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ESTIMATION OF TWO ORDERED TRANSLATION PARAMETERS

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1. Introduction and summary. Let the random variables $X_{i1}, X_{i2}, \cdots, X_{in}$, $i = 1, 2$, be real valued and independent with density functions $f(x - \theta_i)$ (θ_i real), $i = 1, 2$, (with respect to Lebesgue measure). We take $\int_{-\infty}^{\infty} xf(x)\, dx = 0$ with no loss of generality. The problem is to estimate the ordered pair (θ_1, θ_2), under the condition $\theta_2 \geqq \theta_1$, when the loss function is the sum of the squared errors in estimating the individual components. Questions of minimaxity and admissibility of the analogue of the Pitman estimator are considered.

This problem, which represents a two dimensional estimation problem subject to constraints, has received attention in the past. Most of the literature deals with obtaining maximum likelihood estimates for specified densities. (See for example Brunk [3].) Katz [6] considers some aspects of the problem for the binomial and normal densities.

The analogue of the Pitman estimator studied here is the vector estimator which is the *a posteriori* expected value of (θ_1, θ_2) given X_{ij}, $i = 1, 2, j = 1, 2, \cdots, n$, when the generalized prior distribution is the uniform distribution on the half space $\theta_2 \geqq \theta_1$. If we call this estimator $\delta = (\delta_1, \delta_2)$, then

$$\begin{aligned}(1.1)\qquad \delta_i(X_{11}, X_{12}, \cdots, X_{2n}) &= \textstyle\iint_{\theta_2 \geqq \theta_1} \theta_i \prod_{j=1}^n f(X_{1j} - \theta_1) \prod_{j=1}^n f(X_{2j} - \theta_2)\, d\theta_1\, d\theta_2 \\ &\quad \cdot [\textstyle\iint_{\theta_2 \geqq \theta_1} \prod_{j=1}^n f(X_{1j} - \theta_1) \prod_{j=1}^n f(X_{2j} - \theta_2)\, d\theta_1\, d\theta_2]^{-1}, \quad i = 1, 2.\end{aligned}$$

In order to state the main results, it is convenient to introduce some notation. Let

$$(1.2)\qquad X_i = \textstyle\int \theta_i \prod_{j=1}^n f(X_{ij} - \theta_i)\, d\theta_i / \int \prod_{j=1}^n f(X_{ij} - \theta_i)\, d\theta_i, \quad i = 1, 2.$$

Let $Y_i = (Y_{i1}, Y_{i2}, \cdots, Y_{i,n-1})$ where $Y_{ij} = X_{i,j+1} - X_{i1}$. Let $p(x, y)$ be the conditional density of X_i given Y_i when $\theta_i = 0$.

We obtain the following results: (a) If $EE[(X_1^2 + X_2^2) \mid Y_1, Y_2] < \infty$, and if $p(x, y) = p(-x, y)$ then the Pitman estimator (δ_1, δ_2) given in (1.1) is minimax. The normal and uniform densities are examples of when this condition is satisfied. (b) Let $P(x, y)$ denote the cumulative distribution function corresponding to $p(x, y)$. That is, for a fixed y,

$$P(x, y) = \textstyle\int_{-\infty}^{x} p(u, y)\, du. \quad \text{If} \quad EE[(X_1^2 + X_2^2) \mid Y_1, Y_2] < \infty$$

and if $p(x, y)$ is such that for each y, $p(x, y)/(1 - P(x, y))$ increases in x, (i.e.

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increasing hazard rate) and $p(x, y)/P(x, y)$ decreases in x, then the Pitman estimator is minimax. The family of gamma densities, $f(t) = t^\alpha \exp(-t)/\Gamma(\alpha + 1)$ for $\alpha > 0$, are examples of when the above condition is satisfied. The proofs of the minimax results (a) and (b) are based essentially on the method of Farrell [4]. (c) An example is given which indicates that in general the Pitman estimator is not minimax. The example is justified by a computation performed by numerical integration. The numerical integration shows that the risk of the Pitman estimator exceeds the risk of an estimator known to be minimax. The results (a) (b), and (c) indicate that whereas in a related one dimensional problem, namely to estimate a translation parameter θ, subject to $\theta \geqq 0$, (see Farrell [4], Section 7), the Pitman estimator is always minimax (save for moment and continuity conditions), the same is not true for this two dimensional problem. (d) Let

$$(1.3)\quad \rho(y) = \max \{\sup_{-\infty<x<-1} [\int_{-\infty}^{\infty} v\, dv \int_{-\infty}^{\infty} p(u - v, y_1) \cdot p(u + v, y_2)\, du/x \int_{-\infty}^{x} \int_{-\infty}^{\infty} p(u - v, y_1)p(u + v, y_2)\, du\, dv], 2\}.$$

If

$$(1.4)\quad E\rho^2(y)E[(X_1^4 + X_2^4) \cdot(1 + |\log (X_1^2 + X_2^2)|^\beta \mid Y_1, Y_2] < \infty, \quad \text{for some} \quad \beta > 0,$$

then the Pitman estimator given in (1.1) is admissible. The normal density is an example for which (1.4) holds. Whereas Katz [6] stated the admissibility result for the normal case, the proof there is not adequate. The proof given here is based on results of Stein [9], and James and Stein [5].

In the next section we give notation. The minimax results are given in Section 3, and admissibility in Section 4.

2. Notation. The notation of the preceding paper [2] will be adopted completely and references in the sequel to equations (2.1) through (2.16) refer to the corresponding equations in [2]. In addition we add here the equation for the risk of an estimator $(\delta_1(X_1, X_2, Y), \delta_2(X_1, X_2, Y))$ for (θ_1, θ_2)

$$(2.17)\quad R(\theta_1, \theta_2, \delta_1, \delta_2) = \int \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [(\delta_1(x_1, x_2, y) - \theta_1)^2 + (\delta_2(x_1, x_2, y) - \theta_2)^2]p(x_1 - \theta_1, x_2 - \theta_2, y)\, dx_1\, dx_2 \nu(dy).$$

In terms of this notation the estimator (1.1) becomes

$$(2.18)\quad \delta_i(X_1, X_2, Y) = \int \int_{\theta_2 \geqq \theta_1} \theta_i p(X_1 - \theta_1, Y_1)p(X_2 - \theta_2, Y_2)\, d\theta_1\, d\theta_2 \cdot [\int\int_{\theta_2 \geqq \theta_1} p(X_1 - \theta_1, Y_1)p(X_2 - \theta_2, Y_2)\, d\theta_1\, d\theta_2]^{-1}.$$

3. Minimax property of Pitman estimator for estimating both parameters, given $\theta_2 \geqq \theta_1$. In Theorems 3.2 and 3.3 below we give sufficient conditions for the Pitman estimator to be minimax. It appears that, whereas in the analogous one dimensional problem, the Pitman estimator is always minimax, provided the distribution p has finite variance (see Farrell [4], Section 7), the same is not true for the problem considered here. In fact an example is offered in which it appears that the Pitman estimator is not minimax. This proposed counterexample is supported by computing a risk function by numerical integration. The development of this section is based on that of Farrell [4].

