The Annals of Mathematical Statistics
1968, Vol. 39, No. 6, 1825–1843

ON A GENERAL CLASS OF DESIGNS FOR MULTIRESPONSE EXPERIMENTS[1]

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1. Introduction and summary. The purpose of this paper is to present a class of designs suitable for experiments where several responses or characteristics are under study, but all characteristics are not measured on each unit.

Usually, while choosing the design, not much consideration has been given to the multiresponse aspect of the experiment, the choice being often made as if only a single response had been under study. Furthermore, while planning multiresponse experiments, we merely take over such designs and then assume that each experimental unit is studied on all variates or characteristics, and carry out the analysis accordingly. However, in a large number of cases, it is either physically impossible, uneconomic, or inadvisable on account of unequal importance or measuring costs of the various characteristics of interest, to study each of them on each experimental unit. Such situations arise in diverse areas of research in the humanities and the natural sciences. An interesting example will be found in [15] and many others in [14]. These communications bring out the important fact that in many situations we need to have experiments where observations on some of the responses are missing not by accident (as, for example, in [7] or [8]), but by design.

For the sake of illustration, we include an example here too. Often the process of taking measurements is quite time-consuming. Suppose a biologist has a number of growing organisms of a similar kind, on each of which he could observe (say) p responses provided that the process of taking measurements on any given unit were fast enough to enable him to finish in the limited amount of time, during which the experimental conditions remain unchanged. However, if the process is necessarily slow he may have to content himself with fewer (than p) responses on each unit.

The above indicates that every design has two aspects: one relative to the responses and experimental units, and the other relative to treatments and blocks. The first decides for each unit the responses to be studied therein, while the latter tells us for each block the treatments (and units) allocated to it. However, the first aspect may also influence the second in the sense that relative to each response, we may have a different system of blocks. Designs which possess this last property are called multiresponse designs with p block systems. These have been studied in [11] and will not be considered here.

In [11], we have also studied another class called hierarchical designs, which

Received 15 August 1966; revised 22 May 1967.
[1] This research was supported by the Air Force Office of Scientific Research.
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are suitable for those situations in which the various responses could be graded in a descending order of importance, say (V_1, V_2, $\cdots$), such that the response V_i is supposed to be more important than V_j, if $i < j$. The designs are defined essentially by requiring that if V_i is more important than V_j, then V_i should also be observed on each experimental unit on which V_j is observed. However, in many cases such a grading may not exist, or it may be otherwise inadvisable to impose such a hierarchical structure on the experiment. More general classes of designs are then called for.

In this paper we introduce a class of designs (which may be called 'regular incomplete multiresponse designs') that are response-wise incomplete, i.e., in which there are (at least some) experimental units on which all responses are not measured. However, in addition, they may or may not be treatment-wise incomplete (in the sense of an ordinary incomplete block design). As will be evident from the definition in the next section, these designs will be useful in a large variety of multiresponse experiments.

The approach to the analysis of response-wise incomplete designs, by Trawinski (unpublished dissertation, [14]) and Trawinski and Bargmann [15], is through the use of maximum likelihood estimates and the likelihood ratio tests. However, as mentioned by the author in [13], apart from large sample approximations, etc., (inherent in the likelihood ratio tests), the formulae for the estimates of the parameters and for the test statistic that one obtains this way are somewhat cumbersome even for an electronic computer.

In this paper therefore, a different approach which works for regular designs, is adopted. The attempt is to transform the data back into the framework of linear estimation and multivariate analysis of variance. Once this is done, the usual techniques of analysis become available.

It may be stressed that the purpose of the regular designs is not merely to permit the estimation of treatment effects by linear analysis methods (which in itself is important), but also to make available valid (free of nuisance parameters) and exact test regions for testing linear hypotheses on the treatments. As usual, we require normality assumptions for the latter but not the former. It will also be noted that the approach in this paper possesses in a sense a symmetry with respect to the responses, unlike, for example, the one used in the analysis of hierarchical designs in [11] where the responses are arranged in an order of importance, say V_1, V_2, $\cdots$, V_p, and then the analysis for V_{i+1} ($i = 0, \cdots, p-1$) is performed *conditional* to the data observed for V_1, V_2, $\cdots$, V_i. Finally, we remark that though the derivations of the condition for regular designs may seem a little complex, their actual use is simple relative particularly to the other existing techniques.

2. Definition and preliminary discussion of incomplete multiresponse designs. Consider the total set of (say) N experimental units. Suppose this is divided into $u(>1)$ disjoint sets S_1, S_2, $\cdots$, S_u, the set S_i having N_i units. Let the p responses or variates be denoted by V_1, V_2, $\cdots$, V_p. On each of the N_i units ($i = 1, 2, \cdots, u$) in the set S_i, the p_i variates V_{i1}, V_{i2}, $\cdots$, V_{ip_i} are measured, these being selected from the total set of p responses $V_1, V_2, \cdots, V_p$ according to a rule D_1 which may henceforth be called the "response design". Furthermore, we envisage that for each i, there is an ordinary block-treatment design D_{2i} defined over the N_i units in the set S_i. The set of blocks under the design D_{2i} may of course be different for different i. The *multiresponse design* D is then defined to be the total design over all the N units, and is fixed by the $(u + 1)$-tuplet

$$D = (D_1, D_{21}, \cdots, D_{2u}). \tag{2.1}$$

Furthermore, a multiresponse design will be called *incomplete* if and only if there exists an $i(1 \leqq i \leqq u)$ such that $p_i < p$.

We now formulate the model. Let the design D_{2i} consist of a set of blocks $\beta_{i1}, \beta_{i2}, \cdots, \beta_{ib_i}$, which will then correspond to b_i block-parameter vectors, each consisting of p_i elements corresponding to the set of p_i responses measured on each unit in the set S_i. Clearly the block-parameter vectors may differ from one S_i to another. Furthermore, for each S_i, we envisage (as usual) a general effect μ_i $(i = 1, 2, \cdots, u)$. As for block effects, the effect μ_i also may vary with i. Also let there be t treatments $\tau_1, \tau_2, \cdots, \tau_t$. These t treatments are supposed to be the same for all sets S_i, but the blocks are different for each set.

The t treatments will give rise to t treatment effect vectors, each consisting of p elements corresponding to the total of p responses; these vectors are the same for all i, although for some i only a part (corresponding to the p_i responses measured in S_i) will come into picture. Thus the treatment effect vectors are the same everywhere, except that for some sets S_i they may be incomplete. Let the p "true" responses to the different treatments be denoted by the $t \times p$ matrix

$$\xi = \begin{bmatrix} \xi_{11} & \xi_{12} & & & \xi_{1p} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \xi_{t1} & \xi_{t2} & & & \xi_{tp} \end{bmatrix} = [\xi_1, \xi_2, \cdots, \xi_p], \text{ say,} \tag{2.2}$$

where ξ_{jl} denotes the "true" value of the lth response to the jth treatment. Suppose l_{i1}th, l_{i2}th, $\cdots$, l_{ip_i}th variates are measured on each of the units in the set S_i. Then the set of treatment effects related to measurements in S_i are denoted by

$$\xi^{(i)} = [\xi_{l_{i1}}, \xi_{l_{i2}}, \cdots, \xi_{l_{ip_i}}], \qquad i = 1, 2, \cdots, u. \tag{2.3}$$

Let $Y_i(N_i \times p_i)$ be the matrix of the p_i observations on the N_i units in S_i. Let $\eta^{(i)}$ be a $(b_i + 1) \times p_i$ matrix corresponding to the general effect μ_i and the b_i block effects under the design D_{2i}. Further we suppose as usual that there exists a known design matrix A_i of order $N_i \times (1 + b_i + t)$ for the set S_i such that we can write

$$\text{Exp}\,(Y_i) = A_i \begin{bmatrix} \eta^{(i)} \\ \cdots \\ \xi^{(i)} \end{bmatrix} = [A_{i1} \vdots A_{i2}] \begin{bmatrix} \eta^{(i)} \\ \cdots \\ \xi^{(i)} \end{bmatrix}, \quad \text{say,} \tag{2.4}$$

where Exp denotes expected value, and A_{i1} and A_{i2} are respectively of orders

