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MOST POWERFUL TESTS FOR SOME NON-EXPONENTIAL FAMILIES

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1. Introduction and summary. We shall be concerned with the parametric problem of testing hypotheses concerning the value of one parameter when the values of other parameters (nuisance parameters) are not specified. Neyman [6] derived under certain conditions a locally most powerful two-sided test for this problem, i.e., he gave the form of the test maximizing the second derivative of the power function with respect to the parameter of interest at the point specified by the hypothesis. Generalizations of Neyman's results were given by Scheffé [7] and Lehmann [2], using the same technique as Neyman. They were also able to prove that the tests were UMP unbiased. A new technique for dealing with these problems was introduced by Sverdrup [9] and Lehmann and Scheffé [4] where the completeness of the sufficient statistics in an exponential family of densities is used to derive UMP unbiased tests. It is stated by Lehmann and Scheffé [4] that the conditions imposed earlier imply an exponential family of densities.

When no UMP unbiased test exists we have little general theory. The problem is both one of principle and of technique. Most stringest tests exist under general conditions but are difficult to derive in particular cases. Lehmann [3] proposed maximin tests. Spjøtvoll [8] has given an example of the form of a maximin test when no UMP unbiased and invariant test exists.

This paper is an attempt to establish some results for testing hypotheses when the probability density of the observations does not constitute an exponential family under both the hypothesis and the alternative. The assumptions made in Section 2 are satisfied if we have an exponential family under the hypothesis, but do not say anything about the form of the density under the alternative. The results concern most powerful similar or unbiased tests, and under certain conditions the form of these tests for the particular family of densities studied, is given in Section 3.

In Section 4 the theory in Section 3 is applied to the problem of testing serial correlation (not circular) in a first order autoregressive sequence. It is found that the usual tests are nearly UMP invariant.

In Section 5 the problem of testing the value of the ratio of variances in the one-way classification variance components model is considered. Some numerical results are given for the power functions of the maximin test, the locally most powerful test and the standard F-test. The results indicate that the standard F-test performs well compared with the other tests.

2. Assumptions and definitions. We shall consider the case where the prob-

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ability distribution of X belongs to a family

$$\mathcal{P}^X = \{P^X_{\theta,\vartheta} : (\theta, \vartheta) \varepsilon \Omega\}$$

where $P^X_{\theta,\vartheta}$ is defined by

$$dP^X_{\theta,\vartheta}(x) = a(x, \theta, \vartheta)b(t(x), \theta, \vartheta)\, d\mu(x),$$

where μ is a σ-finite measure over a Euclidean space.

Let the parameter θ be real. We shall consider the problem of testing the hypotheses

$$H_1 : \theta = \theta_0 \quad \text{against} \quad \theta > \theta_0,$$

$$H_2 : \theta = \theta_0 \quad \text{against} \quad \theta \neq \theta_0.$$

We shall assume that there exists a value ϑ_0 of ϑ such that the distribution $P^X_{\theta_0,\vartheta_0}$ dominates the family $\mathcal{P}^X$. In that case we may write

$$(2.1) \quad dP^X_{\theta,\vartheta}(x) = a(x, \theta, \vartheta)b(t(x), \theta, \vartheta)/(a(x, \theta_0, \vartheta_0) \cdot b(t(x), \theta_0, \vartheta_0))\, dP^X_{\theta_0,\vartheta_0}(x) \quad \text{a.e. } \mathcal{P}^X,$$

Further it is assumed that the statistic $T = t(X)$ is sufficient when $\theta = \theta_0$ and that the family of distributions for T when $\theta = \theta_0$ is boundedly complete.

We define *locally most powerful tests* of the hypotheses H_1 and H_2 as follows. (Compare Lehmann [3], p. 342.)

DEFINITION. A level α test ϕ_0 of $H_1(H_2)$ is locally most powerful (LMP) if, given any other level α test ϕ, there exists for each ϑ a Δ such that $\beta(\theta, \vartheta, \phi_0) \geqq \beta(\theta, \vartheta, \phi)$ when $\theta_0 \leqq \theta < \theta_0 + \Delta$ $(|\theta - \theta_0| < \Delta)$.

Similarly we may define LMP tests among similar tests (LMPS) and among unbiased tests (LMPU).

A test φ of H_2 satisfying (a) $\beta(\theta_0, \vartheta, \varphi) = \alpha$, (b) $\beta_\theta'(\theta_0, \vartheta, \varphi) = 0$ and $\beta_\theta''(\theta_0, \vartheta, \varphi) =$ maximum among tests satisfying (a) and (b), was denoted test of type B by Neyman [6]. If there exists a unique test of type B, then it is a LMPU test of H_2.

3. Derivation of most powerful tests. Let $P^{X|t}_{\theta_0}$ denote the conditional probability distribution of X given $T = t$ when $\theta = \theta_0$. Since T is sufficient, $P^{X|t}_{\theta_0}$ can be chosen to be independent of ϑ. Let $E^{X|t}_{\theta_0}$ denote expectation taken with respect to $P^{X|t}_{\theta_0}$. Similarly let $E^X_{\theta,\vartheta}$ and $E^T_{\theta,\vartheta}$ denote expectations with respect to the distribution of X and the marginal distribution of T respectively.

A test φ is similar if $E^X_{\theta_0,\vartheta}\varphi(X) = \alpha$ for all ϑ. Since T is sufficient and complete when $\theta = \theta_0$, a test is similar if and only if $E^{X|t}_{\theta_0}\varphi(X) = \alpha$ a.e. $\mathcal{P}^T_{\theta_0}$ where $\mathcal{P}^T_{\theta_0}$ denotes the family of distributions for T when $\theta = \theta_0$.

