psi(X_j)/X_j \quad \text{for} \quad j \geqq i$$

implies

$$X_i/\textstyle\sum_{j=i}^n X_j \geqq \Psi(X_i)/\sum_{j=i}^n \Psi(X_j) = Y_i^*/\sum_{j=i}^n Y_j^*.$$

Since $h(x) = x(1-x)^{1/x-1}$ is increasing in x $(0 \leqq x \leqq 1)$, it follows that

$$\textstyle\prod_{i=1}^{n-1} h(X_i/\sum_i^n X_j) \geqq_{\text{st}} \prod_{i=1}^{n-1} h(Y_i/\sum_i^n Y_j)$$

where $(Y_1 \leqq Y_2 \leqq \cdots \leqq Y_n)$ is an independent ordered sample from F_2. (2.6) follows immediately. □

We say that $F_1 <_* F_2$ (i.e., F_1 is starshaped with respect to F_2) if $F_2^{-1}F_1(x)/x$ is nondecreasing for $x \geqq 0$. From the proof of Theorem 2.1 it follows that $F_1 <_* F_2$ implies

$$P_{F_1}\{\Lambda_n(\mathbf{X}) \leqq c_\alpha\} \leqq P_{F_2}\{\Lambda_n(\mathbf{X}) \leqq c_\alpha\}.$$

Hence the power of the likelihood ratio test is greater at F_2 than at F_1 when $F_1 <_* F_2$.

COROLLARY 2.2. *The test,* ϕ, *based on the likelihood ratio statistic* $\Lambda_n(\mathbf{X})$ *is unbiased at all significance levels for the problem*

$$H_0\text{: } F \in \mathfrak{F}_0 = \{\text{IFRA}\} \qquad \textit{versus} \qquad H_1\text{: } F \in \mathfrak{F}_1 = \{\text{DFRA}\}.$$

The derivation of the maximum likelihood estimate for IFRA distributions is due to Marshall and Proschan (1967). It is interesting, that even though this estimate is *not* consistent, it leads to a test, called ϕ^{**} in Section 6, which seems quite powerful.

3. IFR distributions. Let $\mathfrak{F} = \{F \mid F(0) = 0$ and $-\log [1 - F(x)]$ is convex for $x \geqq 0\}$. This is the class of IFR (for increasing failure rate) distributions. Proschan and Pyke (1965) have proposed a rank test for testing constant versus increasing failure rate based on the normalized spacings. Bickel and Doksum (1967) have shown that this test is asymptotically inadmissable among rank tests based on normalized spacings. Bickel and Doksum (1967) and Nadler and Eilbott show that a uniform conditional test (see Section 6) is asymptotically superior to the Proschan-Pyke test. Sampling experiments performed by Nadler and Eilbott and independently by the author indicate that the uniform conditional test has significantly greater power than the Proschan-Pyke test for small samples against Weibull and gamma distribution alternatives.

M. Boswell (1966) studied a similar problem concerning Poisson type processes. His statistic based on a conditional maximum likelihood ratio test is essentially the same as the likelihood ratio statistic studied in this section. The main result in Boswell's paper is a derivation of the asymptotic distribution of his test statistic under the null hypothesis. In contrast, we concentrate on small sample results.

Since IFR distributions can have a jump at the right-hand end of their interval of support it is clear from Definition 1 that we need only consider estimators absolutely continuous with respect to Lebesgue measure on $[0, X_n)$ with a jump at X_n (see Barlow and Proschan, (1965), p. 26). Hence

$$L_n(\mathbf{X} \mid F) = [\textstyle\prod_{i=1}^{n-1} f(X_i)]F\{X_n\}$$

where f is the density of F on $[0, X_n)$. Since

$$1 - F(x) = \exp [-\textstyle\int_0^x r(u)\, du] \qquad \text{where} \qquad r(u) = f(u)/[1 - F(u)]$$

for $0 \leqq u < X_n$, we may write

$$f(x) = r(x) \exp [-\textstyle\int_0^x r(u)\, du], \qquad 0 \leqq x < X_n$$

and

$$F\{X_n\} = \exp [-\textstyle\int_0^{X_n} r(u)\, du].$$

Hence

$$\log L_n(\mathbf{X} \mid F) = \textstyle\sum_{i=1}^{n-1} \log r(X_i) - \sum_{i=1}^{n} \int_0^{X_i} r(u)\, du. \tag{3.1}$$

