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ON THE NON-CENTRAL DISTRIBUTION OF THE SECOND ELEMENTARY SYMMETRIC FUNCTION OF THE ROOTS OF A MATRIX[1]

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1. Summary and introduction. Let $\mathbf{X}$ be a $p \times f$ matrix variate ($p \leqq f$) whose columns are independently normally distributed with $E(\mathbf{X}) = \mathbf{M}$ and covariance matrix $\mathbf{\Sigma}$. Let $w_1, \cdots, w_p$ be the characteristic roots of $|\mathbf{XX}' - w\mathbf{\Sigma}| = 0$, then the distribution of $\mathrm{W} = \mathrm{diag}\,(w_i)$ is given by [4], [5]

$$(1.1)\quad e^{-\frac{1}{2}\mathrm{tr}\mathbf{\Omega}}{}_0F_1(\tfrac{1}{2}f; \tfrac{1}{4}\mathbf{\Omega}, \mathrm{W})\kappa(p,f) \cdot e^{-\frac{1}{2}\mathrm{tr}\mathrm{W}}\,|\mathrm{W}|^{\frac{1}{2}(f-p-1)} \textstyle\prod_{i>j}(w_i - w_j), \quad 0 < w_1 \leqq \cdots \leqq w_p < \infty,$$

where

$$(1.2)\qquad \kappa(p,f) = \pi^{\frac{1}{2}p^2}/\{2^{\frac{1}{2}pf}\Gamma_p(\tfrac{1}{2}f)\Gamma_p(\tfrac{1}{2}p)\},$$

$\mathbf{\Omega} = \mathrm{diag}\,(\omega_i)$ where $\omega_i, i = 1, \cdots, p,$ are the characteristic roots of $|\mathbf{MM}' - \omega\mathbf{\Sigma}| = 0$ and ${}_0F_1$ is the hypergeometric function of matrix argument (see Section 2) defined in [5]. The above distribution of non-central means with known covariance matrix was obtained by James [4]. But (1.1) has also been shown, [5], to be the limiting distribution as $n \to \infty$ of $n\mathrm{R}^2 = \mathrm{W}$ such that $0 < n\mathbf{P}^2 = \mathbf{\Omega} < \infty$, where $\mathbf{R}^2 = \mathrm{diag}\,(r_i^2)$ and $\mathbf{P}^2 = \mathrm{diag}\,(\rho_i^2)$ and where the canonical correlation coefficients $r_1^2, \cdots, r_p^2$ between a p-set and a q-set of variates ($p \leqq q$) following a $(p+q)$ variate normal distribution, are calculated from a sample of $n+1$ observations and $\rho_1^2, \cdots, \rho_p^2$ are population canonical correlation coefficients. Further $q = f$.

In this paper, the first two non-central moments of $W_2^{(p)}$, the second elementary symmetric function (esf) in $\frac{1}{2}w_1, \frac{1}{2}w_2, \cdots, \frac{1}{2}w_p$ have been obtained first by evaluating certain integrals involving zonal polynomials, and then alternately in terms of generalized Laguerre polynomials, [2], [5]. These moments were used to suggest an approximation to the non-central distribution of $W_2^{(p)}$. The approximation is observed to be good even for small values of f.

2. The moments of $W_2^{(p)}$. First let us recall a lemma due to Constantine [1] which will be used later in this section.