We have the following theorems:

THEOREM 3.1. *For the hypothesis $\theta = \theta_0$ against $(\theta, \vartheta) = (\theta_1, \vartheta_1)$ there exists a* *most powerful similar level* α *test* φ_1 *defined by*

$$\begin{aligned}\varphi_1(x) &= 1 && \text{when} \quad a(x, \theta_1, \vartheta_1)/a(x, \theta_0, \vartheta_0) > c(t)\\ &= \gamma(t) && \text{when} \quad a(x, \theta_1, \vartheta_1)/a(x, \theta_0, \vartheta_0) = c(t)\\ &= 0 && \text{when} \quad a(x, \theta_1, \vartheta_1)/a(x, \theta_0, \vartheta_0) < c(t),\end{aligned}$$

where $c(t)$ *and* $\gamma(t)$ *are determined by* $E_{\theta_0}^{X|t}\varphi_1(X) = \alpha$ *for all* t.

Let a_θ' and b_θ' denote the derivatives with respect to θ of the functions a and b respectively. The next theorem gives the form of the test that maximizes $\beta_\theta'(\theta_0, \vartheta, \varphi)$ locally.

THEOREM 3.2. *Suppose that for any test* φ *the derivative with respect to* θ *of the power function* $\beta(\theta, \vartheta, \varphi)$ *can be computed under the integral sign. Then among similar level* α *tests the following test* φ_2 *maximizes the derivative of the power function at* (θ_0, ϑ_0)

$$\begin{aligned}\varphi_2(x) &= 1 && \text{when} \quad a_\theta'(x, \theta_0, \vartheta_0)/a(x, \theta_0, \vartheta_0) > c(t)\\ &= \gamma(t) && \text{when} \quad a_\theta'(x, \theta_0, \vartheta_0)/a(x, \theta_0, \vartheta_0) = c(t)\\ &= 0 && \text{when} \quad a_\theta'(x, \theta_0, \vartheta_0)/a(x, \theta_0, \vartheta_0) < c(t),\end{aligned}$$

where $c(t)$ *and* $\gamma(t)$ *are determined by* $E_{\theta_0}^{X|t}\varphi_2(X) = \alpha$ *for all* t.

Let a_θ'' and b_θ'' denote the second derivatives with respect to θ of the functions a and b respectively.

THEOREM 3.3. *Suppose that for any test* φ *the first and second derivative with respect to* θ *of* $\beta(\theta, \vartheta, \varphi)$ *can be computed under the integral sign and suppose that* $a_\theta'(x, \theta_0, \vartheta)/a(x, \theta_0, \vartheta) + b_\theta'(t(x), \theta_0, \vartheta)/b(t(x), \theta_0, \vartheta) = k(\vartheta)h(x)$ *for some functions* k *and* h, *with* $k(\vartheta) > 0$ *for all* ϑ. *Then among level* α *tests unbiased at* θ_0 *the following test* φ_3 *maximizes the second derivative of the power function at* (θ_0, ϑ_0)

$$\begin{aligned}\varphi_3(x) &= 1 && \text{when} \quad a_\theta''(x, \theta_0, \vartheta_0)/a(x, \theta_0, \vartheta_0)\\ & && \qquad + c_1(t)a_\theta'(x, \theta_0, \vartheta_0)/a(x, \theta_0, \vartheta_0) > c_2(t)\\ &= \gamma(t) && \text{when} \quad a_\theta''(x, \theta_0, \vartheta_0)/a(x, \theta_0, \vartheta_0)\\ & && \qquad + c_1(t)a_\theta'(x, \theta_0, \vartheta_0)/a(x, \theta_0, \vartheta_0) = c_2(t)\\ &= 0 && \text{when} \quad a_\theta''(x, \theta_0, \vartheta_0)/a(x, \theta_0, \vartheta_0)\\ & && \qquad + c_1(t)a_\theta'(x, \theta_0, \vartheta_0)/a(x, \theta_0, \vartheta_0) < c_2(t),\end{aligned}$$

where $c_1(t)$, $c_2(t)$ *and* $\gamma(t)$ *are determined by* $E_{\theta_0}^{X|t}\varphi_3(X) = \alpha$ *and* $E_{\theta_0}^{X|t}\varphi_3(X)\cdot(a_\theta'(X, \theta_0, \vartheta_0)/a(X, \theta_0, \vartheta_0) + b_\theta'(t(X), \theta_0, \vartheta_0)/b(t(X), \theta_0, \vartheta_0)) = 0$.

PROOF. Theorem 3.1 is Theorem 3 of [9]. The proofs of Theorem 3.2 and Theorem 3.3 follow by the usual technique [3], [9], now demonstrated on the proof of Theorem 3.3.