The problem of maximizing (3.1) subject to $r(x)$ nondecreasing was solved by Grenander (1956) and independently by Marshall and Proschan (1965). They show that the problem can be reduced to maximizing

$$\textstyle\sum_{i=1}^{n-1} \log r(X_i) - \sum_{i=1}^{n-1} (n - i)(X_{i+1} - X_i)r(X_i)$$

subject to $r(X_1) \leqq r(X_2) \leqq \cdots \leqq r(X_{n-1})$. The maximum likelihood estimates are

$$\hat{r}_n(X_i) = \min_{v \geqq i+1} \max_{u \leqq i} [(v - u)((n - u)(X_{u+1} - X_u) + \cdots + (n - v + 1)(X_v - X_{v-1}))^{-1}]$$

for $i = 1, 2, \cdots, n - 1$. The maximum likelihood is

$$L_n(\mathbf{X} \mid \hat{F}) = [\textstyle\prod_{i=1}^{n-1} \hat{r}(X_i)]e^{-(n-1)}. \tag{3.2}$$

The exponent on e can be easily verified using the definition of $\hat{r}$ and observing that

$$\textstyle\sum_{i=1}^{n} \int_0^{X_i} \hat{r}(u)\, du = \sum_{i=1}^{n-1} (n - i) \int_{X_i}^{X_{i+1}} \hat{r}(u)\, du.$$

Let $\mathfrak{F}_0 = \{F \mid F(0) = 0, F(x) = 1 - e^{-\lambda x}$ for $x < T$ and $F(T) = 1, \lambda > 0, T > 0\}$. Then $\mathfrak{F}_0$ denotes the class of exponential distributions with possible truncation on the right. Consider now the problem of testing H_0: $F \varepsilon \mathfrak{F}_0$ versus H_1: $F \varepsilon \mathfrak{F} - \mathfrak{F}_0$. The choice of H_0 was determined by the fact that the MLE's $\hat{F}_0$ and $\hat{F}$ are both absolutely continuous with respect to Lebesgue measure on $[0, X_n)$ and place mass at X_n. The likelihood under H_0 will be

$$L_n(\mathbf{X} \mid F_0) = [\textstyle\prod_{i=1}^{n-1} \lambda e^{-\lambda X_i}]e^{-\lambda X_n}$$

and the maximum likelihood will be

$$L_n(\mathbf{X} \mid \hat{F}_0) = ((n - 1)/\textstyle\sum_1^n X_i)^{n-1}e^{-(n-1)}. \tag{3.3}$$

According to Definition 2, the likelihood ratio statistic for testing for truncated exponentiality versus IFR and not truncated exponentiality will be

$$\Lambda_n^*(\mathbf{X}) = (n - 1)^{n-1}/(\textstyle\sum_1^n X_i)^{n-1} \prod_{i=1}^{n-1} \hat{r}(X_i). \tag{3.4}$$

If $1/(n - 1)(X_2 - X_1) \leqq 1/(n - 2)(X_3 - X_2) \leqq \cdots \leqq 1/(X_n - X_{n-1})$ so that

$$\hat{r}(X_i) = 1/(n - 1)(X_{i+1} - X_i), \quad i = 1, 2, \cdots, n - 1,$$

then (3.4) becomes

$$\Lambda_n^{**}(\mathbf{X}) = ((n - 1)/\textstyle\sum_1^n X_i)^{n-1} \prod_{i=1}^{n-1} (n - i)(X_{i+1} - X_i).$$

As in Section 2 we consider the test, ϕ^*, which rejects H_0 when

$$\Lambda_n^*(\mathbf{X}) \leqq c_\alpha$$

where c_α is determined by

$$P_G\{\Lambda_n^*(\mathbf{Y}) \leqq c_\alpha\} = \alpha.$$

The asymptotic distribution of $-\log \Lambda_n^*(\mathbf{Y})$ can be found in Boswell (1966), p. 1572. A table of percentage points obtained using Monte Carlo methods is contained in Table 2.

4. Unbiasedness of the likelihood ratio test for IFR. The likelihood ratio test has greater power under the alternative than under the null hypothesis. To show this we need to introduce some auxiliary results.

TABLE 2
Percentage points for $-\log \Lambda_n^*(\mathbf{Y})$

Sample Size n	Percentiles					
	.01	.05	.10	.90	.95	.99
2	0.010	0.049	0.10	2.29	2.97	4.58
3	0.017	0.086	0.17	2.89	3.68	5.41
4	0.026	0.12	0.22	3.23	4.09	6.06
5	0.029	0.14	0.26	3.52	4.43	6.38
6	0.038	0.16	0.30	3.70	4.63	6.84
7	0.038	0.18	0.32	3.79	4.73	6.73
8	0.046	0.20	0.35	3.95	4.87	7.14
9	0.051	0.22	0.38	4.08	5.02	7.26
10	0.053	0.22	0.40	4.22	5.14	7.37

