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AN INEQUALITY IN CONSTRAINED RANDOM VARIABLES

BY C. L. MALLOWS

Bell Telephone Laboratories, Incorporated

Suppose the random variables $\{X_{ij}, i = 1, \cdots, m, j = 1, \cdots, n\}$ are independent and identically distributed. Then the event A defined by the inequalities

$$X_{11} + X_{12} + \cdots + X_{1n} \leqq a_1,$$

$$X_{21} + X_{22} + \cdots + X_{2n} \leqq a_2,$$

$$X_{m1} + X_{m2} + \cdots + X_{mn} \leqq a_n$$

has a certain probability, $P(A)$. If the random variables are constrained to satisfy some conditions of the form $X_{11} = X_{21}$, $X_{23} = X_{63}$ etc. (but not of the form $X_{11} = X_{12}$ etc.), then $P(A)$ is increased. This result was conjectured by E. Arthurs.

THEOREM. *If*

(i) $X = \{X_{ij}, i = 1, \cdots, m, j = 1, \cdots, n\}$ *is an array of independent and identically distributed random variables;*

(ii) A *is an event of the form*

$$A = \{\textstyle\sum_{j=1}^{n} X_{ij} \leqq a_i, i = 1, \cdots, m\};$$

(iii) C_1, C_2 *are sets of constraints of the form* $X_{ij} = X_{i'j}$ *with* $C_1 \subset C_2$; *then*

$$P(A \mid C_1) \leqq P(A \mid C_2).$$

PROOF. It is sufficient to prove the result in the case that C_1 does not imply $X_{11} = X_{21}$, while $C_2 = C_1 \cup \{X_{11} = X_{21}\}$. Suppose the dimensionality of X conditioned by C_1 is $k + 2$, where $k \geqq 0$. We can choose k variables $Y_1, Y_2, \cdots, Y_k$ out of X so that $Y_1, Y_2, \cdots, Y_k, X_{11}, X_{21}$ are linearly independent under C_1. Then $Y_1, \cdots, Y_k, X_{11}, X_{21}$ are statistically independent, and

$$P(A \mid C_1) = \int_B \textstyle\prod_{i=1}^{k} dF(y_i) \int_{-\infty}^{b_1} dF(x_{11}) \int_{-\infty}^{b_2} dF(x_{21})$$

where $F(x) = P(X_{11} \leqq x)$, B is some region in R^k (which is the range of $(y_1, \cdots, y_k)$), and b_1, b_2 are certain functions of $y_1, \cdots, y_k$ derived from the inequalities defining the event A. This is to be compared with

$$P(A \mid C_2) = \int_B \textstyle\prod_{i=1}^{k} dF(y_i) \int_{-\infty}^{\min(b_1, b_2)} dF(x).$$

However, for any distribution function F and any b_1, b_2

$$\int_{-\infty}^{b_1} dF(u) \int_{-\infty}^{b_2} dF(v) \leqq \int_{-\infty}^{\min(b_1, b_2)} dF(x)$$

and the theorem is proved.

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Remark 1. If F is Gaussian, or even if X has an arbitrary multivariate Gaussian distribution, the truth of the theorem follows from a result of Slepian (see [1]). Slepian's theorem shows that $P(A)$ is a monotone increasing function of each of the correlations $\rho_{ii'} = \text{corr}\ (\sum X_{ij}, \sum X_{i'j})$; these correlations are increased by the imposition of constraints of the form we assume.

Remark 2. The result fails to hold in general if conditions of the form $X_{ij} = X_{ij'}$ are allowed, since in general it is not true that $P(2X \leqq a) > P(X_1 + X_2 \leqq a)$. However, the theorem remains true provided the conditions are such that no constraints of this type are allowed, even if it is not possible to present them as in the statement of the theorem. For example, the constraints

$$X_1 + X_2 \leqq a, \qquad X_1 + X_3 \leqq b, \qquad X_2 + X_3 \leqq c$$

cannot be put in this form, but the result applies (by the same proof). In general there is no need to have the same number of terms in each of the inequalities defining A; and arbitrary non-negative coefficients can be inserted.

Remark 3. A simple corollary of the theorem and the previous remark in the case $n = m$ is that $P(A) \leqq P(A \mid C)$ where C is the set of conditions $\{X_{ij} = X_{ji}, 1 \leqq i < j \leqq n\}$.

Remark 4. One way of generalizing the result is as follows: Suppose first that X has an arbitrary multivariate Gaussian distribution. Then using Slepian's theorem (see Remark 1), $P(A)$ is a monotone increasing function of each of the $\frac{1}{2}nm(nm - n)$ simple correlations corr $(X_{ij}, X_{i'j'})$ where $i \neq i'$ (since the correlations $\rho_{ii'}$ in Remark 1 are monotone functions of these). Now let X be obtained by monotone coordinate-wise transformation of an arbitrary multivariate Gaussian distribution (i.e., $X_{ij} = f_{ij}(X'_{ij})$ where the $\{f_{ij}\}$ are monotone increasing, and $\{X'_{ij}\}$ is Gaussian). Assuming for simplicity that X' is nonsingular, we can write $P(A)$ as

$$\int_B dG \int_{-\infty}^{b_1} \int_{-\infty}^{b_2} g(x_{11}, x_{21}; \rho_{11,21.\text{rest}})\, dx_{11}\, dx_{21}$$

where G is a $(nm - 2)$-dimensional Gaussian distribution, b_1 and b_2 are functions of the limits defining A and the variables in G (b_1 and b_2 also involve the functions $\{f_{ij}\}$), $g(x, y; \rho)$ is the standard bivariate Gaussian density with correlation ρ, and $\rho_{11,21.\text{rest}}$ is the partial correlation between X'_{11} and X_{21} when all the other (primed) variables are held constant. Now $\rho_{11,21.\text{rest}}$ is a monotone function of the simple correlation $\rho_{11,21}$, so using Slepian's theorem we again have that $P(A)$ is monotone in each of the $\frac{1}{2}nm(nm - n)$ simple correlations of this type.

Remark 5. Now we can ask the following question. Is the result of the previous remark the most general possible, or does there exist another class of distributions, indexed by a set of parameters (at least $\frac{1}{2}nm(nm - n)$ in number), for which the monotonicity result holds? Notice that the k-dimensional "transformed Gaussian" distributions are quite a small family, in that they are determined by only k marginal distributions and $\frac{1}{2}k(k - 1)$ *pairwise* coefficients of dependence. All the joint distributions in three or more dimensions can be determined from the two-dimensional marginal distributions. It is tempting to suppose that it should be possible to define a family of joint distributions in which this is not the case; an attractive possibility is that each k'-dimensional distribution ($k' \leqq k$) involves just one more parameter than do its various marginals. An attempt has been made to generalize the construction of [2] in this way, but without success (so far).

REFERENCES

[1] Gupta, S. S. (1963). Probability integrals of multivariate normal and multivariate t. *Ann. Math. Statist.* **43** 792–828.

[2] Plackett, R. L. (1965). A class of bivariate distributions. *J. Amer. Statist. Assoc.* **60** 516–522.