Unbiasedness in some neighbourhood of θ_0 implies $\beta_\theta'(\theta_0, \vartheta, \varphi) = 0$, hence

$$
\begin{aligned}
0 &= E^X_{\theta_0,\vartheta}\varphi(X)(a_\theta'(X, \theta_0, \vartheta)/a(X, \theta_0, \vartheta) \\
&\quad + b_\theta'(t(X), \theta_0, \vartheta)/b(t(X), \theta_0, \vartheta)) \qquad (3.1)\\
&= E^T_{\theta_0,\vartheta}E^{X|t}_{\theta_0}\varphi(X)(a_\theta'(X, \theta_0, \vartheta)/a(X, \theta_0, \vartheta) \\
&\quad + b_\theta'(t(X), \theta_0, \vartheta)/b(t(X), \theta_0, \vartheta)).
\end{aligned}
$$

We may choose the function $h(x)$ in the theorem equal to $a_\theta'(x, \theta_0, \vartheta_0)/a(x, \theta_0, \vartheta_0) + b_\theta'(t(x), \theta_0, \vartheta_0)/b(t(x), \theta_0, \vartheta_0)$. Hence by (3.1)

$$
\begin{aligned}
0 = E^T_{\theta_0,\vartheta}E^{X|t}_{\theta_0}\varphi(X)(a_\theta'(X, \theta_0, \vartheta_0)/a(X, \theta_0, \vartheta_0) \\
+ b_\theta'(t(X), \theta_0, \vartheta_0)/b(t(X), \theta_0, \vartheta_0)).
\end{aligned}
$$

Completeness of T implies

$$
\begin{aligned}
(3.2) \quad E^{X|t}_{\theta_0}\varphi(X)(a_\theta'(X, \theta_0, \vartheta_0)/a(X, \theta_0, \vartheta_0) \\
+ b'(t(X), \theta_0, \vartheta_0)/b(t(X), \theta_0, \vartheta_0)) = 0 \quad \text{a.e.} \quad \mathcal{P}^T_{\theta_0}.
\end{aligned}
$$

The test must be similar, hence

$$
(3.3) \qquad E^{X|t}_{\theta_0}\varphi(X) = \alpha \quad \text{a.e.} \quad \mathcal{P}^T_{\theta_0}.
$$

We have

$$
\begin{aligned}
\beta_\theta''(\theta_0, \vartheta_0, \varphi) = E^T_{\theta_0,\vartheta_0}E^{X|t}_{\theta_0}\varphi(X)(a_\theta''(X, \theta_0, \vartheta_0)/a(X, \theta_0, \vartheta_0) \\
+ 2a_\theta'(X, \theta_0, \vartheta_0)b_\theta'(t(X), \theta_0, \vartheta_0)/(a(X, \theta_0, \vartheta_0)b(t(X), \theta_0, \vartheta_0)) \\
+ b_\theta''(t(X), \theta_0, \vartheta_0)/b(t(X), \theta_0, \vartheta_0)).
\end{aligned}
$$

Maximum is obtained if for each t the expectation $E^{X|t}_{\theta_0}$ in the above expression is maximized under conditions (3.1) and (3.2). An application of the Neyman-Pearson fundamental lemma gives the test φ_3.

The tests ϕ_1, ϕ_2 and ϕ_3 are not proved to be admissible. Hence there may exists tests ψ_1, ψ_2 and ψ_3 with the same properties as ϕ_1, ϕ_2 and ϕ_3, and which at the same time dominate the ϕ's. But if ϕ_1, ϕ_2 and ϕ_3 are unique, they are admissible.

The following lemma will be useful when establishing uniqueness.

LEMMA 3.1. *Let X be a random variable, $T = t(X)$ a statistic and let E^X, E^T and $E^{X|t}$ denote expectations with self-evident notation. Given a test function φ_0 such that*

$$
\begin{aligned}
\varphi_0(x) &= 1 \quad \textit{when} \quad h(x) > \textstyle\sum_{i=1}^m k_i(t(x))f_i(x) \\
&= 0 \quad \textit{when} \quad h(x) < \textstyle\sum_{i=1}^m k_i(t(x))f_i(x)
\end{aligned}
$$

for some functions $h, f_1, f_2, \cdots, f_m, k_1, k_2, \cdots, k_m$. Then a test function φ satisfying

$$
(3.4) \quad E^{X|t}\varphi(X)f_i(X) = E^{X|t}\varphi_0(X)f_i(X) \quad \text{a.e.,} \quad i = 1, 2, \cdots, m,
$$

satisfies

$$E^X\varphi(X)h(X) = E^X\varphi_0(X)h(X)$$

if and only if $\varphi(x) = \varphi_0(x)$ a.e. *on the set* $\{x : h(x) \neq \sum_{i=1}^m k_i(t(x))f_i(x)\}$. *Otherwise* $E^X\varphi(X)h(X) < E^X\varphi_0(X)h(X)$.

Remark. This lemma establishes the uniqueness of an optimum test constructed by maximizing conditional expectations given a statistic T. The Neyman-Pearson fundamental lemma can be used to prove the uniqueness for each given $T = t$, but the above lemma in addition states that the test is unique unconditionally.

Proof. We have by (3.4)

$$\begin{aligned} E^X\varphi(X)k_i(t(X))f_i(X) &= E^Tk_i(T)E^{X|t}\varphi(X)f_i(X) \\ &= E^Tk_i(T)E^{X|t}\varphi_0(X)f_i(X) \\ &= E^X\varphi_0(X)k_i(t(X))f_i(X). \end{aligned}$$

It follows that

$$\begin{aligned} E^X\varphi_0(X)h(X) - E^X\varphi(X)h(X) = E^X(\varphi_0(X) - \varphi(X)) \\ \cdot(h(X) - \textstyle\sum_{i=1}^m k_i(t(X))f_i(X)). \end{aligned}$$

By the definition of φ_0 the above difference is $\geqq 0$ for all φ. It is $= 0$ if and only if $\varphi(x) = \varphi_0(x)$ a.e. on the set $\{x : h(x) \neq \sum_{i=1}^m k_i(t(x))f_i(x)\}$.