First we state

THEOREM 3.0. *Let* $(\delta_1(X_1, X_2, Y), \delta_2(X_1, X_2, Y))$ *be any estimator of* (θ_1, θ_2) *with the property that*

$$(3.1) \qquad R(\theta_1, \theta_2, \delta_1, \delta_2) \leqq R \quad \text{for} \quad \theta_2 \geqq \theta_1,$$

where R *is the constant risk of the estimator* (X_1, X_2) (we assume $R < \infty$). *Then* (δ_1, δ_2) *is a minimax estimator of* (θ_1, θ_2), *for* $\theta_2 \geqq \theta_1$.

Theorem 3.0 is a special case of the following more general result.

THEOREM 3.1. *Let* Σ_0 *be a subset of Euclidean 2-space* E_2 *such that there exists a sequence* $\{a_n, b_n \mid n \geqq 1\}$ *for which*

$$(3.2) \qquad \liminf_{n\to\infty} \{(\theta_1, \theta_2):(\theta_1 + a_n, \theta_2 + b_n) \,\varepsilon\, \Sigma_0\} = E_2.$$

Let (δ_1, δ_2) *be an estimator with*

$$(3.3) \qquad R(\theta_1, \theta_2, \delta_1, \delta_2) \leqq R \quad \text{if} \quad (\theta_1, \theta_2) \,\varepsilon\, \Sigma_0$$

where R *is as in Theorem* 3.0. *Then* (δ_1, δ_2) *is a minimax estimator of* (θ_1, θ_2), *for* $(\theta_1, \theta_2) \,\varepsilon\, \Sigma_0$.

PROOF. Suppose there exists an estimator $(\tilde{\delta}_1, \tilde{\delta}_2)$ and $\epsilon \geqq 0$, such that

$$(3.4) \qquad R(\theta_1, \theta_2, \tilde{\delta}_1, \tilde{\delta}_2) \leqq R - \epsilon \quad \text{if} \quad (\theta_1, \theta_2) \,\varepsilon\, \Sigma_0.$$

By change of variable in (2.17), we see that if $(\theta_1 + a_n, \theta_2 + b_n) \,\varepsilon\, \Sigma_0$, then

$$(3.5) \quad R(\theta_1, \theta_2, \tilde{\delta}_1(x_1 + a_n, x_2 + b_n) - a_n, \tilde{\delta}_2(x_1 + a_n, x_2 + b_n) - b_n) \leqq R - \epsilon.$$

From the sequence of estimators $[\tilde{\delta}_1(x_1 + a_n, x_2 + b_n) - a_n, \tilde{\delta}_2(x_1 + a_n, x_2 + b_n) - b_n]$ we may take a subsequence which converges regularly (weakly) to a limiting estimator $\delta^* = (\delta_1^*, \delta_2^*)$ with the property that

$$(3.6) \qquad R(\theta_1, \theta_2, \delta_1^*, \delta_2^*) \leqq R - \epsilon, \qquad \text{for all} \quad (\theta_1, \theta_2) \,\varepsilon\, E_2.$$

This last statement follows from (3.5), (3.2), and the fact that for the unrestricted problem the set of decision functions is compact, after compactification of the action space. (See LeCam [7], Remark 6 and see [2], Theorem 3.1). Furthermore, for the unrestricted problem it is well known that (X_1, X_2) is a minimax invariant estimator with risk R. Hence from (3.6) we must have $\epsilon = 0$, which proves the theorem.

REMARKS. (1) The generalization of Theorem 3.1 to K dimensions is obvious. Clearly, (3.2) will not be satisfied when Σ_0 is a bounded set, or when Σ_0 is contained between a pair of parallel lines. For Theorem 3.0, we may take $a_n = 0$, $b_n = n$.

(2) It appears that this theorem would follow also from the results of Peisakoff [8].

By virtue of Theorem 3.0, it follows that if we show that the Pitman estimator satisfies (3.1) then it will be minimax. We proceed to develop sufficient conditions that determine when the Pitman estimator satisfies (3.1). Note that the risk for an estimator (δ_1, δ_2), given in (2.17), may be written as

$$\begin{aligned} &R(\theta_1, \theta_2, \delta_1, \delta_2) \\ (3.7) \qquad &= \tfrac{1}{2} \int_{\mathcal{Y}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [(\delta_1 + \delta_2 - (\theta_1 + \theta_2))^2 + (\delta_2 - \delta_1 - (\theta_2 - \theta_1))^2] \\ &\quad \cdot p(x_1 - \theta_1, x_2 - \theta_2, y)\, dx_1\, dx_2 \nu(dy). \end{aligned}$$

If we let $\gamma_1 = (\delta_1 + \delta_2)/2$, $\gamma_2 = (\delta_2 - \delta_1)/2$ and define Z_0, Z_1, η, and μ as in (2.12), then (3.7) becomes

$$\begin{aligned} &R(\mu, \eta, \gamma_1, \gamma_2) \\ (3.8) \qquad &= 4 \int_{\mathcal{Y}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\gamma_1 - \mu)^2 + (\gamma_2 - \eta)^2] p((z_1 - \mu) - (z_0 - \eta), y_1) \\ &\quad \cdot p((z_1 - \mu) + (z_0 - \eta), y_2)\, dz_0\, dz_1 \nu(dy). \end{aligned}$$

It is convenient to rewrite (3.8) as

$$(3.9) \qquad R(\mu, \eta, \gamma_1, \gamma_2) = 4R_1(\mu, \eta, \gamma_1, \gamma_2) + 4R_2(\mu, \eta, \gamma_1, \gamma_2)$$

where

$$\begin{aligned} (3.9a) \quad R_1 = \int_{\mathcal{Y}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\gamma_1 - \mu]^2 p((z_1 - \mu) - (z_0 - \eta), y_1) \\ \cdot p((z_1 - \mu) + (z_0 - \eta), y_2)\, dz_0\, dz_1 \nu(dy) \end{aligned}$$

and R_2 is defined similarly.

