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A NOTE ON THE WEAK LAW OF LARGE NUMBERS

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Let $\{X_k : k \geqq 1\}$ denote a sequence of independent and identically distributed (iid) random variables. Let $S_n = \sum_{k=1}^n X_k$. If S_n/n converges to zero in probability but not with probability one (wp 1) it is well known that $\limsup_n |S_n|/n = +\infty$ wp 1. The purpose of this note is to show that in fact $\limsup_n S_n/n = +\infty$ wp 1 and $\liminf_n S_n/n = -\infty$ wp 1.

Lemma. *Let* $\{X_k : k \geqq 1\}$ *be* iid *then the following are equivalent.*

(a) $\limsup_n S_n/n = +\infty$ wp 1.

(b) $\sum_{n=1}^\infty n^{-1} P(S_n > nM) = \infty$ *for all* $M > 0$.

Proof. Suppose that (a) holds. Then $\limsup_n (S_n - nM)/n = +\infty$ wp 1 for all positive M. Consequently $\limsup_n (S_n - nM) = \infty$ wp 1 for all positive M. Therefore by (Theorem 4.1, [1]) (b) holds. Conversely if (b) holds it follows, again from (Theorem 4.1, [1]), that $\limsup_n (S_n - nM) = +\infty$ wp 1 for all positive M and therefore (a) holds.

Lemma. *Let* $X_k : k \geqq 1\}$ *be* iid, $\epsilon > 0$, *and suppose that* S_n/n *converges to zero in probability. Then there exists a positive constant* A *such that*

$$P(S_n > n\epsilon) \geqq AnP(X_1 > 2n\epsilon) \quad \textit{for} \quad n \geqq n_0 .$$

Proof. Let $\mu(X)$ denote the median of the random variable X and let $S_n^{\,i} = \sum_{k=1, k\neq i}^n X_k$. Then

$$\begin{aligned} P(S_n > n\epsilon) &\geqq P \textstyle\bigcup_{i=1}^n \{[X_i > n\epsilon - (n-1)\mu(S_n^{\,i}/(n-1))] \\ &\qquad \cap [S_n^{\,i} > (n-1)\mu(S_n^{\,i}/(n-1))]\} \\ &\geqq \textstyle\sum_{i=1}^n [\frac{1}{2} - (i-1)P(T_1)]P(T_i) \end{aligned}$$

where $T_i = [X_i > n\epsilon - (n-1)\mu(S_n^{\,i}/(n-1))]$. Further S_n/n converging to zero in probability implies that $\mu(S_n^{\,i}/(n-1)) \to 0$ and $nP[|X_1| > n\epsilon/2] \to 0$. Therefore there exists n_0 such that if $n \geqq n_0$ it follows that

$$P(X_1 > 2n\epsilon) \leqq P(T_1) \leqq P(X_1 > n\epsilon/2) \quad \text{and} \quad [\tfrac{1}{2} - nP(X_1 > n\epsilon/2)] \geqq A > 0.$$

Thus if $n \geqq n_0$ it follows that $P(S_n > n\epsilon) \geqq AnP(X_1 > 2n\epsilon)$.

Theorem. *Let* $\{X_k : k \geqq 1\}$ *be* iid *and suppose* S_n/n *converges to zero in probability but not* wp 1. *Then* $\limsup_n S_n/n = +\infty$ wp 1 *and* $\liminf_n S_n/n = -\infty$ wp 1.

Proof. First note that $\sum_{n=1}^\infty P(X_1 > 2n\epsilon) = \infty$ for all $\epsilon > 0$. For if $\sum_{n=1}^\infty P(X_1 > 2n\epsilon) < \infty$ for some $\epsilon > 0$ it would follow that $EX_1^+ < \infty$; and

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then, since $\int_{|x|<n} xF(dx) \rightarrow 0$, that $EX_1 = 0$. This is impossible since S_n/n does not converge wp 1. Therefore, by the preceding lemma, $\sum_{n=1}^{\infty} n^{-1}P(S_n > n\epsilon)$ is infinite for all $\epsilon > 0$ and it follows from the first lemma that $\limsup_n S_n/n = +\infty$ wp 1.

Finally since $\{-X_k : k \geqq 1\}$ satisfy the hypotheses of the theorem it follows that $\liminf_n S_n/n = -\infty$.

REFERENCE

[1] SPITZER, F. (1956). A combinatorial lemma and its applications to probability theory. *Trans. Amer. Math. Soc.* **82** 323–339.