We may state the following remark:

Remark. (i) The test φ_1 is unique (a.e. P^X) if

$$P^X_{\theta_0,\vartheta_0}(a(X, \theta_1, \vartheta_1) = c(t(X))a(X, \theta_0, \vartheta_0)) = 0.$$

(ii) The test φ_2 is unique (a.e. P^X) if

$$P^X_{\theta_0,\vartheta_0}(a_\theta'(X, \theta_0, \vartheta_0) = c(t(X))a(X, \theta_0, \vartheta_0)) = 0.$$

(iii) The test φ_3 is unique (a.e. P^X) if

$$P_{\theta_0,\vartheta_0}(a_\theta''(X, \theta_0, \vartheta_0) + c_1(t(X))a_\theta'(X, \theta_0, \vartheta_0) = c_2(t(X))a(X, \theta_0, \vartheta_0)) = 0.$$

Remark (i) is proved by using Lemma 3.1 with $h(x) = a(x, \theta_1, \vartheta_1)b(t(x), \theta_1, \vartheta_1)/(a(x, \theta_0, \vartheta_0)b(t(x), \theta_0, \vartheta_0))$ and $P^X_{\theta_0,\vartheta_0}$ as probability measure. The remarks (ii) and (iii) can be proved in a similar way.

If ϕ_1 is unique and does not depend upon ϑ_1, then it is the unique most powerful similar test for testing $\theta = \theta_0$ against $\theta = \theta_1$.

If $\phi_2(\phi_3)$ does not depend upon ϑ_0 and is unique, then it is the LMPS (LMPU) test of the hypothesis $H_1(H_2)$.

We have tacitly assumed measurability of the functions occurring in the theorems. We shall not prove this, but only note that in each specific case we may try to find (measurable) tests which is of the form given in the theorems. By Lemma 3.1 they will be most powerful.

4. Testing for serial correlation. The model for the observations $X_1, X_2, \cdots, X_n$ is

$$X_i = \rho X_{i-1} + U_i, \qquad i = 2, 3, \cdots, n,$$

where $U_2, U_3, \cdots, U_n$ are independent $N(0, \sigma^2)$, and $X_1, X_2, \cdots, X_n$ have a multinormal distribution with $EX_i = 0$, $\operatorname{Var} X_i = \sigma^2/(1 - \rho^2)$ and $\operatorname{Cov}(X_i, X_j) = \rho^{|i-j|}\sigma^2/(1 - \rho^2)$. The parameters σ and ρ are unknown.

We shall consider the problem of testing the hypotheses

$$H_1 : \rho = \rho_0 \quad \text{against} \quad \rho > \rho_0,$$

$$H_2 : \rho = 0 \quad \text{against} \quad \rho \neq 0.$$

The hypothesis testing problem H_1 is invariant under a common (positive) change of scale of $X_1, X_2, \cdots, X_n$. A maximal invariant is

$$S' = [X_1(\textstyle\sum_{i=1}^n X_i^2)^{-\frac{1}{2}}, X_2(\sum_{i=1}^n X_i^2)^{-\frac{1}{2}}, \cdots, X_n(\sum_{i=1}^n X_i^2)^{-\frac{1}{2}}].$$

The distribution of S depends only upon ρ, hence any invariant test is similar. When considering invariant tests it is therefore no restriction to restrict attention to similar tests.

The probability density of $X_1, X_2, \cdots, X_n$ is

$$(2\pi)^{-\frac{1}{2}n}\sigma^{-n}(1 - \rho^2)^{\frac{1}{2}} \exp(-\tfrac{1}{2}\sigma^{-2}(\textstyle\sum_{i=1}^n x_i^2 - 2\rho \sum_{i=2}^n x_i x_{i-1} + \rho^2 \sum_{i=2}^{n-1} x_i^2)),$$

which can be written in the form $a(x, \rho, \sigma)b(t(x), \rho, \sigma)$ with

$$a(x, \rho, \sigma) = \exp(-\tfrac{1}{2}\sigma^{-2}((\rho^2 - \rho_0^2)\textstyle\sum_{i=2}^{n-1} x_i^2 - 2(\rho - \rho_0)\sum_{i=2}^n x_i x_{i-1}))$$

and

$$b(t(x), \rho, \sigma) = (2\pi)^{-\frac{1}{2}n}\sigma^{-n}(1 - \rho^2)^{\frac{1}{2}} \exp(-\tfrac{1}{2}\sigma^{-2}t(x))$$

where

$$t(X) = \textstyle\sum_{i=1}^n X_i^2 + \rho_0^2 \sum_{i=2}^{n-1} X_i^2 - 2\rho_0 \sum_{i=2}^n X_i X_{i-1}.$$

The probability measure for $\rho = \rho_0$ and $\sigma = \sigma_0$ can be used as a dominating measure for any σ_0. We have $a(x, \rho_0, \sigma) = 1$. $T = t(X)$ is sufficient and complete when $\rho = \rho_0$. Applying Theorem 3.1, the most powerful similar test against an alternative (ρ_1, σ_1) is found to have the rejection region

$$2\textstyle\sum_{i=2}^n X_i X_{i-1} - (\rho_1 + \rho_0)\sum_{i=2}^{n-1} X_i^2 > c(T).$$

Introduce

$$W_1 = (2\textstyle\sum_{i=2}^n X_i X_{i-1} - (\rho_1 + \rho_0)\sum_{i=2}^{n-1} X_i^2) \cdot (\sum_{i=1}^n X_i^2 + \rho_0^2 \sum_{i=2}^{n-1} X_i^2 - 2\rho_0 \sum_{i=2}^n X_i X_{i-1})^{-1}.$$

The distribution of W_1 does not depend upon σ. T is sufficient and complete when $\rho = \rho_0$. Then by a theorem of Basu [1] W_1 and T are independent when $\rho = \rho_0$.