Given a sequence of nonnegative real numbers $\{z_i\}_{i=1}^n$, plot $\sum_1^i z_j$ versus i and interpolate linearly between $(0, 0)$, $(1, z_1)$, $\cdots$, $(n, z_1 + \cdots + z_n)$. Let $\bar{z}_1 \geqq \bar{z}_2 \geqq \cdots \geqq \bar{z}_n$ denote the slopes of the least concave majorant to this graph in successive intervals. This operation converts the original sequence into a nonincreasing sequence and will be useful later on. For convenience, call $\bar{z}_i$ the *monotone regression function*[2] (a function of the index) of the original function z_i (see Brunk et. al. (1955)). Note that $\bar{z}_1 \geqq \bar{z}_2 \geqq \cdots \geqq \bar{z}_n$ can also be obtained by successive averaging of the original sequence until it becomes nonincreasing.

We say that $H(\bar{z}_1, \bar{z}_2, \cdots, z_n)$ is a *Schur function* if

$$(z_i - z_j)(\partial H/\partial z_i - \partial H/\partial z_j) \geqq 0$$

for all vectors $\mathbf{z}$ and $i, j = 1, 2, \cdots, n$. This concept is needed in the following useful lemma.

LEMMA 4.1. *Let* $(z_1, z_2, \cdots, z_n)$ *and* $(z_1', z_2', \cdots, z_n')$ *denote two nonnegative sequences such that*

$$\sum_1^r z_i \geqq \sum_1^r z_i' \quad \text{for} \quad r = 1, 2, \cdots, n-1$$

and

$$\sum_1^n z_i = \sum_1^n z_i'.$$

Then the inequalities are preserved under monotone regression; i.e.,

$$\sum_{i=1}^r \bar{z}_i \geqq \sum_{i=1}^r \bar{z}_i' \qquad (r = 1, 2, \cdots, n-1)$$

$$\text{(i)} \qquad \sum_{i=1}^n \bar{z}_i = \sum_{i=1}^n \bar{z}_i'.$$

If H is a Schur function then

$$\text{(ii)} \qquad H(\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_n) \geqq H(\bar{z}_1', \bar{z}_2', \cdots, \bar{z}_n').$$

PROOF. (i) is obvious since the least concave majorant to the $\{z_i\}$ sequence lies above the least concave majorant to the $\{z_i'\}$ sequence.

[2] Personal communication with Professor Brunk.

Since (i) holds, $\bar{z}_1 \geqq \bar{z}_2 \geqq \cdots \geqq \bar{z}_n$; $\bar{z}_1' \geqq \cdots \geqq \bar{z}_n'$ and $\sum_1^n \bar{z}_i = \sum_1^n \bar{z}_i'$ we have (ii) by the Schur, Ostrowski theorem (see Ostrowski (1952)). □

THEOREM 4.2. *If $F_2^{-1}F_1(x)$ is convex for $x \geqq 0$, $F_1(0) = F_2(0) = 0$ and $\mathbf{X}(\mathbf{Y})$ denotes an ordered sample from $F_1(F_2)$, then*

$$\Lambda_n^*(\mathbf{X}) \leqq_{\mathrm{st}} \Lambda_n^*(\mathbf{Y})$$

where $\leqq_{\mathrm{st}}$ denotes stochastic ordering.

PROOF. Let $Y_i^* = F_2^{-1}F_1(X_i)$ and note $Y_i^* =_{\mathrm{st}} Y_i$. Let

$$z_i = (n - i)(X_{i+1} - X_i)/ \textstyle\sum_1^{n-1} (n - i)(X_{i+1} - X_i)$$

and $$z_i' = (n - i)(Y_{i+1}^* - Y_i^*)/ \textstyle\sum_1^{n-1} (n - i)(Y_{i+1}^* - Y_i^*).$$

Since $Y_i^* = F_2^{-1}F_1(X_i)$ and $F_2^{-1}F_1$ is convex

$$(n - i)(Y_{i+1}^* - Y_i^*)/(n - i)(X_{i+1} - X_i)$$

is increasing in $i = 1, 2, \cdots, n$. It follows from Lemma 3.7 (i) of Barlow and Proschan (1966) that

$$\textstyle\sum_1^r z_i'/\sum_1^r z_i \leqq \sum_1^{n-1} z_i'/\sum_1^{n-1} z_i = 1$$

and hence $\sum_1^r z_i \geqq \sum_1^r z_i'$ for $r = 1, 2, \cdots, n - 1$. Let $\{\bar{z}_i\}$ and $\{\bar{z}_i'\}$ denote the monotone regression estimates of $\{z_i\}$ and $\{z_i'\}$ respectively. Let

$$-H(x_1, x_2, \cdots, x_{n-1}) = \textstyle\prod_{i=1}^{n-1} x_i$$

and note that H is a Schur function. Since $\{z_i\}$ and $\{z_i'\}$ satisfy the hypotheses of Lemma 4.1, it follows that

$$-H(\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_{n-1}) \leqq -H(\bar{z}_1', \bar{z}_2', \cdots, \bar{z}_{n-1}').$$