Lemma 1. *Let* $\mathbf{Z}$:$m \times m$ *be a complex symmetric matrix whose real part* $R(\mathbf{Z})$ *is* p.d. *and let* $\mathbf{T}$:$m \times m$ *be an arbitrary complex symmetric matrix. Then*

$$(2.1)\quad \int_{\mathbf{S}>0} \exp\,(-\mathrm{tr}\,\mathbf{ZS})|\mathbf{S}|^{t-\frac{1}{2}(m+1)}C_\kappa(\mathbf{TS})\,d\mathbf{S} = \Gamma_m(t,\kappa)|\mathbf{Z}|^{-t}C_\kappa(\mathbf{TZ}^{-1}),$$

where $R(t) > \frac{1}{2}(m-1)$ *and* $\Gamma_m(t,\kappa) = \pi^{\frac{1}{4}m(m-1)}\prod_{j=1}^m \Gamma(t + k_j - \frac{1}{2}(j-1))$

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where $\kappa = (k_1, \cdots, k_m)$, $k_1 \geqq k_2 \geqq \cdots \geqq k_m \geqq 0$, $k_1 + \cdots + k_m = k$. (*See Khatri* [6].) Now let us note that

$$C_\kappa(\mathbf{S}) = [\chi[2\kappa](1)2^k k!/(2k)!]Z_\kappa(\mathbf{S}) \tag{2.2}$$

where $\chi[2\kappa](1)$ is defined in [3]. Hence one can either work with the zonal polynomials $C_\kappa(\mathbf{S})$ or $Z_\kappa(\mathbf{S})$ which differ only in their normalizing constants. Now since $Z_{(1^2)} = 2a_2$, where a_2 is the second esf in the roots of S, $W_2^{(p)}$ can be expressed in terms of the zonal polynomials $C_{(1^2)}(\mathbf{W})$ or $\mathbf{Z}_{(1^2)}(\mathbf{W})$. Further let us note that [5]

$$_0F_1(\tfrac{1}{2}f; \tfrac{1}{4}\mathbf{\Omega}, \mathbf{W}) = \textstyle\sum_{k=0}^{\infty} \sum_\kappa C_\kappa(\tfrac{1}{4}\mathbf{\Omega})C_\kappa(\mathbf{W})/\{(\tfrac{1}{2}f)_\kappa C_\kappa(\mathbf{I}_p)k!\}. \tag{2.3}$$

Now since

$$C_\kappa(\mathbf{W})C_\eta(\mathbf{W}) = \textstyle\sum_\delta g_{\kappa,\eta}^{\delta} C_\delta(\mathbf{W}), \tag{2.4}$$

where δ is a partition of $k + n = d$ and g's are constants, it is easy to see that using (2.3) and (2.4) in the product of (1.1) by $(3/4^2)C_{(1^2)}(\mathbf{W})$, we can obtain $E(W_2^{(p)})$ by using Lemma 1. Similarly the higher order moments can be obtained successively. Thus the first moment of $W_2^{(p)}$ is given by

$$E(W^{(p)}) = (3/4^2)e^{-\frac{1}{2}\mathrm{tr}\mathbf{\Omega}} \textstyle\sum_{k=0}^{\infty} \sum_\kappa \sum_\delta 2^\delta[\Gamma_p(\tfrac{1}{2}f, \delta)/\Gamma_p(\tfrac{1}{2}f, \kappa)] \cdot [C_\delta(\mathbf{I}_p)/C_\kappa(\mathbf{I}_p)][g_{\kappa,(1^2)}^{\delta}/k!]C_\kappa(\tfrac{1}{4}\mathbf{\Omega}), \tag{2.5}$$

where $k + 2 = d$ such that $n = 2$. Similarly the rth moment of $W_2^{(p)}$ is given by

$$E(W_2^{(p)})^r = (3/4^2)^r e^{-\frac{1}{2}\mathrm{tr}\mathbf{\Omega}} \textstyle\sum_{k=0}^{\infty} \sum_\kappa \sum_\delta 2^\delta[\Gamma_p(\tfrac{1}{2}f, \delta)/\Gamma_p(\tfrac{1}{2}f, \kappa)] \cdot [C_\delta(\mathbf{I}_p)/C_\kappa(\mathbf{I}_p)][g_{\kappa,\eta}^{\delta}/k!(]C_k(\tfrac{1}{4}\mathbf{\Omega}), \tag{2.6}$$

where now $k + 2r = d$ such that $n = 2r$. The g-coefficients in (2.5) and (2.6) may be computed using (2.4).