The rejection region may be written $W_1 > c(T)/T$ where $c(T)$ is determined by $P(W_1 > c(T)/T \mid T) = \alpha$ when $\rho = \rho_0$. But since W_1 and T are independent when $\rho = \rho_0$ we must have $c(t)/t$ equal to a constant. Hence the rejection region is $W_1 > c$ where c is determined by $P(W_1 > c) = \alpha$ when $\rho = \rho_0$.

Since W_1 does not depend upon σ_1 it is the most powerful similar test for $\rho = \rho_0$ against $\rho = \rho_1$. Since here invariance implies similarity and W_1 is invariant, it is also the most powerful invariant test for $\rho = \rho_0$ against $\rho = \rho_1$. By the Hunt-Stein theorem ([3], p. 336) the test also maximizes the minimum power over the set of alternatives with $\rho = \rho_1$. If we could prove that the power function of the test increases with ρ, then it is proved that it maximizes the minimum power over the set of alternatives with $\rho \geqq \rho_1$.

The following argument will show that the test based on W_1 is almost a UMP invariant test. We have

$$W_1 = [2\sum_{i=2}^n X_iX_{i-1}/\sum_{i=1}^n X_i^2 + (\rho_1 + \rho_0)((X_1^2 + X_n^2)/\sum_{i=1}^n X_i^2 - 1)]$$
$$\cdot[1 + \rho_0^2(1 - (X_1^2 + X_n^2)/\sum_{i=1}^n X_i^2) - 2\rho_0\sum_{i=2}^n X_iX_{i-1}/\sum_{i=1}^n X_i^2]^{-1}.$$

If we neglect the term $(X_1^2 + X_n^2)/\sum_{i=1}^n X_i^2$ which is small even for moderately large values of n, we find that to reject when $W_1 > c$ is equivalent to reject when $W_0 > c'$ where

$$W_0 = \sum_{i=2}^n X_iX_{i-1}(\sum_{i=1}^n X_i^2)^{-1}.$$

For each ρ_1 this is an approximation to the most powerful invariant test for $\rho = \rho_0$ against $\rho = \rho_1$. It does not depend upon ρ_1. Hence it is almost a UMP invariant test for $\rho = \rho_0$ against $\rho > \rho_0$.

Using Theorem 3.2 and reasoning as above it is found that the test which maximizes the derivative of the power function with respect to ρ at the point (ρ_0, σ_0) is based on the statistic

$$W_2 = (\sum_{i=2}^n X_iX_{i-1} - \rho_0\sum_{i=2}^{n-1} X_i^2)$$
$$\cdot(\sum_{i=1}^n X_i^2 + \rho_0^2\sum_{i=2}^{n-1} X_i^2 - 2\rho_0\sum_{i=2}^n X_iX_{i-1})^{-1}$$

with rejection region $W_2 >$ constant.

Since the distribution of W_2 depends only upon ρ, and W_2 does not depend upon σ_0, the test based on W_2 is LMPS.

If we in W_2 neglect the term $(X_1^2 + X_n^2)/\sum_{i=1}^n X_i^2$ it is seen as for W_1 that the test based on W_2 reduces to the test based on W_0. Hence the test based on W_0 may be regarded as an approximation to the LMPS test.

The statistics W_1 and W_2 do not, of course, uniquely reduce to W_0 when we neglect terms of the form $(X_1^2 + X_n^2)/\sum_{i=1}^n X_i^2$. Another possible statistic is

$$W_0' = \sum_{i=2}^n (X_i - X_{i-1})^2(\sum_{i=1}^n X_i^2)^{-1} = 2 - (X_1^2 + X_n^2)(\sum_{i=1}^n X_i^2)^{-1} - 2W_0.$$

See for example [5]. The test with rejection region $W_0' <$ constant can also be regarded as both a nearly UMP invariant test and LMPS test. The difference between the power functions of the two tests can be expected to be small.

If we set $\rho_0 = 0$, then the test based upon W_2 reduces exactly to the test based upon W_0, hence in this case the latter is LMPS. If we set $\rho_0 = 0$ and $\rho_1 = 1$, then the test based upon W_1 reduces exactly to the test based upon W_0', hence the latter is most powerful invariant against the alternative $\rho_1 = 1$. This should give an indication of the difference between the two tests. The test based upon W_0 is a little more powerful than the test based upon W_0' near the hypothesis, and the latter is a little more powerful at alternative near $\rho = 1$.

Finally we shall find a test of the hypothesis $H_2 : \rho = 0$ against $\rho \neq 0$. $[X_1/X_n, X_2/X_n, \cdots, X_{n-1}/X_n]$ is a maximal invariant under a common change of scale of all variables, and $|\rho|$ is a maximal invariant in the parameter space.