Hence

$$(4.1) \quad \{[\textstyle\sum_1^{n-1} (n - i)(X_{i+1} - X_i)]^{n-1} \prod_1^{n-1} \hat{r}(X_i)\}^{-1}$$
$$\leqq \{[\textstyle\sum_1^{n-1} (n - i)(Y_{i+1}^* - Y_i^*)]^{n-1} \prod_1^{n-1} \hat{r}(Y_i^*)\}^{-1}.$$

Since

$$\textstyle\sum_0^r (n - i)(Y_{i+1}^* - Y_i^*)/\sum_0^r (n - i)(X_{i+1} - X_i) \leqq \sum_1^n Y_i^*/\sum_1^n X_i$$

for $(1 \leqq r \leqq n - 1)$ by Lemma 3.7 (i′) of Barlow and Proschan (1966) it follows that

$$(4.2) \quad \textstyle\sum_1^{n-1} (n - i)(X_{i+1} - X_i)/\sum_1^n X_i$$
$$\leqq \textstyle\sum_1^{n-1} (n - i)(Y_{i+1}^* - Y_i^*)/ \sum_1^n Y_i^*.$$

(4.1) and (4.2) together imply

$$\Lambda_n^*(\mathbf{X}) \leqq \Lambda_n^*(\mathbf{Y}^*).$$

The theorem follows from $(Y_1, Y_2, \cdots, Y_n) =_{\mathrm{st}} (Y_1^*, Y_2^*, \cdots, Y_n^*)$. □

COROLLARY 4.3. *The test, ϕ^*, based on the likelihood ratio statistic $\Lambda_n^*(\mathbf{X})$ is unbiased at all significance levels for the problem*

$$H_0: F \varepsilon \{\text{DFR}\} \qquad \textit{versus} \qquad H_1: F \varepsilon \{\text{IFR}\}.$$

5. Distribution of the maximum likelihood ratio statistic under the exponential assumption. From the computations in Boswell (1966) it is clear that the distribution of Λ_n^*, even under the null hypothesis, is exceedingly complicated. For this reason we have had to use Monte Carlo methods to obtain the percentage points tabulated in Table 2. However, the distribution of Λ_n^* under H_0 is quite smooth as we show in

THEOREM 5.1. *The likelihood ratio statistic Λ_n^* has a nonincreasing density on* $(0, 1)$ *under the exponential assumption.*

PROOF. Let $0 \equiv W_0 \leqq W_1 \leqq \cdots \leqq W_n$ denote an ordered sample from the uniform distribution on $(0, 1)$. Let

$$U_i = W_i - W_{i-1}, \qquad i = 1, 2, \cdots, n-1.$$

Then the random vector $(U_1, U_2, \cdots, U_{n-1})$ has joint density

$$h(u_1, u_2, \cdots, u_{n-1}) = (n-1)! \text{ for } u_i \geqq 0, i = 1, 2, \cdots, n-1, 0 \leqq u_1 + u_2 + \cdots + u_{n-1} \leqq 1;$$
$$= 0 \qquad \text{otherwise.}$$

Let $(\bar{U}_1, \bar{U}_2, \cdots, \bar{U}_{n-1})$ denote the modified vector $(U_1, U_2, \cdots, U_{n-1})$ under monotone regression and subject to $\bar{U}_1 \geqq \bar{U}_2 \geqq \cdots \geqq \bar{U}_{n-1}$.

The likelihood ratio statistic

$$\Lambda_n^*(\mathbf{Y}) = \{[\sum_1^n Y_i/(n-1)]^{n-1} \prod_1^{n-1} \hat{r}(Y_i)\}^{-1}$$

is distributed as $(n-1)^{n-1} \prod_1^{n-1} \bar{U}_i$ under the exponential assumption. Notationally it will be convenient to replace $n-1$ by n. Hence we need only prove that $P\{\prod_{i=1}^n \bar{U}_i \leqq z\}$ is concave in $z \varepsilon (0, 1)$. Let I denote the usual indicator set function and observe that

$$P\{\prod_{i=1}^n \bar{U}_i \leqq z\} = n! \int\int \cdots \int_{u_i \geqq 0, 0 \leqq u_1 + \cdots + u_n \leqq 1} I[\bar{u}_1 \cdots \bar{u}_n \leqq z]\, du_1 \cdots du_n.$$