The first two moments of $W_2^{(p)}$ obtained from (2.5) and (2.6) are given at the end of this section. Following are some intermediate results on the expected values of certain expressions in the central case i.e. when $\mathbf{\Omega} = \mathbf{0}$ which have been used to obtain $E(W_2^{(p)})$ and $E(W_2^{(p)})^2$ in the non-central case. These results were obtained with the help of Lemma 1.

Noting that

$$a_2 = \tfrac{1}{2}Z_{(1^2)}; \tag{2.7}$$

$$E(a_2 Z_{(k)}) = B_\kappa(p, f)(p - 1)(f - 1)(pf + 4k) \qquad k = 1, 2, \cdots; \tag{2.8}$$

$$E(a_2 Z_{(k-1,1)}) = B_\kappa(p, f)[(p - 1)(f - 1)(pf + 4k) + 4(2k - 1)] \qquad k = 2, 3, \cdots; \tag{2.9}$$

$$E(a_2 Z_{(k-2,1^2)}) = B_\kappa(p, f)[(p - 1)(f - 1)(pf + 4k) + 4(4k - 3)] \qquad k = 3, 4, \cdots; \tag{2.10}$$

$$E(a_2 Z_{(k-3,1^3)}) = B_\kappa(p, f)[(p - 1)(f - 1)(pf + 4k) + 4(6k - 6)] \tag{2.11}$$

$$k = 4, 5, \cdots ;$$

$$E(a_2 Z_{(2^2)}) = B_\kappa(p,f)[p(p-1)f^2 - (p-1)(p-16)f - 8(2p-7)]; \tag{2.12}$$

where $B_\kappa(p,f) = 2^{2k-1}(\frac{1}{2}p)_\kappa(\frac{1}{2}f)_\kappa$, κ denoting the specific partition of k given on the left side of each equation involving $B_\kappa(p,f)$. Further noting that

$$a_2^2 = (1/24)Z_{(2^2)} + (2/15)Z_{(2,1^2)} + (3/40)Z_{(1^4)} \tag{2.13}$$

$$\begin{aligned} E(a_2^2 Z_{(k)}) = {} & 2^{2k-2}(p/2)_\kappa (f/2)_\kappa (p-1)(f-1)[p^2(p-1)f^3 \\ & - p(p-1)\{p - 8(k+1)\}f^2 \\ & - 4\{2(k+1)p^2 - (4k^2+14k+5)p + 4k(k+3)\}f \\ & - 16\{(k+3)p - (3k+7)\}k] \qquad k = 1, 2, \cdots . \end{aligned} \tag{2.14}$$

$$\begin{aligned} E(a_2^2 Z_{(k-1,1)}) = {} & 2^{2k-2}(p/2)_\kappa (f/2)_\kappa [p^2(p-1)^2 f^4 \\ & - 2p(p-1)^2\{p - 4(k+1)\}f^3 \\ & + (p-1)\{p^3 - (16k+17)p^2 + 4(4k^2+20k+5)p \\ & - 16k(k+3)\}f^2 + 4\{2(k+1)p^3 \\ & - (8k^2+32k+5)p^2 + (40k^2+94k-13)p \\ & - (32k^2+80k-24)\}f + 16\{k(k+3)p^2 \\ & - 2(4k^2+10k-3)p + (15k^2+25k-12)\}] \\ & \qquad k = 2, 3, \cdots \end{aligned} \tag{2.15}$$

where

$$(a)_\kappa = \textstyle\prod_{i=1}^{p}(a - \frac{1}{2}(i-1))_{k_i}, \tag{2.16}$$

$$(a)_k = a(a+1)\cdots(a+k-1). \tag{2.17}$$

In addition, expressions for $E(a_2^2 Z_{(1^3)})$, $E(a_2^2 Z_{(3,2)})$, $E(a_2^2 Z_{(3,1^2)})$, $E(a_2^2 Z_{(2^2,1)})$, $E(a_2^2 Z_{(2,1^3)})$, and $E(a_2^2 Z_{(1^5)})$ were also obtained (which are not presented here) and all these were used to compute the following two moments of $W_2^{(p)}$.