The Pitman estimator is determined by

$$\begin{aligned} (3.10) \quad 0 = \int_{-\infty}^{\infty} \int_0^{\infty} (\gamma_1 - \mu) p((z_1 - \mu) - (z_0 - \eta), y_1) \\ \cdot p((z_1 - \mu) + (z_0 - \eta), y_2)\, d\eta\, d\mu \end{aligned}$$

and

$$\begin{aligned} (3.11) \quad 0 = \int_{-\infty}^{\infty} \int_0^{\infty} (\gamma_2 - \eta) p((z_1 - \mu) - (z_0 - \eta), y_1) \\ \cdot p((z_1 - \mu) + (z_0 - \eta), y_2)\, d\eta\, d\mu. \end{aligned}$$

If we now let $u = z_1 - \mu$ and $v = z_0 - \eta$, then (3.11) becomes

$$(3.12) \quad 0 = \int_{-\infty}^{\infty} \int_{-\infty}^{z_0} (\gamma_2 - z_0 + v) p(u - v, y_1) p(u + v, y_2)\, dv\, du.$$

Call

$$(3.13) \qquad \int_{-\infty}^{\infty} p(u - v, y_1) p(u + v, y_2)\, du = g(v, y).$$

It follows from (3.12) and (3.13) that for (z_0, y) such that $\int_{-\infty}^{z_0} g(v, y)\, dv > 0$,

$$(3.14) \qquad \gamma_2 = z_0 - \left(\int_{-\infty}^{z_0} v g(v, y)\, dv \Big/ \int_{-\infty}^{z_0} g(v, y)\, dv \right).$$

We define $\gamma_2(z_0, y)$ to be z_0 otherwise. Similarly from (3.10) we find that for

(z_0, y) such that $\int_{-\infty}^{z_0} g(v, y)\, dv > 0$,

$$(3.15)\quad \gamma_1 = z_1 - \int_{-\infty}^{z_0}\int_{-\infty}^{\infty} up(u - v, y_1)p(u + v, y_2)\, du\, dv / \int_{-\infty}^{z_0}\int_{-\infty}^{\infty} p(u - v, y_1)p(u + v, y_2)\, du\, dv.$$

From (3.14) we see that γ_2 depends only on (z_0, y) and not on z_1. Hence

$$(3.16)\quad 4R_2(\mu, \eta, \gamma_1, \gamma_2) = \int_{\mathcal{Y}}\int_{-\infty}^{\infty} (\gamma_2 - \eta)^2 g(z_0 - \eta, y)\, dz_0 \nu(dy),$$

which implies by the result of Farrell [4], Section 7, that $4R_2(\mu, \eta, \gamma_1, \gamma_2) \leqq R/2$, and that

$$(3.17)\quad \lim_{\mu\to\infty} \lim_{\eta\to\infty} 4R_2(\mu, \eta, \gamma_1, \gamma_2) = R/2.$$

Also, if

$$(3.18)\quad \int_{-\infty}^{\infty} up(u - v, y_1)p(u + v, y_2)\, du = 0,$$

it follows that $\gamma_1 = z_1$, which in turn implies that $4R_1(\mu, \eta, \gamma_1, \gamma_2) = R/2$. This then gives us

THEOREM 3.2. *If the density p defined in* (2.6) *has finite variance and if it is such that* (3.18) *holds, then the Pitman estimator is minimax.*

We note that when $n = 1$, condition (3.18) is implied by any density f which is symmetric. Also it is clear that (3.18) is satisfied when f is either the normal density or uniform density for arbitrary n. (See Section 3 of [2].)

In order to develop another sufficient condition for the Pitman estimator to be minimax, we introduce some further notation and prove some lemmas.

Let

$$(3.19)\quad G(z_0, y) = \int_{-\infty}^{z_0} g(v, y)\, dv, \quad P(z_0, y_i) = \int_{-\infty}^{z_0} p(x, y_i)\, dx, \quad i = 1, 2.$$

Note

$$(3.20)\quad G(z_0, y) = \tfrac{1}{2}\int_{-\infty}^{\infty} p(x, y_1)P(x + 2z_0, y_2)\, dx = \tfrac{1}{2}\int_{-\infty}^{\infty} p(x, y_2)[1 - P(x - 2z_0, y_1)]\, dx.$$

Also let

$$\begin{aligned}
E(z_0, y) &= \tfrac{1}{4}\int_{-\infty}^{\infty} xp(x, y_2)P(x - 2z_0, y_1)\, dx \\
&= -\tfrac{1}{4}\int_{-\infty}^{\infty} xp(x, y_2)[1 - P(x - 2z_0, y_1)]\, dx, \\
F(z_0, y) &= \tfrac{1}{4}\int_{-\infty}^{\infty} xp(x, y_1)P(x + 2z_0, y_2)\, dx, \\
(3.21)\quad \hat{E}(z_0, y) &= E(z_0, y)/G(z_0, y), \qquad \hat{F}(z_0, y) = F(z_0, y)/G(z_0, y), \\
K(z_0, y) &= -(F(z_0, y) + E(z_0, y))/G(z_0, y), \\
H(z_0, y) &= (F(z_0, y) - E(z_0, y))/G(z_0, y). \\
T(z_0, y) &= \int_{-\infty}^{\infty} xp(x - z_0, y_1)p(x + z_0, y_2)\, dx.
\end{aligned}$$

We note that the Pitman estimator is determined by (3.15) and (3.14) and so it may be written as

$$\gamma_1 = z_1 - H(z_0, y) \tag{3.22}$$

and

$$\gamma_2 = z_0 + K(z_0, y), \tag{3.23}$$

whenever $G(z_0, y) > 0$, and $\gamma_1 = z_1$, $\gamma_2 = z_0$, otherwise. To verify (3.22) we see that the numerator of the integral on the right hand side of (3.15) may be written as

$$\begin{aligned}
&\int_{-\infty}^{\infty} \int_{-\infty}^{z_0} up(u - v, y_1)p(u + v, y_2)\, dv\, du \\
&\quad = \tfrac{1}{4} \int_{-\infty}^{\infty} \int_{-\infty}^{2z_0+s} (t + s)p(s, y_1)p(t, y_2)\, ds\, dt \\
&\quad = \tfrac{1}{4} \int_{-\infty}^{\infty} sp(s, y_1)P(2z_0 + s, y_2)\, ds + \tfrac{1}{4} \int_{-\infty}^{\infty} \int_{t-2z_0}^{\infty} tp(t, y_2)p(s, y_1)\, ds\, dt \\
&\quad = F(z_0, y) + \tfrac{1}{4} \int_{-\infty}^{\infty} tp(t, y_2)[1 - P(t - 2z_0, y_1)]\, dt \\
&\quad = F(z_0, y) - E(z_0, y).
\end{aligned}$$

Thus (3.22) is verified and (3.23) is verified similarly.