$N_i \times (b_i + 1)$ and $N_i \times t$. Also write

$$(2.5) \qquad Y_i = \begin{bmatrix} \mathbf{y}_{i1}' \\ \mathbf{y}_{i2}' \\ \cdots \\ \mathbf{y}_{iN_i}' \end{bmatrix}, \qquad i = 1, 2, \cdots, u,$$

where $\mathbf{y}_{is}$ denotes the vector of p_i observations on the variate nos. $l_{i1}, l_{i2}, \cdots, l_{ip_i}$ taken on the sth unit ($s = 1, 2, \cdots, N_i$) inside the set S_i. Then we assume that

$$(2.6) \qquad \text{Var}(\mathbf{y}_{is}) = \Sigma^{(i)},$$

$$\text{Cov}(\mathbf{y}_{is}, \mathbf{y}_{i's'}) = \mathbf{0}_{p_ip_i'}, \quad \text{if} \quad i = i', s \neq s', \quad \text{or if} \quad i \neq i',$$

where $\Sigma^{(i)}$ is a $p_i \times p_i$ matrix obtained by taking the l_{i1}th, l_{i2}th, $\cdots$, l_{ip_i}th rows and columns of a (covariance) matrix Σ of order $p \times p$, which we shall henceforth call the population dispersion matrix. Of course, the matrix Σ is assumed to be unknown. Also $\mathbf{0}_{\alpha\beta}$ will always denote a matrix of size $(\alpha \times \beta)$ with zero everywhere.

For convenience, we introduce the matrices M_i $(i = 1, 2, \cdots, u)$, such that M_i is a $p \times p_i$ matrix which contains 1 in the cells $(l_{ij}, j), j = 1, 2, \cdots, p_i$, and 0 elsewhere. Then it can be checked that

$$(2.7) \qquad \xi^{(i)} = \xi M_i, \qquad \Sigma^{(i)} = M_i' \Sigma M_i.$$

The reader will note that the use of the symbol M_i here is similar to the post-matrices M_i used by S. N. Roy [10]. Consider now an individual set of units S_i. It is well known that we cannot estimate each treatment effect, so we shall make the usual assumption:

$$(2.8) \qquad J_{1t}\xi_l = 0, \qquad l = 1, 2, \cdots, p,$$

where $J_{t't}$ shall always denote a $(t' \times t)$ matrix with unity in each cell, t' and t being any positive integers.

Now consider a fixed set of units S_i, and the corresponding design D_{2i}. We have b_i blocks, t treatments and p_i variates. Suppose the response V_l is one of these p_i responses. Define

(2.9) T_{ilj} = total (corresponding to V_l) yield for the jth treatment over all units in S_i to which jth treatment was allotted by D_{2i},

B_{ilg} = total (for V_l) yield over all units in the gth block in the set S_i,

n_{ijg} = number of units in the gth block in S_i, to which jth treatment is allotted under D_{2i},

k_{ig} = number of units in the gth block in S_i,

r_{ij} = number of units in S_i under treatment j;

$Q_{ilj} = T_{ilj} - \sum_{g=1}^{b_i} [n_{ijg}B_{ilg}/k_{ig}]$, and

$$(2.10) \qquad \mathbf{Q}_{il} = \begin{bmatrix} Q_{il1} \\ Q_{il2} \\ \cdots \\ Q_{ilt} \end{bmatrix}.$$

Then it is well known [e.g. Kempthorne [5]] from the theory of block treatment designs that

$$(2.11) \qquad \text{Exp}\,(\mathbf{Q}_{il}) = C_i \xi_l,$$

$$(2.12) \qquad \text{Var}\,(\mathbf{Q}_{il}) = C_i,$$

where C_i is a $t \times t$ matrix with

$$C_i(j, j') = -\sum_{g=1}^{b_i} n_{ijg} n_{ij'g}/k_{ig} = -\mu_{ijj'}, \quad \text{say}, \quad j \neq j',$$
$$C_i(j, j) = r_{ij} - \sum_{g=1}^{b_i} (n_{ijg}^2/k_{ig}) = r_{ij} - \mu_{ijj},$$
$$j, j' = 1, 2, \cdots, t; \qquad i = 1, 2, \cdots, u.$$

Now let l vary over the p_i responses measured on S_i and define the $t \times p_i$ matrix

$$(2.13) \qquad Q_i = (\mathbf{Q}_{i1}, \mathbf{Q}_{i2}, \cdots, \mathbf{Q}_{ip_i}).$$

Then

$$(2.14) \qquad \text{Exp}\,(Q_i) = C_i \xi^{(i)}, \qquad i = 1, 2, \cdots, u.$$

We shall call a multiresponse design *homogeneous* provided that there exist known real numbers α_{ir} ($i = 1, 2, \cdots, u$; $r = 1, 2, \cdots, m$) such that

$$(2.15) \qquad C_i = \alpha_{i1} F_1 + \alpha_{i2} F_2 + \cdots + \alpha_{im} F_m, \qquad i = 1, 2, \cdots, u;$$

where $F_1, F_2, \cdots, F_m$ are known real $t \times t$ matrices, and $m < p(p + 1)/2$. The rest of this paper attempts to develop the theory of homogeneous incomplete multiresponse designs. It is clear that 'homogeneity' here essentially implies that the various designs D_{2i} have the same basic structure, as would happen when, for example, the matrices $F_1, \cdots, F_m$ are the association matrices of an m associate class (including the 0th associate class) PBIB design. Thus, in this case, F_1 may be diagonal and correspond to the 0th or self-associate class, and $F_2, F_3, \cdots, F_m$ to the other $(m - 1)$ associate classes. (Note that many authors exclude the 0th associate class, and would call this a PBIB with $(m - 1)$ associate classes.)

To make ideas clear, we shall illustrate the discussion throughout by a simple example.

Let $p = 3$, $N = 96$, $u = 8$, $t = 4$. Thus there are 3 responses V_1, V_2, V_3, and 8 sets S_i ($i = 1, \cdots, 8$) each containing $12(= N_i)$ units. For simplicity, we take $m = 2$, and assume each D_{2i} to be a BIBD with parameters ($v = b = 4, r = k = 3, \lambda = 2$). Thus, on any S_i there are 4 blocks with 3 treatments each; the blocks (apart from randomization) being (1, 2, 3), (1, 2, 4), (2, 3, 4) and (1, 3, 4). Also let D_1 be as shown in the table below:

Set	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8
Response measured	1, 2	1, 2	1, 3	1, 3	2, 3	2, 3	1, 2, 3	1, 2, 3

Thus each response is measured in 6 sets, and each pair of responses in 4 sets. Various matrices occurring above can be easily exemplified. Thus ξ is (4×3),

$\xi^{(6)}$ is (4×2), Y_2 is (12×2), $\eta^{(3)}$ is (5×2), A_i is (12×9) and A_{i2} is (12×4) for all i, etc; $p_1 = \cdots = p_6 = 2, p_7 = p_8 = 3; l_{11} = 1, l_{12} = 2, l_{31} = 1, l_{32} = 3$ etc. Also,

$$M_5 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad M_8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Sigma^{(1)} = \begin{bmatrix} \sigma^{11} & \sigma^{12} \\ \sigma^{21} & \sigma^{22} \end{bmatrix}, \text{etc.}$$

Again, $r_{ij} = 3$, $k_{ig} = 3$, $\mu_{ijj} = \frac{3}{3}$, $\mu_{ijj'} = \frac{2}{3}$, and we can take $F_1 = I_4$, $F_2 = [J_{44} - I_4]$, with $\alpha_{i1} = 2$, $\alpha_{i2} = -\frac{2}{3}$.

To further illustrate the technique, we use the artificial data of Table 1.

It may be stressed that there are 96 units, the 12 units corresponding to one set being completely different from the 12 units for another. Similarly the four blocks change from set to set. To examplify various symbols, we check $\mathbf{y}'_{71} = (2, 7, 2)$, $T_{624} = 18$, etc.