$$E(W_2^{(p)}) = [1/2^3][(p-1)(f-1)(pf+4b_1) + 8b_2], \tag{2.18}$$

$$\begin{aligned} E(W_2^{(p)})^2 = {} & [(p-1)(f-1)/2^6][p^2(p-1)f^3 \\ & - p(p-1)\{p - 8(b_1+1)\}f^2 \\ & - 4\{2(b_1+1)p^2 - (4b_1^2+18b_1+5)p + 4b_1(b_1+4)\}f \\ & - 16b_1\{(b_1+4)p - (3b_1+10)\}] + (b_2/4)[p(p-1)f^2 \\ & - \{p^2 - (4b_1+17)p + 4(b_1+5)\}f - 4\{(b_1+5)p \\ & - 3(b_1+4)\}] + b_2^2 + 6b_3, \end{aligned} \tag{2.19}$$

where b_i is the ith esf in $\frac{1}{2}\omega_1, \cdots, \frac{1}{2}\omega_p$.

It may be pointed out that (2.18) and (2.19) were obtained after summation of infinite series arising from the use of (2.5) or (2.6). For example, (2.18) was obtained from the following expression:

$$(2.20)\quad E(W_2^{(p)}) = e^{-\frac{1}{2}\mathrm{tr}\mathbf{\Omega}}\{\textstyle\sum_{i=0}^{\infty} b_1^i[(p-1)(f-1)(pf+4i)/2^3 i!] + b_2[\sum_{i=0}^{\infty}(b_1^i/i!)]\}.$$

The coefficients of b_1^i and $b_2b_1^i$ in (2.20) were obtained by the use of (2.8) to (2.12) since computing the coefficients for a few small values of i easily yielded the generalization. It was further observed that the coefficients of terms $b_1^{\nu_1}b_2^{\nu_2}b_3^{\nu_3}\cdots$ other than given in (2.20) reduced to zero. The method used for obtaining (2.19) was similar.

However, it may be pointed out that the above method only suggests the results (2.18) and (2.19) and does not prove them. But the intermediate results (2.8)–(2.15) should be of additional interest. Now, a second method is given which proves (2.18) and (2.19).

Alternately, the moments of $W_2^{(p)}$ may be obtained in terms of the generalized Laguerre polynomials in the sense of Constantine [2]. For obtaining the first two moments of $W_2^{(p)}$ this method is simpler since for these cases, certain $a_{\kappa,\tau}$ coefficients involved in the generalized Laguerre polynomials are available in Constantine [2]. Hence using Equations (13), (14) and (20) of Constantine [2] we get

$$(2.21)\quad E(W_2^{(p)}) = \tfrac{3}{4}L_{(1^2)}^{\gamma}(-\tfrac{1}{2}\mathbf{\Omega})$$

and

$$(2.22)\quad E(W_2^{(p)})^2 = (1/16)[5L_{(2^2)}^{\gamma}(-\tfrac{1}{2}\mathbf{\Omega}) + 4L_{(2,1^2)}^{\gamma}(-\tfrac{1}{2}\mathbf{\Omega}) + 9L_{(1^4)}^{\gamma}(-\tfrac{1}{2}\mathbf{\Omega})],$$

where $\gamma = \frac{1}{2}(f-p-1)$, and L_κ^γ is the generalized Laguerre polynomial [2].