Integrating out on u_n we have

$$P\{\prod_1^n \bar{U}_i \leqq z\} = n! \int\int \cdots \int_{u_i \geqq 0, u_1 + \cdots + u_{n-1} \leqq 1} f(u_1, \cdots, u_{n-2}; z)\, du_1 \cdots du_{n-1}$$

where

$$f(u_1, \cdots, u_{n-1}; z) = \int_{0 \leqq u_n \leqq 1 - u_1 - \cdots - u_{n-1}} I[\bar{u}_1 \cdots \bar{u}_n \leqq z]\, du_n$$
$$= \min[1 - u_1 - \cdots - u_{n-1}, u_n(z)]$$

and $u_n(z)$ is the solution of $z = \bar{u}_1 \cdots \bar{u}_{n-1}\bar{u}_n$ for fixed $u_1, u_2, \cdots, u_{n-1}$.

We claim that z is a strictly increasing convex function of u_n and, therefore, that u_n is a strictly increasing *concave* function of z. It follows that $f(u_1, \cdots,$

$u_{n-1}\,; z)$ is a concave function of z for fixed $(u_1, u_2, \cdots, u_{n-1})$. Hence

$P\{\prod_1^n \bar{U}_i \leqq z\} = n! \int\int \cdots \int_{u_i \geqq 0, u_1+\cdots+u_{n-1} \leqq 1} f(u_1, \cdots, u_{n-1}\,; z\, du \cdots du_{n-1}$

is a concave function of z.

To show $z = \bar{u}_1 \cdots \bar{u}_{n-1}\bar{u}_n$ is a convex function of u_n, define $(\bar{u}_1^*, \cdots, \bar{u}_{n-1}^*)$ to be the monotone regression modification of $(u_1, u_2, \cdots, u_{n-1})$ subject to $\bar{u}_1^* \geqq \bar{u}_2^* \geqq \cdots \geqq \bar{u}_{n-1}^*$. Clearly z is piecewise convex for u_n in the intervals $[0, \bar{u}_{n-2}^*]$, $[\bar{u}_{n-2}^*, \bar{u}_{n-3}^*]$, $\cdots$, $[1 - \bar{u}_1^* - \cdots - \bar{u}_1^*]$. It is therefore sufficient to show that z has a continuous derivative in u_n. We show that the right and left hand derivatives at $u_n = \bar{u}_{n-1}^*$ are equal. For $u_n < \bar{u}_{n-1}^*$

$$(dz/du_n) = \bar{u}_1^* \cdots \bar{u}_j^*(u_{j+1} + \cdots + u_{n-1})/(n - 1 - j))^{n-j-1}.$$

For $u_{n-1}^* < u_n < u_{n-2}^*$

$$(dz/du_n) = \bar{u}_1^* \cdots \bar{u}_j^*((u_{j+1} + \cdots + u_n)/(n - j))^{n-j-1}.$$

For $\bar{u}_n = \bar{u}_{n-1}^*$, obviously

$(u_{j+1} + \cdots + u_{n-1})/(n - 1 - j) = (u_{j+1} + \cdots + u_{n-1} + u_n)/(n - j)$. □

For $n = 2$ and $n = 3$ it is a straightforward computation to obtain the distribution of $\Lambda_n^*(\mathbf{Y})$. Clearly, for $n = 2$, $\Lambda_2^*(\mathbf{Y}) = U_1$ and the likelihood ratio is uniformly distributed on $(0, 1)$. For $n = 3$

$$\Lambda_3^*(\mathbf{Y}) = 4\bar{U}_1\bar{U}_2 = 4U_1U_2 \qquad \text{if} \qquad U_1 \geqq U_2$$
$$= (U_1 + U_2)^2 \qquad \text{if} \qquad U_1 \leqq U_2.$$

Hence

$P_G\{\Lambda_3^*(\mathbf{Y}) \leqq u\} = \int\int_{\{u_1u_2 \leqq \frac{1}{4}u, u_1 \geqq u_2\}} 2\, du_1\, du_2 + \int\int_{\{1+u_2 \leqq u^{\frac{1}{2}}, u_1 \leqq u_2\}} 2\, du_1\, du_2$

and

$P_G\{\Lambda_3^*(\mathbf{Y}) \leqq u\} = \frac{1}{4}3u + \frac{1}{2}u[\log((1 + (1 - u)^{\frac{1}{2}})u^{-\frac{1}{2}}] + (\frac{1}{2}(1 - (1 - u)^{\frac{1}{2}}))^2.$

The density is $g_3(u) = \frac{1}{2} + \frac{1}{2} \log[(1 + (1 - u)^{\frac{1}{2}})u^{-1}]$.