3. Transforming the data to a form suitable for MANOVA. We shall consider now the possibility of analysing a homogeneous incomplete multiresponse

TABLE 1

Unit No.		1	2	3	4	5	6	7	8	9	10	11	12
Block		1	1	1	2	2	2	3	3	3	4	4	4
Treatment		1	2	3	1	2	4	2	3	4	1	3	4
Set	Response												
S_1	1	6	6	8	3	6	8	8	12	11	11	15	19
S_1	2	5	1	3	9	3	5	8	9	8	15	11	12
S_2	1	4	6	7	8	9	13	2	4	6	11	14	17
S_2	2	9	6	4	13	8	9	1	2	2	14	12	10
S_3	1	1	3	5	3	5	7	8	9	10	6	9	12
S_3	3	3	3	6	4	7	9	8	7	10	8	11	13
S_4	1	3	4	8	6	10	12	10	8	9	2	8	6
S_4	3	5	3	4	5	9	13	7	7	13	5	2	5
S_5	2	4	1	0	9	4	6	2	5	4	10	7	7
S_5	3	2	3	0	2	7	10	4	5	7	5	8	13
S_6	2	6	0	0	7	5	7	5	3	2	11	7	9
S_6	3	0	2	1	3	5	11	5	4	7	10	8	14
S_7	1	2	6	6	2	2	5	7	8	10	5	6	9
S_7	2	7	4	4	5	1	2	6	6	7	8	5	4
S_7	3	2	3	3	0	1	7	7	7	14	3	7	12
S_8	1	3	5	5	1	3	7	6	6	10	5	9	9
S_8	2	6	3	1	5	1	0	5	5	5	11	8	8
S_8	3	3	2	5	2	0	5	6	2	10	4	5	11

design by making suitable transformations which pool up the data in such a way that the standard procedure of MANOVA becomes applicable. We thus first try to combine the data from the u sets.

Using (2.7) and (2.14) one obtains

$$\text{Exp}\,(Q_i) = C_i\xi M_i = \textstyle\sum_{r=1}^m F_r\xi(\alpha_{ir}M_i). \tag{3.1}$$

Define $Q(t \times \sum p_i)$ and $L_r(p \times \sum p_i)$ by

$$Q = (Q_1, Q_2, \cdots, Q_u), \qquad L_r = (\alpha_{1r}M_1, \alpha_{2r}M_2, \cdots, \alpha_{ur}M_u). \tag{3.2}$$

One then gets

$$\text{Exp}\,(Q) = \textstyle\sum_{r=1}^m F_r\xi L_r = (F_1\xi, F_2\xi, \cdots, F_m\xi)L, \tag{3.3}$$

where $L(mp \times \sum p_i)$ is defined by $L' = (L_1', L_2', \cdots, L_m')$.
In order to transfer L to the left side of (3.3), we first observe that

$$\text{Exp}\,(QL') = (F_1\xi, F_2\xi, \cdots, F_m\xi)(LL') \tag{3.4}$$

$$LL' = \begin{pmatrix} L_1 L_1' & \cdots & L_1 L_m' \\ \cdot & \cdots & \cdot \\ L_m L_1' & \cdots & L_m L_m' \end{pmatrix}, \tag{3.5}$$

$$L_r L_{r'}' = \textstyle\sum_{i=1}^u \alpha_{ir}\alpha_{ir'}(M_iM_i'). \tag{3.6}$$

Using the definition of M_i given before (2.7), one can easily check that

$$M_iM_i' = D_i \tag{3.7}$$

where D_i is a $(p \times p)$ diagonal matrix containing 1 in the l_{i1}th, l_{i2}th, $\cdots$, l_{ip_i}th diagonal cells and zero elsewhere. Thus the matrix (3.6) is diagonal and is given by $\sum_{i=1}^u \alpha_{ir}\alpha_{ir'}D_i$.

Consider now the nonsingularity of (LL'). It is obvious that in order that LL' may be nonsingular it is necessary that

$$mp \leqq \textstyle\sum_{i=1}^u p_i. \tag{3.8}$$

To push our investigation further suppose that Δ is an $mp \times mp$ matrix, such that it has m row blocks and m column blocks, and at the (r, r') block-cell, it has the $(p \times p)$ matrix $\Delta_{rr'}$. Let $\Delta_{rr'}(s, s')$ be the element in the (s, s') cell of $\Delta_{rr'}$. Consider the p^2 matrices $\Delta^{ss'}$ each of size $m \times m$, such that the (r, r') cell of $\Delta^{ss'}$ contains the element $\Delta_{rr'}(s, s')$. Let Δ^* be the $mp \times mp$ matrix which contains $\Delta^{ss'}$ at the intersection of sth row block and s'th column block. Thus

$$\Delta = \begin{pmatrix} \Delta_{11} & \cdots & \Delta_{1m} \\ \cdot & \cdots & \cdot \\ \Delta_{m1} & \cdots & \Delta_{mm} \end{pmatrix}, \qquad \Delta^* = \begin{pmatrix} \Delta^{11} & \cdots & \Delta^{1p} \\ \cdot & \cdots & \cdot \\ \Delta^{p1} & \cdots & \Delta^{pp} \end{pmatrix}. \tag{3.9}$$

It is easy to see that Δ^* can be obtained from Δ by an appropriate permutation of rows and columns. Hence

$$\Delta^* = E\Delta E, \tag{3.10}$$

where E is a permutation matrix, which is necessarily nonsingular. Taking inverses, one obtains

$$\Delta^{-1} = E\Delta^{*-1}E. \tag{3.11}$$

Hence Δ^{-1} is obtained from Δ^{*-1} by the same permutation by which Δ^* is obtained from Δ.

Now if we take $\Delta = LL'$ with $\Delta_{rr'} = L_r L'_{r'}$, then one can easily check that the matrices $\Delta^{ss'}$ are all zero, for $s \neq s'$, and that for $s = 1, 2, \cdots, p$,

$$\Delta^{ss} = \begin{bmatrix} \sum'_r \alpha_{r1}^2 & \cdots\cdot & \sum'_r \alpha_{r1}\alpha_{rm} \\ \cdots\cdots\cdots & \cdots\cdot & \cdots\cdots\cdots \\ \sum'_r \alpha_{rm}\alpha_{r1} & \cdots\cdot & \sum'_r \alpha_{rm}^2 \end{bmatrix} = (\Delta_s\text{, say}) \tag{3.12}$$

where r in $\sum'_r$ ranges over the collection U_s of all sets S_i (of experimental units) on which the sth response is measured. From (2.10) it follows then that LL' is nonsingular if and only if Δ^* is so which in turn is equivalent to Δ_s being nonsingular for all s.

However, as we now prove, the matrix Δ_s is singular for all s, whatever the multiresponse design D may be. For this purpose, recall from (2.15) that

$$C_i = \alpha_{i1}F_1 + \alpha_{i2}F_2 + \cdots + \alpha_{im}F_m,$$

where $F_1, F_2, \cdots, F_m$ are linearly independent known matrices.