Now we prove

LEMMA 3.1. *Let $\gamma_1(z_0, z_1, y)$ and $\gamma_2(z_0, z_1, y)$ be the components determining the Pitman estimator. Then for each (z_1, y),*

$$\lim_{z_0 \to \infty} (\gamma_1 - z_1) = 0, \tag{3.24}$$

$$\lim_{z_0 \to \infty} (\gamma_2 - z_0) = 0. \tag{3.25}$$

For all (z_0, y) such that $\int_{-\infty}^{z_0} g(v, y)\, dv > 0$, $\gamma_2 - z_0$ is a nonincreasing function of z_0, and

$$|H(z_0, y)| \leqq -K(z_0, y). \tag{3.26}$$

Also,

$$\lim_{\mu \to \infty} \lim_{\eta \to \infty} R(\mu, \eta, \gamma_1, \gamma_2) = R. \tag{3.27}$$

PROOF. Statements (3.24) through (3.26) are obvious from the definitions (3.14), (3.15), and (3.21). Using (3.9) and (3.17), then (3.22) and (3.26) in (3.9a) with the monotone convergence theorem gives (3.27).

LEMMA 3.2. *Suppose for each y there are real numbers $b(y)$ and $c(y)$ satisfying*

$$b(y) < c(y) \quad \text{and} \quad 1 = \int_{b(y)}^{c(y)} g(z_0, y)\, dz_0. \tag{3.28}$$

Suppose also that for each (z_0, y) such that $G(z_0, y) > 0$

$$\hat{E}(z_0, y) \quad \text{and} \quad \hat{F}(z_0, y) \tag{3.29}$$

are monotone functions of z_0, for each y. If $R(\mu, \eta, \gamma_1, \gamma_2)$ is the risk for the estimator (γ_1, γ_2) given in (3.22) and (3.23), then

$$\text{if } \eta \geqq 0, \quad R(\mu, \eta, \gamma_1, \gamma_2) \leqq R; \qquad R(0, 0, \gamma_1, \gamma_2) = R. \tag{3.30}$$

PROOF. Consider the functions

$$R(\mu, \eta, \gamma_1, \gamma_2, y)$$
$$(3.31) \qquad = 4 \int_{b(y)}^{c(y)} \int_{-\infty}^{\infty} [(\gamma_1(z_0 + \eta, z_1, y) - \mu)^2 + (\gamma_2(z_0 + \eta, z_1, y) - \eta)^2] \cdot p((z_1 - \mu) - z_0, y_1)p((z_1 - \mu) + z_0, y_2)\, dz_1\, dz_0$$

and

$$(3.32) \quad R(y) = 4 \int_{b(y)}^{c(y)} \int_{-\infty}^{\infty} (z_0^2 + z_1^2) p(z_1 - z_0, y_1) p(z_1 + z_0, y_2)\, dz_1\, dz_0 .$$

From the proof of Lemma 3.1 we note that if $z_0 > c(y)$ then $\gamma_1 = z_1$ and $\gamma_2 = z_0$ which in turn implies that

$$(3.33) \quad \text{if } \eta > c(y) - b(y) \quad \text{then} \quad R(\mu, \eta, \gamma_1, \gamma_2, y) = R(y).$$

Let the function

$$(3.34) \qquad \int_0^\infty (R(y) - R(\mu, \eta + d, \gamma_1, \gamma_2, y))\, d\eta = \Gamma(d).$$

If $\Gamma(d)$ is a decreasing function of d, it will follow that if $\eta, d \geqq 0$, $R(y) \geqq R(\mu, \eta + d, \gamma_1, \gamma_2, y)$. Suppose $0 < d_1 < d_2$. From (3.22), (3.23), and (3.34) and some simplification we find

$$\Gamma(d_1) - \Gamma(d_2)$$
$$= 4 \int_0^\infty d\eta \int_{-\infty}^{\infty} \{[H^2(z_0 + d_2, y) - H^2(z_0 + d_1, y)]g(z_0 - \eta, y)$$
$$(3.35) \qquad + [K^2(z_0 + d_2, y) - K^2(z_0 + d_1, y)]g(z_0 - \eta, y)$$
$$- 2[H(z_0 + d_2, y) - H(z_0 + d_1, y)]T(z_0 - \eta, y)$$
$$+ 2[K(z_0 + d_2, y) - K(z_0 + d_1, y)](z_0 - \eta)g(z_0 - \eta, y)\}\, dz_0 .$$

It follows from (3.28) and (3.33) that the double integral in (3.35) is absolutely convergent. The order of integration may be interchanged. Note That

$$\int_0^\infty g(z_0 - \eta)\, d\eta = G(z_0, y),$$
$$\int_0^\infty T(z_0 - \eta, y)\, d\eta = \int_{-\infty}^{z_0} T(v, y)\, dv$$
$$= G(z_0, y)H(z_0, y),$$
$$\int_0^\infty (z_0 - \eta)g(z_0 - \eta, y)\, d\eta = -K(z_0, y)G(z_0, y).$$

Therefore (3.35) becomes

$$4 \int_{-\infty}^{\infty} \{[H^2(z_0 + d_2, y) - H^2(z_0 + d_1, y)]G(z_0, y)$$
$$- 2[H(z_0 + d_2, y) - H(z_0 + d_1, y)]H(z_0, y)G(z_0, y)$$
$$+ [K^2(z_0 + d_2, y) - K^2(z_0 + d_1, y)]G(z_0, y)$$
$$(3.36) \qquad - 2[K(z_0 + d_2, y) - K(z_0 + d_1, y)]K(z_0, y)G(z_0, y)\}\, dz_0$$
$$= 4 \int_{-\infty}^{\infty} G(z_0, y)\{[H(z_0 + d_2, y) - H(z_0 + d_1, y)]$$
$$\cdot [H(z_0 + d_2, y) + H(z_0 + d_1, y) - 2H(z_0, y)]$$
$$+ [K(z_0 + d_2, y) - K(z_0 + d_1, y)]$$
$$\cdot [K(z_0 + d_2, y) + K(z_0 + d_1) - 2K(z_0, y)]\}\, dz_0 .$$

If we substitute the expressions for $H(z_0, y)$ and $K(z_0, y)$ given in (3.21) into the right hand side of (3.36), and simplify we find that

$$\begin{aligned}\Gamma(d_1) &- \Gamma(d_2)\\ &= 4\int_{-\infty}^{\infty} G(z_0, y)\{[\hat{E}(z_0 + d_2, y) - \hat{E}(z_0 + d_1, y)]^2\\ &\quad + [\hat{F}(z_0 + d_2, y) - \hat{F}(z_0 + d_1, y)]^2\\ &\quad + 2[\hat{E}(z_0 + d_2, y) - \hat{E}(z_0 + d_1, y)][\hat{E}(z_0 + d_1, y) - \hat{E}(z_0, y)]\\ &\quad + 2[\hat{F}(z_0 + d_2, y) - \hat{F}(z_0 + d_1, y)]\\ &\quad\quad \cdot[\hat{F}(z_0 + d_1, y) - \hat{F}(z_0, y)]\}\, dz_0 .\end{aligned} \tag{3.37}$$

Now by hypothesis it follows from (3.37) that for all $0 \leqq d_1 < d_2$, $\Gamma(d_1) - \Gamma(d_2) \geqq 0$. It is clear from the proof of (3.27) that for each y, $R(\mu, \eta, \gamma_1, \gamma_2, y)$ is a continuous function of η. Now we may proceed exactly as in Farrell [4], p. 985, to complete the proof of the lemma.