It is easy to check that g is decreasing, $g_3(0) = +\infty$, $g_3(1) = \frac{1}{2}$ and $g_3'(0) = g_3'(1) = -\infty$. It is tempting to conjecture that this behavior is true in general, i.e., $g_n(0) = +\infty$, $g_n'(0) = g_n'(1) = -\infty$ for $n \geqq 3$.

6. Tests for constant versus monotone failure rate. In Section 3 we obtained the likelihood ratio test, ϕ^*, for testing truncated exponentiality versus IFR. By Theorem 4.1 this test is clearly also unbiased for the problem of testing constant versus increasing failure rate.

Marshall, Walkup and Wets (1966) have characterized the class of unbiased tests for constant failure rate versus nondecreasing failure rate. These are based on functions $h(x_1, x_2, \cdots, x_n)$ satisfying the conditions

(6.1) (i) h is homogeneous;

(ii) $\sum_{i=1}^{j} (x_i - x_{j+1})\partial h(x_1, \cdots, x_n)/\partial x_i \geqq 0$, $j = 1, 2, \cdots, n - 1$, for all $x_1 > x_2 > \cdots > x_n > 0$. The corresponding test consists of rejecting exponentiality if $h(X_1, X_2, \cdots, X_n) \leqq c$ where c is a suitable critical number and $X_1 \geqq X_2 \geqq \cdots \geqq X_n$ are the order statistics labelled in reverse order. Whether or not a UMP unbiased test exists in this class of tests is unknown.

In Section 2 we proved that $F <_* G$ implies

$$\Lambda_n(\mathbf{X}) \geqq_{st} \Lambda_n(\mathbf{Y}).$$

Hence we can define a test, ϕ^{**}, which rejects exponentiality in favor of the IFRA hypothesis when $\Lambda_n(\mathbf{X}) \geqq c_{1-\alpha}$ where

$$P_G\{\Lambda_n(\mathbf{Y}) \geqq c_{1-\alpha}\} = 1 - P_G\{\Lambda_n(\mathbf{Y}) \leqq c_{1-\alpha}\} = \alpha.$$

For this test we would use the upper percentile points of $-\log \Lambda_n(\mathbf{Y})$ given in Table 1. By Theorem 2.1, this test is unbiased.

Note that although the test ϕ^{**} is not a likelihood ratio test it is based on a statistic, $\Lambda_n(\mathbf{X})$, which is essentially the maximum likelihood under the IFRA assumption. This may help to explain the fact that ϕ^{**} performs better (see Figure 1) than the test ϕ^* even though the latter test is (almost) the likelihood ratio test for this problem.

Of course there are also many unbiased tests for constant failure rate versus IFRA. Marshall, Walkup and Wets (1966) have characterized the class of all such tests. These are just the tests based on functions $f(x_1, x_2, \cdots, x_n)$ having the properties:

(6.2) (1) f is homogeneous;

(2) $\sum_{i=1}^{j} x_i \partial f(x_1, x_2, \cdots, x_n)/\partial x_i \geqq 0$ for $j = 1, 2, \cdots, n-1$ and all

$$x_1 \geqq x_2 \geqq \cdots \geqq x_n \geqq 0.$$

The test associated with f would reject exponentiality if $f(X_1, X_2, \cdots, X_n) \leqq c$ where c is some suitable critical number and $X_1 \geqq X_2 \geqq \cdots \geqq X_n$ are the usual order statistics labelled in reverse order.

The class of tests based on statistics satisfying (6.2) is of course smaller than the class of tests based on statistics satisfying (6.1) since IFR implies IFRA but not conversely. Whether or not a UMP unbiased test exists in the class of tests defined by (6.2) is unknown.

An important test because of its simplicity and good power is a *uniform conditional test* (see Cox and Lewis (1966), p. 153) based on the mean of the rectangular distribution. This has been described by Bartholomew as the oldest known statistical test [see discussion in Cox (1955)]. Epstein (1960) adapted this test to the life testing problem and called it test 3. The test is based on the *total time on test* up to the ith order statistic ($i = 1, 2, \cdots, n$), i.e.,

$$T(X_i) = \sum_{j=1}^{i} (n - j + 1)(X_j - X_{j-1}).$$

If $r(1 \leqq r \leqq n)$ failures are observed from a sample of size n, then the *total time*