Let $f_{\theta jj'}$ be the element in cell (j, j') of F_θ, $\theta = 1, 2, \cdots, m$. Then from (2.12)

$$C_i(j, j) = r_{ij} - \sum_{g=1}^{b_i} n_{igj}^2/k_{ig} = \sum_{\theta=1}^m \alpha_{i\theta}f_{\theta jj},$$

and

$$C_i(j, j') = -\sum_{g=1}^{b_i} n_{igj}n_{igj'}/k_{ig} = \sum_{\theta=1}^m \alpha_{i\theta}f_{\theta jj'}.$$

Hence

$$\begin{aligned}\sum_{\theta=1}^m \sum_{j'=1}^t \alpha_{i\theta}f_{\theta jj'} &= r_{ij} - \sum_{g=1}^{b_i} n_{igj}k_{ig}^{-1}\left(\sum_{j'=1}^t n_{igj'}\right) \\ &= r_{ij} - \sum_{g=1}^{b_i} n_{igj} = r_{ij} - r_{ij} = 0.\end{aligned}$$

This implies that

$$\sum_{\theta=1}^m \alpha_{i\theta}\left(\sum_{j'=1}^t f_{\theta jj'}\right) = 0, \qquad \text{for all} \quad i = 1, 2, \cdots, u; \qquad j = 1, 2, \cdots, t.$$

Now multiplying the θth row of Δ_s by $(\sum_{j'=1}^t f_{\theta jj'})$ and adding, we get

$$\begin{aligned}&\left(\sum_{r\varepsilon U_s} \alpha_{r1}\left(\sum_{\theta=1}^m \alpha_{r\theta}\left(\sum_{j'=1}^t f_{\theta jj'}\right)\right), \cdots, \sum_{r\varepsilon U_s} \alpha_{rm}\left(\sum_{\theta=1}^m \alpha_{r\theta}\left(\sum_{j'=1}^t f_{\theta jj'}\right)\right)\right) \\ &\qquad = \left(\sum'_r \alpha_{r1}\cdot 0, \sum'_r \alpha_{r2}\cdot 0, \cdots, \sum'_r \alpha_{rm}\cdot 0\right) \\ &\qquad = 0_{1m}, \qquad \text{for all} \quad j = 1, 2, \cdots, t,\end{aligned}$$

which completes the proof of our assertation.

Define

$$f_{\theta j} = \sum_{j'=1}^t f_{\theta jj'}, \qquad F = \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1t} \\ \cdot & \cdot & \cdots & \cdot \\ f_{m1} & f_{m2} & \cdots & f_{mt} \end{pmatrix}. \tag{3.13}$$

Then the last result implies that

$$\Delta_s F = 0_{mt} . \tag{3.14}$$

Now let

$$\rho = \text{rank } F, \tag{3.15}$$

so that

$$\text{Rank } (LL') = \text{Rank } (\Delta) = \textstyle\sum_{s=1}^{p} \text{Rank } (\Delta_s) \leqq p(m - \rho).$$

As is usual in ordinary experimental designs, here too we shall take recourse to the conditions (2.8) on the matrix ξ so that we may possibly "remove" the singularity of Δ.

For any fixed i, we can replace C_i by C_{i0} where

$$C_{i0} = C_i + \sigma_i J_{tt} , \tag{3.16}$$

σ_i being an arbitrary constant. We shall have by (2.8) and (2.14),

$$C_{i0}\xi^{(i)} = C_i\xi^{(i)} + \sigma_i J_{tt}\xi^{(i)} = C_i\xi^{(i)} = \text{Exp } (Q_i).$$

Thus the replacement of C_i by C_{i0} does not alter the equations of expectation, though it changes $\alpha_{i\theta}$ to $(\alpha_{i\theta} + \sigma_i)$, in case $\sum_{\theta=1}^{m} F_\theta = J_{nn}$. Before investigating further the effects of this change we shall show that if $u > 1$, and there is some variate (say sth) which is measured on just one set S_i then Δ_s is singular, and the singularity is not removable by introducing the change proposed at (3.16). We are considering the case $u > 1$, for if $u = 1$, we are reduced to the customary problem when all variates are measured on each unit, and in which the consideration of the matrix Δ does not arise.

When $u > 1$, we consider Δ. Then we obtain under (3.16),

$$\Delta_s = \boldsymbol{\alpha}_i\boldsymbol{\alpha}_i',$$

where $\boldsymbol{\alpha}_i' = [(\alpha_{i1} + \sigma_i), (\alpha_{i2} + \sigma_i), \cdots , (\alpha_{im} + \sigma_i)]$, which is of rank 1 for all σ_i , which proves our assertion.

In Section 5 we consider the conditions under which the matrix Δ and hence LL' could be made nonsingular by altering the equations of expectations (2.14) to the form (3.16) and appropriately choosing σ_i's. Throughout this paper we assume that for some real number ν

$$\text{(i)} \quad \textstyle\sum_{\theta=1}^{m} F_\theta = \nu J_{nn} , \qquad \text{(ii)} \quad \rho = \text{Rank } F = 1. \tag{3.17}$$

These conditions are satisfied for example when $F_1 , F_2 , \cdots , F_m$ are the association matrices of a partially balanced design with m associate classes (including the 0th associate class).

Example. (Continued from last section.) For our design, it is easily checked that

$$LL' = \begin{bmatrix} L_1 L_1' & L_1 L_2' \\ L_2 L_1' & L_2 L_2' \end{bmatrix} = \begin{bmatrix} 24I_4 & (-8)I_4 \\ (-8)I_4 & (24/9)I_4 \end{bmatrix}; \qquad \Delta_s = \begin{bmatrix} 24 & -8 \\ -8 & (24/9) \end{bmatrix},$$

for $s = 1, 2, 3$; and $|\Delta_s| = 0$. Also

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 \end{bmatrix}; \qquad \rho = \text{Rank}(F) = 1; \qquad F_1 + F_2 = J_{4,4},$$

so that (3.17) is satisfied. Finally, the matrix Q, defined by (3.2) and (2.13) can be easily obtained:

$$Q = \left[\begin{array}{rr|rr|rr|rr|rr|rr|rrr|rrr} -7.3 & 7.7 & -6.7 & 7.7 & -7.0 & -6.3 & -8.7 & -2.0 & 7.0 & -7.7 & 6.7 & -5.0 & -5.3 & 6.7 & -7.7 & -6.7 & 7.7 & -3.3 \\ -2.7 & -5.0 & -2.7 & -3.0 & -1.0 & -1.0 & 0.7 & -3.0 & -4.7 & 0.7 & -1.7 & -0.7 & -1.0 & -3.0 & -3.7 & -1.3 & -1.3 & -3.7 \\ 3.0 & -1.0 & 1.3 & -2.0 & 2.0 & 1.0 & 4.7 & -4.0 & -1.3 & -2.7 & -4.3 & -4.0 & 0.3 & -2.0 & -2.3 & 0.7 & -3.3 & -4.0 \\ 7.0 & -1.7 & 8.0 & -2.7 & 6.0 & 6.3 & 3.3 & 9.0 & -1.0 & 9.7 & -0.7 & 9.7 & 6.0 & -1.7 & 13.7 & 7.3 & -3.0 & 11.0 \end{array}\right].$$

4. Regular multiresponse designs. Suppose now, that by suitable choice of $\sigma_1, \sigma_2, \cdots, \sigma_u$, the matrix Δ and hence LL' has been made nonsingular. This means that we now have

$$(4.0) \quad \Delta_s = \begin{bmatrix} \sum_r' (\alpha_{r1} + \sigma_r)^2 & \cdots & \sum_r' (\alpha_{r1} + \sigma_r)(\alpha_{rm} + \sigma_r) \\ \cdot & \cdots & \cdot \\ \sum_r' (\alpha_{rm} + \sigma_r)(\alpha_{r1} + \sigma_r) & \cdots & \sum_r' (\alpha_{rm} + \sigma_r)^2 \end{bmatrix}$$

where r in $\sum_r'$ runs over the collection U_s of sets of units in which the sth characteristic is measured. Also, the new (LL') is defined as before by (3.11), with the only change that the new value of Δ_s given above be used in the formula for Δ^* at (3.9). Define

$$(4.1) \qquad \Pi_s = \Delta_s^{-1} = \begin{bmatrix} \overset{s}{\pi}_{11} & \overset{s}{\pi}_{12} & \cdots & \overset{s}{\pi}_{1m} \\ \cdot & \cdot & \cdots & \cdot \\ \overset{s}{\pi}_{m1} & \overset{s}{\pi}_{m2} & \cdots & \overset{s}{\pi}_{mm} \end{bmatrix}, \qquad s = 1, 2, \cdots, p;$$

$$(4.2) \qquad H_{\theta\theta'} = \text{diag}\,(\overset{1}{\pi}_{\theta\theta'}, \overset{2}{\pi}_{\theta\theta'}, \cdots, \overset{p}{\pi}_{\theta\theta'}); \qquad \theta, \theta' = 1, 2, \cdots, m.$$

Then we can write, using (3.9), (3.10), and (3.11):

$$(4.3) \qquad (LL')^{-1} = \begin{bmatrix} H_{11} & H_{12} & \cdots & H_{1m} \\ H_{21} & H_{22} & \cdots & H_{2m} \\ \cdot & \cdot & \cdots & \cdot \\ H_{m1} & H_{m2} & \cdots & H_{mm} \end{bmatrix} = H, \quad \text{say.}$$

Define

$$(4.4) \qquad H_i' = (H_{1i}, H_{2i}, \cdots, H_{mi}), \qquad i = 1, 2, \cdots, m.$$

Then

$$(4.5) \qquad (LL')^{-1} = H = (H_1, H_2, \cdots, H_m).$$

Going back to (3.4), we then get

$$\text{Exp}\,(QL'H) = (F_1\xi, F_2\xi, \cdots, F_m\xi). \tag{4.6}$$

We shall now investigate how (4.6) could be put into a form to which the customary model of multivariate analysis of variance is directly applicable. For this purpose we first proceed to obtain the variance matrix of the left hand side in (4.6).