We now generalize Lemma 3.2.

THEOREM 3.3. *If the density p defined in* (2.6) *has finite variance and if for each* (z_0, y) *such that* $G(z_0, y) > 0$, $\hat{E}(z_0, y)$ *and* $\hat{F}(z_0, y)$, *defined in* (3.21), *are monotone functions* of z_0, *for each y, then the Pitman estimator is minimax.*

PROOF. The proof follows from the argument in Farrell [4], pgs. 985–986, by letting, for each integer $n \geqq 1$, and for all y,

$$\begin{aligned} p_n(z_0, z_1, y) &= p_n(z_1 - z_0, y_1)p_n(z_1 + z_0, y_2)\\ &= p(z_1 - z_0, y_1)p(z_1 + z_0, y_2) \quad \text{for} \quad |z_0| \leqq n.\\ &= 0 \qquad \text{otherwise.}\end{aligned}$$

REMARK. Setting $d_1 = 0$ in (3.37), it is immediate that $\Gamma(d) \leqq \Gamma(0)$ for any $d > 0$, and letting $d \to \infty$, it is then clear that $R(\mu, \eta, \gamma_1, \gamma_2) \leqq R$ for η sufficiently near zero, without any conditions on $\hat{E}$ or $\hat{F}$. Thus, if the conclusion of Theorem 3.3 is to be false, the misbehavior of $R(\mu, \eta, \gamma_1, \gamma_2)$ will occur at "moderate" η values—as it does in the example given at the end of this section. (For large η, we have (3.27).)

COROLLARY 3.1. *If the density* $p(x, y)$ *defined in* (2.6) *is such that for each* y_1 *and* y_2, $p(x, y_1)$ *has increasing hazard rate,* i.e.

$$q(x, y_1) = p(x, y_1)/(1 - P(x, y_1)) \tag{3.38}$$

increases with x, and if $p(x, y_2)$ *is such that*

$$r(x, y_2) = p(x, y_2)/P(x, y_2) \tag{3.39}$$

decreases with x, then the Pitman estimator is minimax.

PROOF. If $p(x, y)$ satisfies (3.38) and (3.39) then it follows from Theorem 3.1 of Barlow, Marshall, and Proschan [1], that $1 - P(x, y_1)$ and $P(x, y_2)$ are Pólya frequency functions of order 2, i.e., (PF_2). We show that this implies that

$\hat{E}(z_0, y)$ and $\hat{F}(z_0, y)$ are monotone, and so the corollary follows from Theorem 3.3. For $z_0^{(1)} < z_0^{(2)}$, look at the difference $\hat{F}(z_0^{(1)}) - \hat{F}(z_0^{(2)})$, which by (3.20) and (3.21) is

$$
\begin{aligned}
&\hat{F}(z_0^{(1)}) - \hat{F}(z_0^{(2)}) \\
(3.40) \quad &= \tfrac{1}{2}[\textstyle\int_{-\infty}^{\infty} xp(x, y_1)P(x + 2z_0^{(1)}, y_2)\, dx / \int_{-\infty}^{\infty} p(x, y_1)P(x + 2z_0^{(1)}, y_2)\, dx \\
&\quad - \textstyle\int_{-\infty}^{\infty} xp(x, y_1)P(x + 2z_0^{(2)}, y_2)\, dx / \\
&\quad \textstyle\int_{-\infty}^{\infty} p(x, y_1)P(x + 2z_0^{(2)}, y_2)\, dx].
\end{aligned}
$$

Clearly $\hat{F}$ is nonincreasing if $(\hat{F}(z_0^{(1)}) - \hat{F}(z_0^{(2)})) \geqq 0$, which by (3.40) is equivalent to

$$
\begin{aligned}
(3.41) \quad &\textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x[P(x + 2z_0^{(1)}, y_2)P(u + 2z_0^{(2)}, y_2) \\
&\quad - P(x + 2z_0^{(2)}, y_2)P(u + 2z_0^{(1)}, y_2)]p(x, y_1)p(u, y_1)\, du\, dx \geqq 0
\end{aligned}
$$

The left hand side of (3.41) can be written as

$$
\begin{aligned}
&\textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{x} x[P(x + 2z_0^{(1)}, y_2)P(u + 2z_0^{(2)}, y_2) \\
&\quad - P(x + 2z_0^{(2)}, y_2)P(u + 2z_0^{(1)}, y_2)]p(x, y_1)p(u, y_1)\, du\, dx \\
&\quad + \textstyle\int_{-\infty}^{\infty}\int_{u}^{\infty} u[P(u + 2z_0^{(1)}, y_2)P(x + 2z_0^{(2)}, y_2) \\
&\quad - P(u + 2z_0^{(2)}, y_2)P(x + 2z_0^{(1)}, y_2)]p(u, y_1)p(x, y_1)\, dx\, du \\
(3.42) \quad &= \textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{x} x[P(x + 2z_0^{(1)}, y_2)P(u + 2z_0^{(2)}, y_2) \\
&\quad - P(x + 2z_0^{(2)}, y_2)P(u + 2z_0^{(1)}, y_2)p(x, y_1)p(u, y_1)\, du\, dx \\
&\quad + \textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{x} u[P(u + 2z_0^{(1)}, y_2)P(x + 2z_0^{(2)}, y_2) \\
&\quad - P(u + 2z_0^{(2)}, y_2)P(x + 2z_0^{(1)}, y_2)]p(u, y_1)p(x, y_1)\, du\, dx \\
&\quad \textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{x} (x - u)[P(u + 2z_0^{(2)}, y_2)P(x + 2z_0^{(1)}, y_2) \\
&\quad - P(u + 2z_0^{(1)}, y_2)P(x + 2z_0^{(2)}, y_2)]p(u, y_1)p(x, y_1)\, du\, dx.
\end{aligned}
$$

Now if we utilize the PF_2 property of $P(x, y_2)$ in the right hand side of (3.42) it follows that $F(z_0^{(1)}) - F(z_0^{(2)}) \geqq 0$. The monotonicity of $\hat{E}(z_0, y)$ is demonstrated in the same way and so the corollary is proved.