3. An approximation to the non-central distribution of $W_2^{(p)}$. In a previous paper the authors [8] had suggested an approximation to the central distribution of $W_2^{(p)}$ in the following form after obtaining the first four moments:

$$(3.1)\quad f(W_2^{(p)}) = [\alpha^\nu/2\Gamma(\nu)]e^{-\alpha(W_2^{(p)})^{\frac{1}{2}}}(W_2^{(p)})^{\frac{1}{2}\nu-1}\qquad 0 < W_2^{(p)} < \infty$$

where

$$(3.2)\quad \nu = \tfrac{1}{2}pf$$

and

$$(3.3)\quad \alpha^2 = 2(pf+2)/[(p-1)(f-1)].$$

From a comparison of the exact and approximate moments and moment quotients the approximation (3.1) was recommended for $f = 14$ and above when $p = 3, f = 11$ and above when $p = 4, f = 10$ and above when $p = 5, f = 8$ and above for $p = 7$ and all values of f and p beyond 7. However, since the lowest value f can take is p and small values of f are quite important from a practical point of view, the approximation to the non-central distribution of $W_2^{(p)}$ given below is believed to serve that purpose. The new approximation satisfies (3.1) with

$$\nu = 2[2(\mu_1')^3/\mu_2((\mu_2')^{\frac{1}{2}} - \mu_1')]^{\frac{1}{2}} \tag{3.4}$$

and

$$\alpha^2 = \nu(\nu + 1)/\mu_1', \tag{3.5}$$

where μ_1' and μ_2' are the first two (non-central) moments given in (2.18) and (2.19) respectively and μ_2 is the variance of $W_2^{(p)}$.

4. Accuracy comparisons. The approximation to the non-central distribution of $W_2^{(p)}$ has the first moment the same as that of the exact. An idea of the closeness of the approximate to the exact second moment can be had from Table 1.

The values of the exact and approximate variances tend to be closer for larger values of f for a given p and hence the tabulation has been confined to the smallest value of f in each case. It may further be noted from Table 1 that ratios of the (approximate to exact) variances are closer to unity for larger values of p, for example, in the null case.

Further, the approximation to the non-central distribution is better even in the null case than that given earlier just for the null case [8] which is the same as in (3.1)-(3.3). The third and fourth moments when $\mathbf{\Omega} = \mathbf{0}$ which were evaluated earlier [8] are presented in a much simpler form below.

TABLE 1

Values of exact and approximate variances

$\frac{1}{2}\omega_1$	$\frac{1}{2}\omega_2$	$\frac{1}{2}\omega_3$	$p = 3$ $f = 3$ Exact	μ_2 Approx.	Ratio (A/E)
0	0	0	24.75	24.56	.9924
1	2	0	156.75	155.89	.9945
25	0	0	1824.75	1820.46	.9976
5	5	5	5154.75	5148.41	.9988
5	5	25	31194.75	31180.31	.9995
15	15	15	98214.75	98194.92	.9998

$\frac{1}{2}\omega_1$	$\frac{1}{2}\omega_2$	$\frac{1}{2}\omega_3$	$\frac{1}{2}\omega_4$	$\frac{1}{2}\omega_5$	$p = 5$ $f = 5$ Exact	μ_2 Approx.	Ratio (A/E)
0	0	0	0	0	875.00	874.27	.9992
1	2	1	2	1	4821.00	4817.81	.9993
25	0	2	2	10	85963.00	85948.64	.9998
10	10	10	10	10	256875.00	256846.38	.9999

$$\mu_3^{(0)}\{W_2^{(p)}\} = [p(p-1)f(f-1)/2^4] \cdot [5(p-1)^2 f^2 - (10p^2 - 40p + 33)f + 5p^2 - 33p + 49], \tag{4.1}$$

and

$$\begin{aligned} \mu_4^{(0)}\{W_2^{(p)}\} = [3p(p-1)f(f-1)/2^8][4p(p-1)^3 f^4 - 4(p-1)^2 \\ \cdot (3p^2 - 34p + 28)f^3 \\ + (12p^4 - 396p^3 + 1917p^2 - 2893p + 1384)f^2 \\ - (4p^4 - 360p^3 + 2893p^2 - 7129p + 5192)f \\ - (112p^3 - 1384p^2 + 5192p - 5864)], \end{aligned} \tag{4.2}$$

where $\mu_3^{(0)}$ and $\mu_4^{(0)}$ denote the third and fourth moments in the central case. The moments (4.1) and (4.2) were obtained by evaluating linear compounds of certain determinants [7], [8]. That the approximation suggested for the non-central case (see eqns. (3.1), (3.4) and (3.5)) works very well for the null case for all values of p and f, can be inferred from Table 2.