If we apply Theorem 3.3 it is found that the test which maximizes the second derivative of the power function at $(0, \sigma_0)$ subject to the restriction of unbiasedness and similarity, rejects when

$$-\sum_{i=2}^{n-1} X_i^2 + \sigma_0^{-2}\left(\sum_{i=2}^{n} X_iX_{i-1}\right)^2 + c_1(T)\sum_{i=2}^{n} X_iX_{i-1} > c_2(T)$$

where in this case $T = \sum_{i=1}^{n} X_i^2$. This can be written as

$$(X_1^2 + X_n^2)\left(\sum_{i=1}^{n} X_i^2\right)^{-2} + \sigma_0^{-2}\left(\sum_{i=2}^{n} X_iX_{i-1}\left(\sum_{i=1}^{n} X_i^2\right)^{-1} + c_3(T)\right)^2 > c_4(T).$$

Neglecting the term $(X_1^2 + X_n^2)/\left(\sum_{i=1}^{n} X_i^2\right)^2$ and reasoning as before we get the rejection region

$$W_0 < -c \qquad \text{and} \qquad W_0 > c$$

where c is determined from the condition of level α.

This test is an approximation to the test which maximizes the second derivative of the power function at $(0, \sigma_0)$. Since the power function of the former test depends only upon $|\rho|$ and W_0 does not depend upon σ_0 it is an approximation to the LMPU test at $\rho = 0$.

5. Variance components models. In a previous paper [8] the author has studied the unbalanced one-way classification variance components model.

$$X_{ij} = \mu + U_i + V_{ij}, \qquad j = 1, 2, \cdots, n_i, i = 1, 2, \cdots, r,$$

where μ is an unknown constant, and where the U_i and V_{ij} are all independently normally distributed with expectations zero and variances τ^2 and σ^2 respectively.

The hypothesis to be tested is

$$H : \Delta = \Delta_0 \quad \text{against} \quad \Delta > \Delta_0$$

where $\Delta = \tau^2/\sigma^2$.

In [8] it is shown that a maximal invariant under a group of translations, changes of scale and orthogonal transformations is

$$[Z_1Q^{-\frac{1}{2}}, Z_2Q^{-\frac{1}{2}}, \cdots, Z_{r-1}Q^{-\frac{1}{2}}]$$

where $Q = \sum_{i=1}^{r}\sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2$ and $Z_i = n_i^{\frac{1}{2}}(\bar{X}_i - \bar{X}_r)$ $(i = 1, 2, \cdots, r-1)$.

The family of probability distributions of $Z' = [Z_1, Z_2, \cdots, Z_{r-1}]$ and Q can be written in the form (2.1) with

$$a(z, q, \Delta, \sigma) = \exp\left(-(2\sigma^2)^{-1}(z'A(\Delta)^{-1}z - z'A(\Delta_0)^{-1}z)\right)q^{\frac{1}{2}(n-r)-1}$$

for any σ_0, where $A(\Delta)\sigma^2$ is the covariance matrix of Z and $n = \sum_{i=1}^{r} n_i$.

In [8] it is shown that the test which maximizes the minimum power over the set of alternatives with $\Delta \geqq \Delta_1$, has a rejection region of the form $W_1 >$ constant where

$$W_1 = (Z'A(\Delta_0)^{-1}Z - Z'A(\Delta_1)^{-1}Z)(Z'A(\Delta_0)^{-1}Z + Q)^{-1}.$$

A limiting form of W_1 is obtained when $\Delta_1 \to \infty$. Then we shall reject when $T >$ constant where $T = Z'A(\Delta_0)^{-1}ZQ^{-1}$. From the identity

$$Z'A(\Delta)^{-1}Z = \sum_{i=1}^{r} n_i(n_i\Delta + 1)^{-1}(\bar{X}_i - \bar{X})^2 \tag{5.1}$$

where $\bar{X} = (\sum_{i=1}^{r} n_i(n_i\Delta + 1)^{-1})^{-1} \sum_{i=1}^{r} n_i(n_i\Delta + 1)^{-1}\bar{X}_i$ the statistics T and W_1 may be computed by the observations X_{ij} [8].

Since the distribution of the invariant statistic depends only upon Δ, any invariant test has constant power when $\Delta = \Delta_0$. Hence similarity represents no restriction when considering invariant tests. We shall now find the form of the locally most powerful invariant (LMPI) test. Derivation gives

$$a_\Delta'(z, q, \Delta_0, \sigma) = (2\sigma^2)^{-1}z'A^*(\Delta_0)zq^{\frac{1}{2}(n-r)-1}$$

where $A^*(\Delta) = -\partial A(\Delta)^{-1}/\partial\Delta$. The statistic $Z'A(\Delta_0)^{-1}Z + Q$ is sufficient and complete when $\Delta = \Delta_0$ [8]. Using Theorem 3.3 and arguing as in Section 4 it is found that the LMPI test has rejection region $W_2 >$ constant where

$$W_2 = Z'A^*(\Delta_0)Z(Z'A(\Delta_0)^{-1}Z + Q)^{-1}.$$

From the identity (5.1) it is found by derivation that

$$Z'A^*(\Delta)Z = \sum_{i=1}^{r} (n_i(n_i\Delta + 1)^{-1})^2(\bar{X}_i - \bar{X})^2.$$

It is seen that the LMPI test puts more weight to the group means with many observations than the other tests. It should be noted that the tests based on T, W_1 and W_2 reduce to the usual test when $n_1 = n_2 = \cdots = n_r$. The same is the case if $r = 2$.

It is of interest to compare the three tests by means of their power functions. In [8] it is proved that W_1 and T are distributed as ratios of linear combinations of chi-square distributed random variables. The exact distribution is not known. In the case $r = 3$ the following lemma can be used to obtain a relatively simple expression for the cumulative distribution of the three statistics.