REMARKS. (1) From the proof of the corollary it is clear that if $(1 - P(x, y_1))$ and $P(x, y_2)$ are PF_2 then $\hat{E}$ and $\hat{F}$ are monotone. From Barlow, Marshall, and Proschan [1] we note that if $p(x, y_1)$ is PF_2 then so is $1 - P(x, y_1)$ and it is clear that if $p(x, y_2)$ is PF_2 then so is $P(x, y_2)$. This latter fact follows since we can write

$$(r(x))^{-1} = \int_0^{\infty} p(x - \Delta)/p(x)\, d\Delta,$$

and if $p(x)$ is PF_2 then the integrand is increasing in x.

(2) If $n = 1$, then the sufficient conditions for minimaxity that refer to the densities $p(x, y_1)$ and $p(x, y_2)$ reduce to conditions on the original density $f(x)$.

Hence, for example, if $n = 1$, and f is PF_2 then the Pitman estimator is minimax. For arbitrary n, if $p(x, y)$ is independent of y, then the conditions refer to the marginal density of the Pitman estimator. An example of where this is the case and the marginal density satisfies the conditions is the exponential.

(3) We now offer the gamma distribution as another example of where the original density is PF_2 and this property is maintained for $p(x, y)$. That is, let $f(x) = x^\alpha e^{-x}/\Gamma(\alpha + 1)$, for $x > 0$, $\alpha > 0$. Then using the notation of Section 2 we get

$$(3.43) \quad r_0(y)p(x, y) \propto (x + r_1(y))^\alpha e^{-(x+r_1(y))}(x + r_1(y) + y_1)^\alpha e^{-(x+r_1(y)+y_1)} \cdots (x + r_1(y) + y_{n-1})^\alpha e^{-(x+r_1(y)+y_{n-1})}$$

for $x + r_1(y) + y_i > 0$, all $i = 0, \cdots, n - 1$; and $y_0 = 1$.

Let $s = x + f(y) + \min_{0 \leq i \leq n-1} y_i$, and let $\Delta_i = y_i - \min y_i$. Then we may write (3.43) as

$$(3.44) \quad r_0(y)p(s, y) = (s + \Delta_0)^\alpha e^{-(s+\Delta_0)}(s + \Delta_1)e^{-(s+\Delta_1)} \cdots (s + \Delta_{n-1})^\alpha e^{-(s+\Delta_{n-1})},$$

for $s + \Delta_i > 0$, $i = 0, \cdots, n - 1$.

Hence if $\xi_2 > \xi_1$,

$$(3.45) \quad r_0(y)p(s - \xi_1, y)/r_0(y)p(s - \xi_2, y) = e^{n(\xi_1-\xi_2)}\{\textstyle\prod_{i=0}^{n-1} (s - \xi_1 + \Delta_i)/(s - \xi_2 + \Delta_i)\}^\alpha$$

for $(s - \xi_2 + \Delta_i) > 0$, $i = 0, 1, \cdots, n - 1$.

Clearly the ratio on the left hand side of (3.45) is infinite if for any i we have $s - \xi_2 + \Delta_i < 0$ and $s - \xi_1 + \Delta_i > 0$. Thus $p(x, y)$ is PF_2 provided the ratio in (3.45) is a decreasing function of s. But if we differentiate the right hand side of (3.45) we find that the derivative is

$$(3.46) \quad \begin{aligned} &-\alpha(\xi_2 - \xi_1)[\textstyle\sum_{i=0}^{n-1} [1/(s - \xi_2 + \Delta_i)^2] \\ &\cdot \textstyle\prod_{j\neq i, j=0}^{n-1} [(s - \xi_1 + \Delta_i)/(s - \xi_2 + \Delta_i)] \\ &\cdot \textstyle\prod_{i=0}^{n-1} [(s - \xi_1 + \Delta_1)/(s - \xi_2 + \Delta_i)]^{\alpha-1} e^{n(\xi_1-\xi_2)}, \end{aligned}$$

which is negative. This demonstrates that the Pitman estimator for this problem, given the original density, is the Gamma, is minimax.

(4) If the marginal density of the Pitman estimator X is PF_2 and X is sufficient for the observations (X, Y), then it is easily seen that $p(X, Y)$ is also PF_2. In fact, the joint density of (X, Y) is $g(\theta)h(X - \theta)K(X, Y)$, so that the marginal $p(x)$ is just $g(\theta)h(X - \theta)I(X)$, where $I(X) = \int_Y K(X, y) \prod_{i=1}^{n-1} dy_i$. Setting $L(y) = g(0) \int h(x)K(x, y)\, dx$, we see that $p(X, Y) = g(\theta)h(X - \theta) \cdot K(T, Y)/L(Y)$ so that $(p(X_1 - \theta_1, y)/p(X_1 - \theta_2, y)) = (p(X_1 - \theta_1)/p(X_1 - \theta_2))$, and the PF_2 property of $p(X)$ holds also for $p(X, Y)$.

We conclude this section with an example which indicates that the Pitman estimator is not in general minimax. Let $n = 1$ and consider the density, with finite variance,

$$f(x) = 3/x^4, \qquad 1 \leqq x \leqq \infty,$$
$$= 0, \qquad \text{otherwise.}$$

It can be verified that for $z_0 > 0$,

$$G(z_0) = (z_0^8 + 2z_0^7 + z_0^6 - z_0^5 + \tfrac{5}{2}z_0^4 - 10z_0^3 - 45z_0^2 - 30z_0 + 30(1 + z_0)^2 \ln(1 + z_0))/z_0^6(1 + z_0)^2$$

and

$$E(z_0) = (3/2)G(z_0) - (3/2)(z_0^6 + z_0^5 - z_0^4 + 2z_0^3 - 6z_0^2 - 12z_0 + 12(1 + z_0)\ln(1 + z_0))/z_0^5(1 + z_0).$$

These quantities enable us by numerical methods, to evaluate the quantity

(3.47) $$R - R(\mu, \eta, \gamma_1, \gamma_2). \qquad \text{(See (3.31) and (3.32)).}$$

At $\eta = 1.75$, the numerical integration yielded a value less than $-.5$ for (3.47), indicating that the Pitman estimator cannot be minimax for this problem.

The numerical integration was based on the 96 point Gaussian quadrature integration routine. As a check on the computation, we calculated (3.47) for $\mu = 0$, $\eta = 0$. In this case it can be verified, by writing out the integrand for (3.47), for $\mu = 0$, $\eta = 0$, in terms of $E(z_0, y)$, $F(z_0, y)$, and $G(z_0, y)$, and performing the integration, that (3.47) is zero. The numerical integration gave a value of $-.01$ for (3.47) when $\mu = \eta = 0$. We wish to acknowledge Mr. Kenneth E. Larsen and the Western Electric Research Center, Princeton, New Jersey, for the computations.