TABLE 2

Ratios of moments of $W_2^{(p)}$ ($\Omega = 0$) from the exact and approximate distributions for $p = 3$ and $f = 3$ and 10

Moments	$f = 3$			$f = 10$		
	Exact	Approximate	Ratio (A/E)	Exact	Approximate	Ratio (A/E)
μ_1'	$.45000000 \times 10$	$.45000000 \times 10$	1.0000	$.67500000 \times 10^2$	$.67500000 \times 10^2$	1.0000
μ_2	$.24750000 \times 10^2$	$.24561937 \times 10^2$	.9924	$.13162500 \times 10^4$	$.13154799 \times 10^4$	.9994
μ_3	$.37350000 \times 10^3$	$.36307500 \times 10^3$	.9721	$.66318750 \times 10^5$	$.65752496 \times 10^5$	.9915
μ_4	$.12067312 \times 10^5$	$.11468028 \times 10^5$	.9503	$.10954364 \times 10^8$	$.10835868 \times 10^8$	.9892
$\mu_2^{1/2}$	$.49749371 \times 10$	$.49560001 \times 10$	.9962	$.36280159 \times 10^2$	$.36269546 \times 10^2$	.9997
β_1	$.92014357 \times 10$	$.88962059 \times 10$	.9668	$.19286681 \times 10$	$.18992046 \times 10$	.9847
β_2	$.19699724 \times 10^2$	$.19009186 \times 10^2$	.9649	$.63228139 \times 10$	$.62617429 \times 10$	.9903

Table 3 provides some comparison of the closeness of the approximate to the exact moments when $\mathbf{\Omega} = \mathbf{0}$ for a) the earlier approximation for the null case (Eqns. (3.1)-(3.3)) and b) the new approximation for the non-null case (Eqns. (3.1), (3.4) and (3.5)).

Thus it may be seen that the new approximation can be used in the null case even for the very small values of f for which the earlier approximation was not recommended.

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TABLE 3

Ratios of moments of $W_2^{(p)}$ ($\Omega = 0$) from the exact and approximate distributions using (a) *earlier approximation* (b) *new approximation for $p = 4$ and $f = 5$*

Moments	Exact	(a) earlier approximation		(b) new approximation	
		Approximation	Ratio (A/E)	Approximation	Ratio (A/E)
μ_1'	$.30000000 \times 10^2$	$.30000000 \times 10^2$	1.0000	$.30000000 \times 10^2$	1.0000
μ_2	$.40500000 \times 10^3$	$.37636364 \times 10^3$	.9293	$.40445819 \times 10^3$	.9987
μ_3	$.14354999 \times 10^3$	$.12228099 \times 10^5$	.8518	$.14153057 \times 10^5$	.9859
μ_4	$.13799024 \times 10^7$	$.11130166 \times 10^7$	.8066	$.13501215 \times 10^7$	.9784
$\mu_2^{\frac{1}{2}}$	$.20124611 \times 10^2$	$.19400093 \times 10^2$	.9640	$.20111145 \times 10^2$	.9993
β_1	$.31019966 \times 10$	$.28047550 \times 10$	.9042	$.30274682 \times 10$	.9760
β_2	$.84127571 \times 10$	$.78575356 \times 10$	.9340	$.82532614 \times 10$	.9810

carried out on the IBM 7094 Computer, Purdue University's Computer Science's Center.

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