LEMMA 5.1. *Let X_1, X_2, X_3 be independently distributed chi-square random variables with ν_1, ν_2, ν_3 degrees of freedom respectively, and let a_1 and a_2 be two constants. Then*

$$U = (a_1X_1 + a_2X_2)(X_1 + X_2 + X_3)^{-1}$$

is distributed as Y_1Y_2 where Y_1 and Y_2 are independent and Y_1 has a beta distribution with $\nu_1 + \nu_2$ and ν_3 degrees of freedom and $(Y_2 - a_1)/(a_2 - a_1)$ has a beta distribution with ν_2 and ν_1 degrees of freedom.

PROOF. Define Y_1 and Y_2 by $Y_1 = (X_1 + X_2)/(X_1 + X_2 + X_3)$ and $Y_2 = (a_1X_1 + a_2X_2)/(X_1 + X_2)$. Then Y_2 is independent of $X_1 + X_2$, and hence independent of Y_1. Also $(Y_2 - a_1)/(a_2 - a_1) = X_2/(X_1 + X_2)$.

We shall use Lemma 5.1 with $\nu_1 = \nu_2 = 1$ and $\nu_3 = n - 3$ where $n = \sum_{i=1}^{3} n_i$. By integration it is found that if $0 < a_1 < a_2$ then

$$\begin{aligned} P(U > u) &= \tfrac{1}{2}(1 - u/a_1)^{\frac{1}{2}(n-3)} + \tfrac{1}{2}(1 - u/a_2)^{\frac{1}{2}(n-3)} \\ &\quad + (2\pi)^{-1}(n - 3) \int_{u/a_2}^{u/a_1} (1 - x)^{\frac{1}{2}(n-5)} \\ &\quad \cdot \text{Arc sin } (1 - 2(u/x - a_1)(a_2 - a_1)^{-1})\, dx \end{aligned} \tag{5.2}$$

for $u \leqq a_1$, and for $a_1 \leqq u \leqq a_2$

$$\begin{aligned} P(U > u) &= \tfrac{1}{2}(1 - u/a_2)^{\frac{1}{2}(n-3)} \\ &\quad + (2\pi)^{-1}(n - 3) \int_{u/a_2}^{1} (1 - x)^{\frac{1}{2}(n-5)} \\ &\quad \cdot \text{Arc sin } (1 - 2(u/x - a_1)(a_2 - a_1)^{-1})\, dx. \end{aligned} \tag{5.3}$$

To avoid complicated formulas we shall in the following consider only the case $\Delta_0 = 0$.

Let β_0, β_1 and β_2 denote the power functions of the tests based upon T, W_1 and W_2 respectively, and let c_0, c_1 and c_2 denote the corresponding constants used in the tests. In [8] it is shown that W_1 is distributed as

$$W_1(\Delta) = \sum_{i=1}^{r-1} (\Delta\lambda_i + 1 - (\Delta\lambda_i + 1)/(\Delta_1\lambda_i + 1))S_i^2 \cdot [\sum_{i=1}^{r-1} (\Delta\lambda_i + 1)S_i^2 + Q]^{-1}$$

where $S_1^2, S_2^2, \cdots, S_{r-1}^2$ are independently chi-square distributed with 1 degree of freedom, independent of Q which has a chi-square distribution with $n - r$ degrees of freedom. The λ_i are the roots of the equation $|B - \lambda C| = 0$ where B and C are determined from $A(\Delta) = B\Delta + C$.

We find

$$\begin{aligned} \beta_1(\Delta) &= P(W_1(\Delta) > c_1) \\ &= P(\sum_{i=1}^{r-1} \lambda_i(\Delta_1(\Delta\lambda_i + 1)/(\Delta_1\lambda_i + 1) - \Delta c_1)S_i^2(\sum_{i=1}^{r-1} S_i^2 + Q)^{-1} > c_1) \end{aligned}$$

where the statistic is in the form of U in Lemma 5.1.

Regarding the test T we may use the fact that T is distributed as

$$T(\Delta) = \sum_{i=1}^{r-1} (\Delta\lambda_i + 1)S_i^2Q^{-1}$$

to write

$$\beta_0(\Delta) = P(\sum_{i=1}^{r-1} (\Delta\lambda_i + 1 + c_0)S_i^2(\sum_{i=1}^{r-1} S_i^2 + Q)^{-1} > c_0).$$