4. Admissibility for $\theta_2 \geqq \theta_1$. In this section, we demonstrate the admissibility of the estimator $(\delta_1(\cdot), \delta_2(\cdot))$ given by (1.1) and (2.18), under certain moment restrictions. This result includes as a special case the normal distribution. We shall use the notation of (3.8) and prove the admissibility of the equivalent estimator (γ_1, γ_2) given by (3.10) and (3.11). The method is an adaptation of that of [5] and of Section 4 of [2]. The major difference is that whereas it was possible to bound from below the denominator of (4.16) in [2] by means of the Markov inequality, the restricted parameter space here does not allow the use of that device in the corresponding expression (4.4). The subsequent necessity of using the Schwarz inequality in (4.5) to eliminate that denominator causes the moment conditions of Theorem 4.1 to be somewhat stronger than those of Theorem 4.2 of [2] or of Theorem 2 of [5].

By adopting Theorem 3.1 of [5] to this problem and making computations similar to (4.4) in [2], it is easily shown that almost admissibility of (γ_1, γ_2) is implied by

(4.1)
$$\lim_{\sigma\to\infty} (1/\pi_\sigma(0)) \int \nu(d_y) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dz_1\, dz_2 \cdot \{\textstyle\sum_{i=1}^2 [\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\gamma_i(z_1, z_2, y) - \xi_i) p_0(z_1 - \xi_1, z_2 - \xi_2, y) \cdot \pi_\sigma(\xi_1, \xi_2) h(\xi_1, \xi_2)\, d\xi_1\, d\xi_2]^2 [\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_0(z_1 - \xi_1, z_2 - \xi_2, y)\pi_\sigma(\xi_1, \xi_2) h(\xi_1, \xi_2)\, d\xi_1\, d\xi_2]^{-1}\} = 0,$$

where we write z_1 for z_1 of (2.12a), z_2 for z_0 of (2.12a), ξ_1 for μ of (2.12b), ξ_2 for η of (2.12b), $\pi_\sigma(\xi_1, \xi_2)$ is given by (4.8) of [2], $p_0(x_1, x_2, y)$ is given by $p(x_1 + x_2, x_2 - x_1, y)$ and

$$h(\xi_1, \xi_2) = 1 \quad \text{if} \quad -\infty < \xi_1 < \infty, \quad 0 \leqq \xi_2 < \infty, \tag{4.2}$$
$$= 0 \quad \text{if} \quad -\infty < \xi_1 < \infty, \quad -\infty < \xi_2 < 0.$$

THEOREM 4.1. *Let the observed variables* (X_1, X_2, Y) *be distributed so that for some* $\theta = (\theta_1, \theta_2)$, $(X_1 - \theta_1, X_2 - \theta_2, Y)$ *has a probability density* $p(x_1, x_2, y)$ *satisfying* (2.7) *and* (2.8). *Let* $g(v, y)$ *be given by* (3.13) *and*

$$\rho(y) = \max \{\sup_{-\infty<x<-1} [\textstyle\int_{-\infty}^{x} vg(v, y)\, dv/x \int_{-\infty}^{x} g(v, y)\, dv], 2\}. \tag{4.3}$$

If

$$\int_{\mathcal{Y}} \nu(dy)\rho^2(y) \int\int (x_1^4 + x_2^4)(1 + |\log (x_1^2 + x_2^2)|^\beta) \cdot p(x_1, x_2, y)\, dx_1\, dx_2 < \infty \tag{4.4}$$

then (δ_1, δ_2) *given by* (2.18) *is an admissible estimator of* (θ_1, θ_2) *given* $(\theta_2 \geqq \theta_1)$.

PROOF. As in Theorem 4.1 of [2] it is sufficient to show almost admissibility. To prove the theorem then we demonstrate (4.1) with $\pi_\sigma(\xi_1, \xi_2)$ given by (4.8) of [2]. Call the inner integral on the left hand side of (4.1), $I(y)$, i.e.

$$\begin{aligned} I(y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dz_1\, dz_2 \{ \textstyle\sum_{i=1}^{2} [\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\gamma_i(z_1, z_2\ y) - \xi_i) \\ \cdot p_0(z_1 - \xi_1, z_2 - \xi_2, y)\pi_\sigma(\xi_1, \xi_2)h(\xi_1, \xi_2)\, d\xi_1\, d\xi_2]^2 \\ \cdot [\textstyle\int\int p_0(z_1 - \xi_1, z_2 - \xi_2, y)\pi_\sigma(\xi_1, \xi_2)h(\xi_1, \xi_2)\, d\xi_1\, d\xi_2]^{-1}\}. \end{aligned} \tag{4.5}$$

Using (3.10) and (3.11) in (4.5), followed by the Schwarz inequality and a change of variables, we obtain

$$\begin{aligned} I(y) &= \int\int dz_1\, dz_2 \{\textstyle\sum [\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\gamma_i(z_1, z_2, y) - \xi_i)p_0(z_1 - \xi_1, z_2 - \xi_2, y) \\ &\quad \cdot (\pi_\sigma(\xi_1, \xi_2) - \pi_\sigma(z_1, z_2))h(\xi_1, \xi_2)\, d\xi_1\, d\xi_2]^2 \\ &\quad \cdot [\textstyle\int\int p_0(z_1 - \xi_1, z_2 - \xi_2, y)\pi_\sigma(\xi_1, \xi_2)\, d\xi_1\, d\xi_2]^{-1}\} \\ &\leqq \int\int dz_1\, dz_2 \textstyle\sum \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\gamma_i(z_1, z_2, y) - \xi_i)^2 p_0(z_1 - \xi_1, z_2 - \xi_2, y) \\ &\quad \cdot [\pi_\sigma(\xi_1, \xi_2) - \pi_\sigma(z_1, z_2)]^2[\pi_\sigma(\xi_1, \xi_2)]^{-1}h(\xi_1, \xi_2)\, d\xi_1\, d\xi_2 \\ &= \int\int p_0(x_1, x_2, y)\, dx_1\, dx_2 \\ &\quad \cdot \textstyle\sum \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\gamma_i(x_1 + \xi_1, x_2 + \xi_2, y) - \xi_i)^2 \\ &\quad \cdot [\pi_\sigma(\xi_1, \xi_2) - \pi_\sigma(\xi_1 + x_1, \xi_2 + x_2)]^2[\pi\sigma(\xi_1, \xi_2)]^{-1}(h(\xi_1, \xi_2)\, d\xi_1\, d\xi_2). \end{aligned} \tag{4.6}$$