From (2.9), (2.10) and (2.13), we can write

$$Q_i = Q_i^* Y_i, \tag{4.7}$$

where Q_i^* is a $(t \times N_i)$ matrix such that the element in the cell (j', j) of Q_i^* is given by $q^*_{ij'j}$ as defined below:

(i) $q^*_{ij'j} = 0$, if the jth row in Y_i corresponds to an experimental unit e_j^i (say), such that e_j^i lies in a block in which the j'th treatment does not occur at all.

(ii) $q^*_{ij'j} = 1 - (n_{ij'g}/k_{ig})$, if e_j^i (as defined above) is in the gth block and has the j'th treatment allotted to itself.

(iii) $q^*_{ij'j} = -(n_{ij'g}/k_{ig})$, if e_j^i is in the gth block, and the j'th treatment occurs in this block but is not allotted to e_j^i.

It can be easily checked that

$$Q_i^* Q_i^{*\prime} = C_i, \tag{4.8}$$

Now using (2.6), (2.7) and (2.5) one could write (under an obvious convention)

$$\text{Var}\,(Y_i) = I_{N_i} \otimes \Sigma^{(i)} = I_{N_i} \otimes (M_i'\Sigma M_i), \tag{4.9}$$

where $\otimes$ denotes Kronecker product. Hence from (4.7) and (4.8),

$$\text{Var}\,(Q_i) = C_i \otimes (M_i'\Sigma M_i). \tag{4.10}$$

Also, if $i \neq i'$, then Q_i and $Q_{i'}$ refer to two different sets of units S_i and $S_{i'}$ respectively. Hence we have

$$\text{Cov}\,(Q_i, Q_{i'}) = 0_{pt,pt}. \tag{4.11}$$

Next we consider the left hand side of (4.6). We have using (3.3) and (3.2), and then (2.13),

$$QL' = \left[\sum_{i=1}^{u} \alpha_{i1}\, Q_i\, M_i', \quad \cdots, \quad \sum_{i=1}^{u} \alpha_{im}\, Q_i\, M_i'\right] = \begin{bmatrix} \sum_{i=1}^{u} \alpha_{i1}\, Q_{i1}\, M_i', & \cdots, & \sum_{i=1}^{u} \alpha_{im}\, Q_{i1}\, M_i' \\ \cdot & \cdots & \cdot \\ \sum_{i=1}^{u} \alpha_{i1}\, Q_{it}\, M_i', & \cdots, & \sum_{i=1}^{u} \alpha_{im}\, Q_{it}\, M_i' \end{bmatrix}. \tag{4.12}$$

Define the $(p \times p)$ diagonal matrices $G_{i\theta}$ $(i = 1, 2, \cdots, u; \theta = 1, 2, \cdots, m)$ by

$$G_{i\theta} = \textstyle\sum_{\theta'=1}^{m} \alpha_{i\theta'} H_{\theta'\theta}. \tag{4.13}$$

Then using (4.3) we get, for $\theta = 1, 2, \cdots, m$,

$$(4.14)\qquad QL'H_\theta = \begin{bmatrix} \sum_{\theta'=1}^{m} \sum_{i=1}^{u} \alpha_{i\theta'} Q_{i1} M_i' H_{\theta'\theta} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \sum_{\theta'=1}^{m} \sum_{i=1}^{u} \alpha_{i\theta'} Q_{it} M_i' H_{\theta'\theta} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{u} Q_{i1} M_i' G_{i\theta} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \sum_{i=1}^{u} Q_{it} M_i' G_{i\theta} \end{bmatrix}.$$

In order to obtain the variance matrix for the expression under expectation sign, in the left hand side, we start with the expressions in (4.14). For convenience we define the $(1 \times p)$ vectors

$$(4.15)\qquad \mathbf{Z}_{j\theta}' = \textstyle\sum_{i=1}^{u} Q_{ij} M_i' G_{i\theta}; \qquad j = 1, 2, \cdots, t; \qquad \theta = 1, 2, \cdots, m.$$

Note that in $\mathbf{Z}_{j\theta}$, the only matrices involving the observations are the Q_{ij}'s. Hence using (4.11), we immediately get

$$(4.16)\qquad \mathrm{Var}\,(\mathbf{Z}_{j\theta}) = \textstyle\sum_{i=1}^{u} G_{i\theta} M_i [\mathrm{Var}\,(Q_{ij})] M_i' G_{i\theta},$$

$$\mathrm{Covar}\,(\mathbf{Z}_{j\theta}, \mathbf{Z}_{j'\theta'}) = \textstyle\sum_{i=1}^{u} G_{i\theta} M_i [\mathrm{Covar}\,(Q_{ij}, Q_{ij'})] M_i' G_{i\theta'}.$$

To obtain the values of the quantities inside the bracket in (4.16), we use (4.10), (2.12), and (3.7) to get for all permissible j and θ,

$$(4.17)\qquad \mathrm{Var}\,(\mathbf{Z}_{j\theta}) = \textstyle\sum_{i=1}^{u} G_{i\theta} M_i [(r_{ij} - \mu_{ijj}) \cdot M_i' \Sigma M_i] M_i' G_{i\theta}$$
$$= \textstyle\sum_{i=1}^{u} (r_{ij} - \mu_{ijj})(G_{i\theta} D_i \Sigma D_i G_{i\theta}).$$

Similarly for all j, and $\theta \neq \theta'$,

$$(4.18)\qquad \mathrm{Cov}\,(\mathbf{Z}_{j\theta}, \mathbf{Z}_{j\theta'}) = \textstyle\sum_{i=1}^{u} (r_{ij} - \mu_{ijj})(G_{i\theta} D_i \Sigma D_i G_{i\theta'}).$$

Also for all θ, θ', and for $j \neq j'$, one gets

$$(4.19)\qquad \mathrm{Cov}\,(\mathbf{Z}_{j\theta}, \mathbf{Z}_{j'\theta'}) = -\textstyle\sum_{i=1}^{u} (\mu_{ijj'})(G_{i\theta} D_i \Sigma D_i G_{i\theta'}).$$

To evaluate the matrices occurring in (4.17)–(4.19), we have from (4.13) and (4.2),

$$(4.20)\qquad G_{i\theta} = \mathrm{diag}\,(\textstyle\sum_{\theta'=1}^{m} \alpha_{i\theta'} \overset{1}{\pi}_{\theta'\theta}, \sum_{\theta'=1}^{m} \alpha_{i\theta'} \overset{2}{\pi}_{\theta'\theta}, \cdots, \sum_{\theta'=1}^{m} \alpha_{i\theta'} \overset{p}{\pi}_{\theta'\theta})$$
$$= \mathrm{diag}\,(g_{i\theta}^{1}, g_{i\theta}^{2}, \cdots, g_{i\theta}^{p}), \qquad \text{where}$$

$$(4.21)\qquad g_{i\theta}^{l} = \textstyle\sum_{\theta'=1}^{m} \alpha_{i\theta'} \overset{l}{\pi}_{\theta'\theta}, \qquad l = 1, 2, \cdots, p; \qquad i = 1, 2, \cdots, u;$$
$$\theta = 1, 2, \cdots, m.$$

It can then be easily checked that the element in the cell (l, l'), $l, l' = 1, 2, \cdots, p$, of the matrix $(G_{i\theta} D_i \Sigma D_i G_{i\theta'})$ occurring in (4.17)—(4.19) is

(4.22) (i) $\sigma_{ll'} g_{i\theta}^{l} g_{i\theta'}^{l'}$, if both the variates l and l' are measured in the set S_i, and (ii) zero, otherwise.