The power function of the test based on W_2 is

TABLE 1

Δ	$n_1 = n_2 = 2, n_3 = 26$			$n_1 = 5, n_2 = 10, n_3 = 15$			$n_1 = 8, n_2 = 10, n_3 = 12$			$n_1 = n_2 = n_3 = 10$
	β_0	$\beta_1 - \beta_0$	$\beta_2 - \beta_0$	β_0	$\beta_1 - \beta_0$	$\beta_2 - \beta_0$	β_0	$\beta_1 - \beta_0$	$\beta_2 - \beta_0$	β
0	.010	.000	.000	.010	.000	.000	.010	.000	.000	.010
.01	.011	.001	.001	.014	.000	.000	.014	.000	.000	.014
.02	.013	.001	.001	.019	.001	.001	.019	.000	.000	.019
.03	.015	.001	.001	.024	.001	.002	.025	.000	.000	.025
.04	.017	.001	.001	.030	.002	.002	.032	.000	.000	.032
.05	.018	.002	.002	.036	.003	.003	.039	.000	.000	.039
.06	.021	.002	.002	.043	.003	.003	.046	.001	.001	.047
.07	.023	.002	.002	.051	.003	.003	.055	.000	.000	.055
.08	.025	.003	.003	.059	.003	.003	.063	.001	.000	.064
.09	.027	.004	.003	.067	.004	.003	.072	.000	.000	.073
.1	.030	.003	.003	.075	.004	.003	.081	.001	.000	.082
.2	.058	.006	.006	.163	.004	−.003	.177	.001	−.001	.180
.3	.091	.007	.006	.246	.002	−.012	.267	.000	−.003	.271
.4	.125	.007	.005	.319	.002	−.021	.343	.000	−.004	.348
.5	.160	.004	.001	.381	−.005	−.031	.408	.001	−.005	.413
.6	.193	.002	−.003	.433	−.006	−.033	.462	−.001	−.006	.466
.7	.225	−.001	−.007	.478	−.008	−.037	.507	−.001	−.006	.512
.8	.255	−.004	−.012	.516	−.008	−.039	.546	−.001	−.006	.551
.9	.283	−.007	−.016	.550	−.010	−.041	.579	−.001	−.006	.584
1	.310	−.010	−.021	.579	−.010	−.042	.608	−.001	−.006	.613
2	.503	−.032	−.054	.746	−.011	−.041	.768	−.001	−.005	.772
3	.614	−.040	−.068	.818	−.009	−.034	.836	−.001	−.004	.839
4	.685	−.042	−.073	.859	−.008	−.029	.873	−.001	−.003	.875
5	.734	−.042	−.073	.884	−.006	−.025	.896	.000	−.002	.898
10	.850	−.032	−.059	.940	−.004	−.015	.946	.000	−.001	.947

$$\begin{aligned}\beta_2(\Delta) &= P(Z'A^*(0)Z(Z'A(0)^{-1}Z + Q)^{-1} > c_2)\\ &= P(Z'(A^*(0) - c_2A(0)^{-1} + c_2A(\Delta)^{-1})Z(Z'A(\Delta)^{-1}Z + Q)^{-1} > c_2)\\ &= P(\textstyle\sum_{i=1}^{r-1} \mu_i S_i^2(\sum_{i=1}^{r-1} S_i^2 + Q)^{-1} > c_2)\end{aligned}$$

where the μ_i are the roots of

$$|A^*(0) - c_2A(0)^{-1} + c_2A(\Delta)^{-1} - \mu A(\Delta)^{-1}| = 0.$$

By means of Lemma 5.1 and the expressions (5.2) and (5.3) the power of the tests can be computed for $r = 3$. The results for some combinations of n_1, n_2 and n_3 are given in Table 1 and Table 2. For fixed n_1, n_2 and n_3 the second and third column show how much must be added to the power function β_0 to get the power functions β_1 and β_2 respectively. The last column in each table gives the power of the F-test when $n_1 = n_2 = n_3 = n/3$. The level is 1% and the value of of Δ_1 is chosen to be 0.1. The reason for choosing $\Delta_1 = 0.1$ is that for larger

TABLE 2

Δ	$n_1 = n_2 = 5,\ n_3 = 80$			$n_1 = 5,\ n_2 = 30,\ n_3 = 55$			$n_1 = n_2 = n_3 = 30$
	β_0	$\beta_1 - \beta_0$	$\beta_2 - \beta_0$	β_0	$\beta_1 - \beta_0$	$\beta_2 - \beta_0$	β
0	.010	.000	.000	.010	.000	.000	.010
.01	.014	.001	.001	.023	.005	.005	.027
.02	.020	.002	.002	.042	.009	.010	.053
.03	.026	.003	.003	.065	.012	.013	.083
.04	.033	.004	.004	.088	.016	.016	.115
.05	.041	.005	.004	.113	.017	.017	.149
.06	.049	.006	.005	.137	.019	.018	.182
.07	.058	.006	.005	.161	.019	.017	.214
.08	.067	.007	.006	.184	.019	.016	.244
.09	.077	.006	.005	.206	.019	.015	.274
.1	.086	.007	.006	.227	.019	.013	.301
.2	.184	.007	.000	.391	.010	−.008	.502
.3	.272	.003	−.010	.498	.001	−.028	.617
.4	.346	−.001	−.020	.572	−.005	−.042	.689
.5	.408	−.005	−.028	.627	−.009	−.052	.739
.6	.460	−.008	−.034	.670	−.012	−.060	.775
.7	.504	−.010	−.039	.703	−.013	−.065	.802
.8	.541	−.011	−.042	.731	−.014	−.069	.824
.9	.573	−.012	−.044	.753	−.014	−.071	.841
1	.602	−.013	−.046	.773	−.015	−.074	.855
2	.761	−.014	−.046	.872	−.013	−.071	.923
3	.829	−.012	−.039	.911	−.010	−.062	.948
4	.867	−.010	−.033	.932	−.009	−.053	.961
5	.891	−.008	−.028	.945	−.007	−.047	.968
10	.943	−.005	−.017	.972	−.004	−.028	.984

values of Δ_1 the difference between β_0 and β_1 vanishes. For $\Delta_1 = 3.0$ for example the two power functions were identical to three decimal places. The power functions were also computed for 5% level, but the results did not in tendency differ much from those for 1% level, though the differences in power were smaller.

It is seen from the tables that the difference between β_0 and β_1 is small, and very little is gained by the LMPI test near the hypothesis as compared with the loss of power for moderate values of Δ. It is also seen that we may have a serious loss of power compared with the situation where $n_1 = n_2 = n_3 = n/3$.

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