We shall show below that for $\xi_2 > 0$,

$$\textstyle\sum_{i=1}^{2} (\gamma_i(x_1 + \xi_1, x_2 + \xi_2, y) - \xi_i)^2 \leqq C_1\rho^2(y)(x_1^2 + x_2^2 + 1) \tag{4.7}$$

with $\rho(y)$ given by (4.3), and C_1 independent of y. Using (4.7) in (4.5) we get

$$\begin{aligned} I(y) \leqq C_1 \int\int [\rho^2(y)(x_1^2 + x_2^2 + 1)]p_0(x_1, x_2, y)\, dx_1\, dx_2 \\ \cdot \textstyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\pi_\sigma(\xi_1, \xi_2) - \pi_\sigma(\xi_1 + x_1, \xi_2 + x_2)]^2 \\ \cdot [\pi_\sigma(\xi_1, \xi_2)]^{-1}h(\xi_1, \xi_2)\, d\xi_1\, d\xi_2. \end{aligned} \tag{4.8}$$

Note that

$$(4.9)\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left([\pi_\sigma(\xi_1,\xi_2)-\pi_\sigma(\xi_1+x_1,\xi_2+x_2)]^2)/\pi_\sigma(\xi_1,\xi_2)\right)h(\xi_1,\xi_2)\,d\xi_1\,d\xi_2$$
$$\leqq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{[\pi_\sigma(\xi_1,\xi_2)-\pi_\sigma(\xi_1+x_1,\xi_2+x_2)]^2/\pi_\sigma^*(\|\xi\|+2^{\frac{1}{2}})\}\,d\xi_1\,d\xi_2,$$

where $\pi^*(\|\xi\|)$ is given by (4.18) of [2]

$$(4.10)\qquad \lambda = \lambda(y) = \int\int (x_1^2+x_2^2)p_0(x_1,x_2,y)\,dx_1\,dx_2.$$

The right side of (4.9) is the inner integral of (4.17) of [2] and has been bounded by Stein [9], equation (2.21), subject to $\lambda(y) \leqq C_3\sigma$. Using the Stein bound and (4.9) in (4.8) gives after some reduction

$$(4.11)\quad I(y) \leqq C_4[\rho^2(y)(\log\sigma)/\sigma][\int\int(x_1^4+x_2^4)[1+|\log(x_1^2+x_2^2)|^\beta]$$
$$\cdot p_0(x_1,x_2,y)\,dx_1\,dx_2],\quad \text{if } \lambda(y)\leqq C_3\sigma.$$

Further, applying the Schwarz inequality directly to (4.5) and using (4.7), it is easily shown that without any restriction on $\lambda(y)$,

$$(4.12)\qquad I(y) \leqq \rho^2(y)[\lambda(y)+1].$$

It is now possible using (4.11) and (4.12) to proceed as in James and Stein [5], pps. 374 and 375, to complete the proof of the theorem (noting that $\lambda^2(y) \leqq 2\int\int(x_1^4+x_2^4)p(x_1,x_2,y)\,dx_1\,dx_2$), subject to

$$(4.13)\quad \int \nu(dy)\rho^2(y)\int\int(x_1^4+x_2^4)(1+|\log^\beta(x_1^2+x_2^2)|)$$
$$\cdot p(x_1,x_2,y)\,dx_1\,dx_2 < \infty.$$

Clearly (4.13) is equivalent to (4.4).

It remains to prove (4.7). To do this we will refer to (3.14), (3.22), (3.23), and (3.26). First observe that if the second moment of $g(v, y)$ given by (3.13) exists, (for $z < 0$)

$$\int_{-\infty}^{z} dx\int_{-\infty}^{x} vg(v,y)\,dv$$
$$= z\int_{-\infty}^{z} vg(v,y)\,dv - \int_{-\infty}^{z} v^2g(v,y)\,dv$$
$$(4.14)\qquad = z\int_{-\infty}^{z} vg(v,y)\,dv - z^2\int_{-\infty}^{z} g(v,y)\,dv + \int_{-\infty}^{z} 2v\int_{-\infty}^{v} g(x,y)\,dx\,dv$$
$$= z\int_{-\infty}^{z}(v-z)g(v,y)\,dv + \int_{-\infty}^{z} 2v\int_{-\infty}^{v} g(x,y)\,dx\,dv$$
$$\geqq \int_{-\infty}^{z} 2x\int_{-\infty}^{x} g(v,y)\,dv\,dx.$$

Since the inequality (4.14) is true for all $z < 0$, it follows that

$$(4.14a)\qquad \lim_{x\to-\infty}\int_{-\infty}^{x} vg(v,y)/x\int_{-\infty}^{x} g(v,y)\,dv \leqq 2.$$

Thus $\rho(y)$ will be well defined, and we will have

$$(4.15)\quad x > \{\int_{-\infty}^{x} vg(v,y)\,dv/\int_{-\infty}^{x} g(v,y)\,dv\} > x\rho(y),\ (x \leqq -1).$$

By virtue of (2.8), $\int_{-\infty}^{\infty} vg(v,y)\,dv = 0$, so that the ratio in (4.15) is always negative and goes to zero as $x \to \infty$. Further, it is easily seen that this ratio increases monotonically for all x (see Lemma 3.1) so that in particular

$$(4.16)\quad \{\textstyle\int_{-\infty}^{x} vg(v, y)\, dv/\int_{-\infty}^{x} g(v, y)\, dv\} \geqq \{\int_{-\infty}^{-1} vg(v, y)\, dv/\int_{-\infty}^{-1} g(v, y)\, dv\} \geqq \rho(y) \qquad (x \geqq -1).$$

Now using (4.15) and (4.16) in (3.14) (with z_2 in place of z_0), we find that

$$(4.17)\qquad (\gamma_2(x_1 + \xi_1, x_2 + \xi_2, y) - \xi_2)^2 \leqq 2\rho^2(y)(1 + 2x_2^2).$$

Using (3.22), (3.23), and (3.26), it is easily found that

$$(4.18)\quad (\gamma_1(x_1 + \xi_1, x_2 + \xi_2, y) - \xi_1)^2 \leqq 2(x_1^2 + 2\rho^2(y)(1 + 2x_2^2)).$$

Combining (4.17) and (4.18), we get (4.7) and complete the proof of Theorem 4.1.

REMARK. If the Pitman estimator X_i is independent of Y_i, the condition (4.4) reduces to $E\{(X_1^4 + X_2^4)[1 + |\log (X_1^2 + X_2^2)|^{\beta}]\} < \infty$, which includes both the normal and exponential distributions for $n \geqq 1$.

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