Let $U_{ll'}$ be the collection of sets S_i, such that the variates l and l' both are measured on each unit in a set S_i if and only if $S_i \varepsilon U_{ll'}$. Then using (4.22) one gets for the (l, l') element of Var $(\mathbf{Z}_{j\theta})$,

$$[\textstyle\sum_i' (r_{ij} - \mu_{ijj}) g_{i\theta}^l g_{i\theta}^{l'}] \sigma_{ll'} , \tag{4.23}$$

with similar expressions for the other two cases. Here $\sum_i'$ indicates that the sum is to be taken over all values of i such that $S_i \varepsilon U_{ll'}$.

Let us now consider the vectors $\mathbf{Z}_{j\theta}$ again. Since $\sum_{j=1}^t Q_{ij} = 0_{1p_i}$, it can be checked using (4.15) that $\sum_{j=1}^t \mathbf{Z}_{j\theta}' = 0_{1p}$, for all θ. Thus the vectors $\mathbf{Z}_{j\theta}$ are linearly dependent, and the rank of the vector space generated by them is at most $m(t - 1)$. Apart from having this linear dependence they are in general correlated, too, as is shown by (4.23), (4.19) and (4.22). To be able to transform them (by a *known* linear transformation) to a set of vectors which are uncorrelated, one way appears to be to require that the expressions like the one in curly brackets in (4.23) be factorisable into two factors, one of which depends on (l, l') and the other does not. This motivates the following definition: A multiresponse design D is called *regular*, provided (i) D is homogeneous, (ii) the corresponding (LL') could be made nonsingular by using (3.16), and provided that there exists a positive definite $p \times p$ matrix $\Gamma = ((\gamma_{ll'}))$, such that for all $j, j' = 1, 2, \cdots, t$; $l, l' = 1, 2, \cdots, p$; and $\theta, \theta' = 1, 2, \cdots, m$; the following factorisation is possible.

$$\text{(a)} \quad \textstyle\sum_{i\varepsilon(U_l \cap U_{l'})} (r_{ij} - \mu_{ijj}) g_{i\theta}^l g_{i\theta'}^{l'} = w_{\theta\theta'}^{jj} \cdot \gamma_{ll'} , \tag{4.24}$$

$$\text{(b)} \quad \textstyle\sum_{i\varepsilon(U_l \cap U_{l'})} (-\mu_{ijj'}) g_{i\theta}^l g_{i\theta'}^{l'} = w_{\theta\theta'}^{jj'} \gamma_{ll'} ,$$

where $W = ((w_{\theta\theta'}^{jj'}))$ is an $(mt \times mt)$ matrix, which is positive semidefinite, with rank $m(t - 1)$. Let $\sigma_{ll'}^* = \gamma_{ll'} \sigma_{ll'}$, and $\Sigma^* = ((\sigma_{ll'}^*))$. Then, for a regular design,

$$\begin{aligned} \text{Var}(Z) &\equiv \text{Var}\,[(\mathbf{Z}_{11} \cdots \mathbf{Z}_{t1} \mathbf{Z}_{12} \cdots \mathbf{Z}_{t2} \cdots \mathbf{Z}_{1m} \cdots \mathbf{Z}_{tm})'] \\ &= W \otimes \Sigma^* \end{aligned} \tag{4.25}$$

where Z is defined by (4.25) above, $\otimes$ denotes Kronecker product, and where clearly Σ^* must be positive definite. Thus we are in the possession of *mpt* random variables (with *mp* linear relations between them), whose variance is given by (4.25) and expectation by

$$\text{Exp}(Z) \equiv \text{Exp}\,[(\mathbf{Z}_{11} \cdots \mathbf{Z}_{tm})'] = [F_1' \mid F_2' \mid \cdots \mid F_m']'\xi. \tag{4.26}$$

If W were positive definite, the usual multivariate analysis of variance model would apply to the above after making an obvious and known transformation. However we are now faced with the singular case. A general solution to the MANOVA problem, when the dispersion matrix is unknown but known to be singular, will be considered elsewhere. But the problem here is simpler, since the dispersion matrix splits into W and Σ^*, and the singularity is in W which is known. Consider W. There exists an orthogonal $(mt \times mt)$ matrix T such that TWT' is a diagonal matrix D_W which contains the characteristic roots of W. Then we have Var $(TZ) = (TWT') \otimes \Sigma^* = D_W \otimes \Sigma^*$. Since W is only of rank $m(t-1)$ exactly m roots in D_W will be zero. Also the singularity in W is caused by the relations $\sum_{j=1}^{t} Z'_{j\theta} = 0_{1p}$ $(\theta = 1, 2, \cdots, m)$. Hence it is clear that without loss of generality we can assume T to be of the form $T' = [T_1' \vdots T_2']$, where T_1' and T_2' have respectively m and $m(t-1)$ columns, and where T_1 is of the form:

$$(4.27) \qquad T_1 = \begin{bmatrix} J_{1t} & 0_{1t} & 0_{1t} & \cdots & 0_{1t} \\ 0_{1t} & J_{1t} & 0_{1t} & \cdots & 0_{1t} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0_{1t} & 0_{1t} & 0_{1t} & \cdots & J_{1t} \end{bmatrix}.$$

Thus,

$$(4.28) \qquad D_W \otimes \Sigma^* = \text{Var}\,(TZ) = \text{Var}\begin{bmatrix} T_1 Z \\ T_2 Z \end{bmatrix} = \begin{bmatrix} 0_{mm} & 0 \\ 0 & T_2 W T_2' \end{bmatrix} \otimes \Sigma^*$$

$$= \begin{bmatrix} 0_{mm} & 0 \\ 0 & D_0 \end{bmatrix} \otimes \Sigma^*, \quad \text{say,}$$

where D_0 is diagonal with all elements positive. Defining

$$(4.29) \qquad T_3 = D_0^{\frac{1}{2}} T_2, \qquad X = T_3 Z,$$

we finally have

$$(4.30) \qquad \text{Exp}\,(X) = \text{Exp}\,(T_3 Z) = T_3 \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_m \end{bmatrix} \xi = \phi\xi, \quad \text{say,}$$

$$\text{Var}\,(X) = \text{Var}\,(T_3 Z) = I_{m(t-1)} \otimes \Sigma^*,$$

to which the usual MANOVA model becomes applicable ([9], [1]). An estimate of ξ is

$$(4.31) \qquad \hat{\xi} = [\phi'\phi]^*\phi' X = \left([F_1' \cdots F_m'](T_3'T_3) \begin{bmatrix} F_1 \\ \vdots \\ F_m \end{bmatrix} \right)^* [F_1' \cdots F_m'](T_3'T_3)Z,$$

where * denotes a conditional inverse (for definition, see [2] or [9]). As is the case even for ordinary linear models, the estimate is not unique unless $[\phi'\phi]$ is non-singular.