26

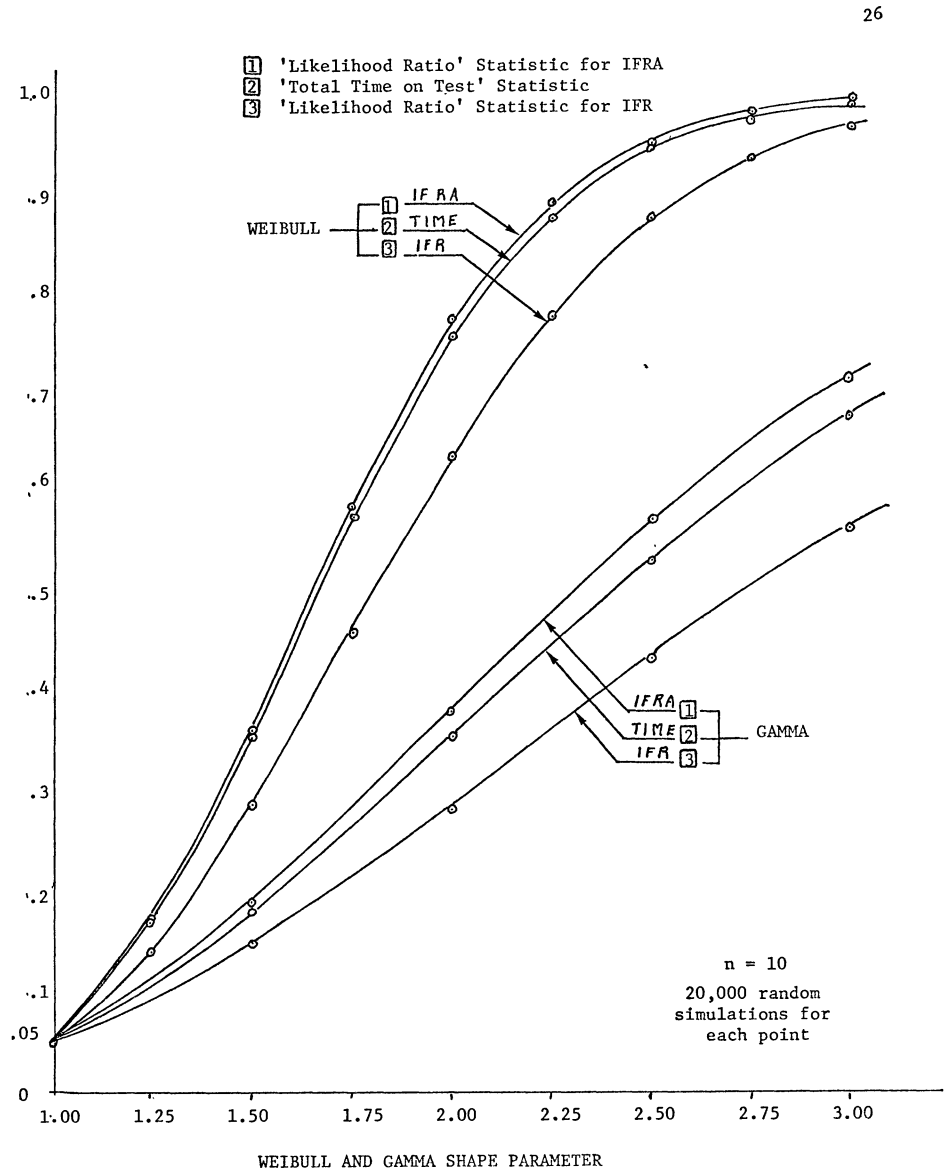

FIG. 1

on test statistic is $\sum_{i=1}^{r-1} T(X_i)/T(X_r)$. Under the exponential hypothesis

$$Z = [\sum_1^{r-1} T(X_i) - \tfrac{1}{2}(r-1)T(X_r)][T(X_r)(12)^{-\frac{1}{2}}(r-1)^{\frac{1}{2}}]^{-1}$$

is approximately $N(0, 1)$ even for relatively small r. If $F <_* G$, then it follows from Lemma 3.7 (i′) [Barlow and Proschan (1966)] that

$$\sum_1^{r-1} T(X_i)/T(X_r) \geqq_{\text{st}} \sum_1^{r-1} T(Y_i)/T(Y_r).$$

Hence a natural test, ϕ^{***}, rejects exponentiality in favor of IFRA if

$$\sum_1^{r-1} T(X_i)/T(X_r) \geqq c_\alpha$$

where

$$P_G\{\sum_1^{r-1} T(Y_i)/T(Y_r) \geqq c_\alpha\} = \alpha.$$

Investigations by Cox (1955) show that the analogue of this test for randomness in a sequence of events is the most powerful test of the Poisson hypothesis against the alternative of a time-dependent Poisson process with occurrence rate