We can formally summarise the above results in

THEOREM 4.1. *For a regular multiresponse design, there exists a linear transformation of the original data, such that the standard techniques of* MANOVA *regarding estimation or hypotheses testing on the parameter matrix* ξ *become applicable.*

To compute $\hat{\xi}$, there is an easier way than is evident by (4.31). We calculate $(T_3'T_3)$ directly in terms of W, as indicated below. We have by (4.29) and (4.28),

$$(4.32) \qquad T_3'T_3 = T_2' D_0^{-1} T_2 = T_2'(T_2 W T_2')^{-1} T_2.$$

Now let

$$J_0 = T_1'T_1, \qquad W_0 = W + \delta J_0, \tag{4.33}$$

where δ is a nonzero real number. Then W_0 is nonsingular. In fact after a rearrangement of the rows and columns, of W, one can write

$$TW_0 T' = TWT' + \delta(TJ_0 T') = \left[\begin{array}{c|c} (\delta t^2)I_m & 0 \\ \hline 0 & D_0 \end{array}\right].$$

Then

$$TW_0^{-1}T' = \left[\begin{array}{c|c} (\delta t^2)^{-1}I_m & 0 \\ \hline 0 & D_0^{-1} \end{array}\right],$$

$$W_0^{-1} = [T_1' \vdots T_2'] \left[\begin{array}{c|c} (\delta t^2)^{-1}I_m & 0 \\ \hline 0 & D_0^{-1} \end{array}\right] \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$$

$$= (\delta t^2)^{-1}J_0 + (T_2'D_0^{-1} T_2).$$

Hence from (4.32), we get the useful result

$$(T_3'T_3) = (W + \delta J_0)^{-1} - (\delta t^2)^{-1} J_0. \tag{4.34}$$

If we want to test any (testable) hypothesis regarding ξ, we can use any of the three tests (largest root, trace or likelihood) based on the roots of $(S_h S_e^{-1})$, where S_h and S_e are respectively the hypothesis and error dispersion matrices (starting from the model (4.30)). The formulae for S_h and S_e are easily available, (e.g., [1], [10]), and will not be reproduced for brevity. It should be remarked however that in S_h and S_e also, W would enter in the form of $(T_3'T_3)$ which is given by (4.34).

We illustrate the above with the earlier

EXAMPLE. To make LL' nonsingular, we choose $\sigma_i = (\frac{2}{3})$, (and hence $\alpha_{i1}' = (\frac{8}{3})$, $\alpha_{i2}' = 0$) for $i = 1, 3, 5, 7$, and $\sigma_i = 0$ (and hence $\alpha_{i1}' = 2$, $\alpha_{i2}' = (-\frac{2}{3})$) otherwise. We then get

$$\Delta_s = (4/3)\begin{bmatrix} 25 & -3 \\ -3 & 1 \end{bmatrix}, \quad \Pi_s = (3/64)\begin{bmatrix} 1 & 3 \\ 3 & 25 \end{bmatrix}, \qquad s = 1, 2, 3;$$

and

$$(LL')^{-1} = \left[\begin{array}{c|c} (3/64)I_3 & (9/64)I_3 \\ \hline (9/64)I_3 & (75/64)I_3 \end{array}\right] = \left[\begin{array}{c|c} H_{11} & H_{12} \\ \hline H_{21} & H_{22} \end{array}\right] = [H_1 \mid H_2];$$

and $g_{i1}^l = (\frac{1}{8})$, $g_{i2}^l = (\frac{3}{8})$, i odd; $g_{i1}^l = 0$, $g_{i2}^l = -(\frac{1}{2})$, i even; for all l. Thus equations (4.24) are satisfied with $\Gamma = (\frac{1}{3})I_3 + (\frac{2}{3})J_{33}$, and

$$W = (1/32)\begin{bmatrix} J^* & 3J^* \\ 3J^* & 25J^* \end{bmatrix}, \quad \text{where } J^* = 4I_4 - J_{44}, \text{ and Rank } (W) = 6.$$

Hence our multiresponse design is regular, and the usual techniques of multivariate estimation and hypothesis testing can be employed using the matrix Z

(defined by (4.25) and (4.15)), which is given by

$$Z' = \left[\begin{array}{cccc|cccc} -2.46 & -0.58 & 0.67 & 2.38 & 3.63 & -0.08 & -1.33 & -2.21 \\ 2.67 & -1.58 & -0.54 & -0.54 & -3.00 & -1.75 & 3.21 & 1.54 \\ -2.71 & -0.50 & -0.50 & 3.71 & -2.96 & 2.17 & 4.50 & -3.71 \end{array}\right].$$

For example, as mentioned earlier, the matrices T_3 and X do not need to be obtained to get $\hat{\xi}$, which from (4.31) is

$$\hat{\xi} = \begin{bmatrix} -0.77 & 0.73 & -0.06 \\ -0.08 & -0.03 & -0.32 \\ 0.25 & -0.44 & -0.58 \\ 0.59 & -0.25 & 0.96 \end{bmatrix}.$$

5. Removing the singularity of LL'. In the beginning of the last section, we assumed that (LL') can be made nonsingular. We now study a necessary and a sufficient condition under which this can be done.

Consider Δ_s given by (4.0). Suppose Δ_s is singular. Then there exist constants $c_{s1}, c_{s2}, \cdots, c_{sm}$, such that

$$(5.1) \qquad \sum_{\theta=1}^{m} c_{s\theta} \sum_r' (\alpha_{r\theta} + \sigma_r)(\alpha_{r\theta'} + \sigma_r) = 0, \qquad \theta' = 1, 2, \cdots, m.$$

Hence for all θ', we have

$$(5.2) \qquad \sum_r' (\alpha_{r\theta'} + \sigma_r)\left(\sum_{\theta=1}^{m} c_{s\theta}\alpha_{r\theta} + \sigma_r \sum_{\theta=1}^{m} c_{s\theta}\right) = 0.$$

Let the collection U_s contain m_s sets, say $S_{s1}, \cdots, S_{sm_s}$. For the design over the set S_{sj}, let the α's be denoted by $\alpha_{sj1}, \alpha_{sj2}, \cdots, \alpha_{sjm}$, or in vector notation by $\boldsymbol{\alpha}_{sj}$. Let the vector space generated by the vectors $\boldsymbol{\alpha}_{s1}, \boldsymbol{\alpha}_{s2}, \cdots, \boldsymbol{\alpha}_{sm_s}$ and J_{m1} be V_s with rank m_s' $(\leqq m_s + 1)$. Then it is clear that if $m > m_s'$ and R_s is the vector space of rank $(m - m_s')$ orthogonal to V_s, we can satisfy the condition (5.2) by letting $(c_{s1}, c_{s2}, \cdots, c_{sm}) = \mathbf{c}_s'$ belong to R_s, whatever the values of σ's may be. Thus we have proved the following necessary condition.

THEOREM 5.1. *If $m > \min(m_1', m_2', \cdots, m_p')$, then the singularity of Δ is not removable by using* (3.16).

Our next result will show that there also exist sufficient conditions under which the singularity of Δ is removable. For simplicity we shall assume

$$(5.3) \qquad m_1 = m_2 = \cdots = m_p = m_0, \quad \text{say,} \qquad \text{and}$$

$$(5.4) \qquad m = m_0.$$

Consider Δ_s, and suppose that the $(m \times m)$ matrix $A_s = ((\alpha_{s\theta\theta'}))$ is nonsingular. Also let

$$(5.5) \quad g_{sr} = \sum_{\theta=1}^{m} c_\theta \alpha_{sr\theta}, \qquad c_s = \sum_{\theta=1}^{m} c_{s\theta}, \quad \text{and} \quad \boldsymbol{\sigma}_s' = (\sigma_{s1}, \cdots, \sigma_{sm})$$

where σ_{sj} is the value of σ chosen for the set S_{sj}. Furthermore let

$$(5.6) \qquad \mathbf{g}_s' = (g_{s1}, \cdots, g_{sm}).$$

Then equations (5.2) can be written

$$(5.7) \qquad A_s\mathbf{g}_s + (\textstyle\sum_{\theta=1}^{m} g_{s\theta}\sigma_{s\theta})J_{m1} + c_sA_s\boldsymbol{\sigma}_s + c_s(\textstyle\sum_{\theta=1}^{m}\sigma_{s\theta}^2)J_{m1} = \theta_{m1}.$$

We now prove

LEMMA 5.1. *If* A_s *and* $\boldsymbol{\sigma}_s$ *are such that*

$$(5.8) \qquad 1 + \boldsymbol{\sigma}_s'A_s^{-1}J_{m1} \neq 0,$$

and Δ_s *is singular, then* $C_s \neq 0$.

PROOF. Let $C_s = 0$. Then (5.7) gives

$$A_s\mathbf{g}_s + (\boldsymbol{\sigma}_s'\mathbf{g}_s)J_{m1} = 0_{m1}.$$

Premultiplying by A_s^{-1}, and then by $\boldsymbol{\sigma}_s'$ and simplifying one gets

$$(\boldsymbol{\sigma}_s'\mathbf{g}_s)(1 + \boldsymbol{\sigma}_s'A_s^{-1}J_{m1}) = 0.$$

Noting (5.8) and also that $\mathbf{g}_s' = A_s'\mathbf{c}_s$, one obtains $A_s\mathbf{g}_s = 0_{m1}$, or equivalently $(A_sA_s')\mathbf{c}_s = 0_{m1}$, i.e., $\mathbf{c}_s = 0_{m1}$, showing that Δ_s is nonsingular. This contradiction proves the lemma.

THEOREM 5.2. *If the arbitrary constants* $\sigma_1, \sigma_2, \cdots, \sigma_u$ *are so chosen that for all* s, $\boldsymbol{\sigma}_s'$ *is such that* (5.8) *is satisfied, then the matrices* Δ_s *as at* (4.0) *and hence the corresponding* Δ *are all nonsingular.*

PROOF. Consider Δ_s again. On substituting $\mathbf{g}_s' = A_s'\mathbf{c}_s$, we have from (5.7)

$$(A_sA_s')\mathbf{c}_s + (\boldsymbol{\sigma}_s'A_s'\mathbf{c}_s)J_{m1} = -c_s[A_s\boldsymbol{\sigma}_s + (\boldsymbol{\sigma}_s'\boldsymbol{\sigma}_s)J_{m1}], \quad \text{or}$$

$$(5.9) \quad \mathbf{c}_s = -(\boldsymbol{\sigma}_s'A_s'\mathbf{c}_s)(A_sA_s')^{-1}J_{m1} - c_s(A_sA_s')^{-1}[A_s\boldsymbol{\sigma}_s + (\boldsymbol{\sigma}_s'\boldsymbol{\sigma}_s)J_{m1}]$$
$$= -d_s\omega_1 - c_s\omega_2, \quad \text{say,} \qquad \text{where} \quad d_s = (\boldsymbol{\sigma}_s'A_s'\mathbf{c}_s).$$

Multiplying by J_{1m}, we get

$$(5.10) \quad c_s = -d_sJ_{1m}\omega_1 - c_sJ_{1m}\omega_2, \quad \text{or} \quad c_s(1 + J_{1m}\omega_2) = -d_s(J_{1m}\omega_1).$$

Now from (5.9),

$$1 + J_{1m}\omega_2 = 1 + J_{1m}A_s'^{-1}\boldsymbol{\sigma}_s + (\boldsymbol{\sigma}_s'\boldsymbol{\sigma}_s)J_{1m}(A_sA_s')^{-1}J_{m1}$$
$$= 1 + (J_{1m}A_s'^{-1})\boldsymbol{\sigma}_s + (\boldsymbol{\sigma}_s'\boldsymbol{\sigma}_s)(J_{1m}A_s'^{-1})(A_s^{-1}J_{m1})$$
$$\geqq 1 + (J_{1m}A_s'^{-1}\boldsymbol{\sigma}_s) + (J_{1m}A_s'^{-1}\boldsymbol{\sigma}_s)^2, \quad \text{by Cauchy's inequality.}$$

But this last expression is obviously always >0. Hence

$$(5.11) \qquad 1 + J_{1m}\omega_2 > 0 \quad \text{and} \quad c_s = -d_s(J_{1m}\omega_1)/(1 + J_{1m}\omega_2);$$

$$(5.12) \qquad \mathbf{c}_s = -d_s[\omega_1 - (J_{1m}\omega_1)(1 + J_{1m}\omega_2)^{-1}\omega_2] = -d_s\omega_3, \quad \text{say.}$$

Hence

$$(5.13) \qquad d_s = \boldsymbol{\sigma}_s'A_s'\mathbf{c}_s = -d_s(\boldsymbol{\sigma}_s'A_s'\omega_3), \quad \text{or} \quad d_s(1 + \boldsymbol{\sigma}_s'A_s'\omega_3) = 0.$$

However, from (5.9) and (5.12) one gets

$$1 + \mathbf{\delta}_s' A_s' \mathbf{\omega}_3$$
$$= (1 + \mathbf{\delta}_s' A_s^{-1} J_{m1}) - (J_{1m}\mathbf{\omega}_1)(1 + J_{1m}\mathbf{\omega}_2)^{-1}[(\mathbf{\delta}_s'\mathbf{\delta}_s) + (\mathbf{\delta}_s'\mathbf{\delta}_s)(\mathbf{\delta}_s' A_s^{-1} J_{m1})]$$
$$= (1 + \mathbf{\delta}_s' A_s^{-1} J_{m1})^2/(1 + J_{1m}\mathbf{\omega}_2),$$

which is greater than zero. Hence from (5.13), $d_s = 0$, and from (5.12), $\mathbf{c}_s = 0$. Thus Δ_s is nonsingular. This completes the proof.

Due to lack of space here, a general study of the situations under which Δ could be made nonsingular is deferred to later communications.

6. Discussion of results. Let us now consider the nature of the preceding development, the problems emerging from it and possible generalisations.

Consider the variance-covariance matrix Σ^* for the new variables and compare it with Σ. Define

$$(6.1) \qquad \rho^*_{ll'} = \sigma^*_{ll'}/[\sigma^*_{ll}\sigma^*_{l'l'}]^{\frac{1}{2}}, \qquad \lambda_{ll'} = \gamma_{ll'}\,[\gamma_{ll}\gamma_{l'l'}]^{\frac{1}{2}}, \qquad \rho_{ll'} = \sigma_{ll'}/[\sigma_{ll}\sigma_{l'l'}]^{\frac{1}{2}}$$

Then $\rho^*_{ll'} = \lambda_{ll'}\rho_{ll'}$ and

$$(6.2) \qquad |\rho^*_{ll'}| \leqq \lambda_{ll'} \leqq 1,$$

which shows that unlike Σ, the matrix Σ^* is not a general covariance matrix, but has the known restrictions (6.2). How to modify the MANOVA test when Σ is singular (the case of dependent variables), or has restrictions of the type (6.2) (a shift towards independence) is not known. However, the author feels that the Hotelling-Lawley trace criterion ([4], [6]) should be less sensitive to restrictions of the type (6.2), than the likelihood ratio or the largest root.

For a regular multiresponse design, we find that one can transform the $t(\sum_1^u p_i)$ random variables Q_{ijl} to $mp(t - 1)$ new variables X, where from (3.8), $mp \leqq \sum_1^u p_i$. The number of new variables is thus proportional to m, which denotes the number of the matrices $\{F_i\}$. It might be fruitful to stress here that we have not assumed that the matrices $F_1, F_2, \cdots, F_m$ are necessarily the association matrices of any PBIB design. This holds out the possibility of discovering new designs D_{2i} such that the D_{2i} themselves may be complicated, but the overall design D is simple to analyse, and is, in some sense, balanced. These matrices determine how the treatments stand relative to each other in the overall design D.

The regular designs considered herein could be of two broad types. The first is when the D_{2i} are designs (used at separate locations for example) which themselves have some importance, the problem being to choose the response design D_1 in such a way that (i) D_1 is convenient or useful to the experimenter, and (ii) the overall design D is regular, so that the whole data could be pooled, and a simple analysis could be carried out. The other case arises when we essentially wish to lay out a single experiment but do not desire to measure each response on each experimental unit. While many examples of the first type can easily be constructed by trial and error, the structure of the regular designs of the second type may be more complicated.

The approach in this paper can perhaps be further generalized to weaken the various requirements for 'regularity' of a design. Thus if the singularity of (LL') is not 'removable' even by using (3.16) one could consider taking a basis of LL', and then attempting to transform the data. The effect of this will in general be to decrease the number of final variables to $m'p(t-1)$ where $m' < m$. These and other generalisations will be considered in later papers.

7. Acknowledgment. I am thankful to Mr. Lyman McDonald for help in the numerical calculations, and to the referee for pointing out some misprints in the earlier version, and to the need of the inclusion of an example.

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