$$\lambda(t) = e^{\alpha+\beta t}.$$

TABLE 3

Percentage points for the total time on test statistic

No. of Failtures r	Percentiles					
	.01	.05	.10	.90	.95	.99
2	0.01	0.05	0.10	0.90	0.95	0.99
3	0.14	0.32	0.45	1.55	1.68	1.85
4	0.39	0.68	0.84	2.15	2.33	2.61
5	0.69	1.04	1.25	2.75	2.95	3.30
6	1.02	1.43	1.65	3.34	3.57	4.00
7	1.41	1.83	2.08	3.90	4.15	4.60
8	1.77	2.24	2.52	4.49	4.75	5.24
9	2.12	2.65	2.94	5.06	5.35	5.88
10	2.52	3.06	3.38	5.62	5.92	6.47

See Bartholomew (1956), Bickel and Doksum (1967) and Nadler and Eilbott for further results concerning this test.

It should be noted that the test described above and whose power is plotted in Figure 1 against Weibull and gamma alternatives is *not* a conditional test as such. Percentage points for this test are tabulated in Table 3.

From Figure 1 it can be seen that although ϕ^{**} has greater power than the test based on the "total time on test" statistic, they are fairly close. ϕ^* is distinctly inferior and the Proschan-Pyke test, if plotted, would be seen to be distinctly inferior to ϕ^*. Unfortunately, asymptotic comparisons of the total time on test statistic with ϕ^{**} are not available at present.

There are many additional unbiased tests of exponentiality versus IFRA or IFR which should perhaps be considered. Recall that all of the associated statistics are necessarily homogeneous. A statistic related to the IFR likelihood ratio statistic is

$$\Lambda_n^{**}(\mathbf{X}) = (n/\sum_1^n X_i) \prod_{i=1}^n (n - i + 1)(X_i - X_{i-1}).$$

If there are no reversals of the normalized differences (they should decrease under IFR alternatives) then Λ_n^* and Λ_n^{**} agree except for the factor nX_1 and a constant. If $G^{-1}F$ is convex, then

$$\Lambda_n^{**}(\mathbf{X}) \leqq_{st} \Lambda_n^{**}(\mathbf{Y}).$$

The test which rejects exponentiality when $\Lambda_n^{**}(\mathbf{X})$ is sufficiently large is related to a test derived by Moran (1951) for a problem concerning renewal processes. Under the assumption of exponentiality

$$W = -2 \log \Lambda_n^{**}(\mathbf{Y})/[1 + (n + 1)/6n]$$

is asymptotically distributed as a χ^2 variable with $n - 1$ degrees of freedom. Epstein's (1960) test 8 uses this statistic. Monte Carlo experiments by Zelen (1961) indicate that the power of this test is poor for small samples against Weibull distribution alternatives.

7. Concluding remarks. All Monte Carlo calculations used to produce Tables 1, 2, 3 and Figure 1 were based on 20,000 simulations.

It is perhaps worth noting that the percentage points in Table 2 and the results of Section 5 also apply to the Boswell test for trend in a stochastic process of Poisson type. However, if the sample size is n and one is using the Boswell statistic then one should locate percentage points in Table 2 corresponding to the number $n + 1$. A proof for unbiasedness of the Boswell test can be made, patterned after the techniques of Section 4.

The number of possible likelihood ratio tests which may be constructed using the definitions in section 1 is fairly large. Recall that the DFR (for decreasing failure rate) maximum likelihood estimate is absolutely continuous when $F(0) = 0$ [Marshall and Proschan (1965)]. Hence one can construct a likelihood ratio test for the following problems:

(1) H_0: F a truncated exponential versus H_1: F DFR and then IFR ($F(0) = 0$);

(2) H_0: F IFR versus H_1: F DFR and then IFR ($F(0) = 0$).

Note that the maximum likelihood estimates under both the hypothesis and the alternative in each case will be absolutely continuous except at the largest observation, X_n, if we impose the additional restriction $F(0) = 0$.

Clearly we can also construct a maximum likelihood test for

(3) H_0: F truncated DFR versus H_1: F DFR and then IFR.

There is no difficulty in constructing maximum likelihood tests for the problems:

(4) H_0: F exponential versus H_1: F DFR ($F(0) = 0$)

and

(5) H_0: F DFR ($F(0) = 0$) versus H_1: F has decreasing density ($F(0) = 0$) and F not DFR.

The maximum likelihood estimate assuming a decreasing density is given by Grenander (1956). Recall that if F is DFR, then it has a decreasing density.

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