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THE TAIL FIELD OF A MARKOV CHAIN[1]

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1. Introduction. In [1] Blackwell characterized the invariant field of a Markov chain in terms of subsets of the state space called *almost closed sets*. We generalize Blackwell's results, and obtain a similar characterization of the tail field of the chain. Our discussion is modeled upon Chung's exposition ([3], Part I, Sec. 17) of Blackwell's results, and many of our techniques are simple extensions of those to be found in Chung's book.

Let I denote a subset of the integers, which will be the state space of the Markov chain we are going to construct. Let I^∞ denote the space of all sequences $\mathbf{j} = (j_0, j_1, \cdots)$ of elements of I. Let $x_n : I^\infty \to I$ denote the nth coordinate function, $x_n(\mathbf{j}) = j_n$ $(n = 0, 1, \cdots)$. Let $\mathfrak{F}$ denote the smallest Borel field of subsets of I^∞ with respect to which all the functions $x_0, x_1, \cdots$ are measurable.

The shift function $T : I^\infty \to I^\infty$ is defined by setting

$$T(j_0, j_1, \cdots) = (j_1, j_2, \cdots).$$

A set $\Lambda \varepsilon \mathfrak{F}$ is said to be *invariant* if $T^{-1}\Lambda = \Lambda$. The class of invariant sets, denoted by $\mathcal{G}$, is a σ-field, called the *invariant field*.

If $Y_1, Y_2, \cdots$ is a sequence of functions defined on I^∞, let $\mathcal{B}(Y_1, Y_2, \cdots)$ denote the smallest Borel field with respect to which these functions are measurable. For $n \geqq 0$, let $\mathfrak{F}_n = \mathcal{B}(x_n, x_{n+1}, \cdots)$. We note that $\mathfrak{F} = \mathfrak{F}_0$. Let $\mathfrak{F}_\infty = \bigcap_{n \geqq 0} \mathfrak{F}_n$. $\mathfrak{F}_\infty$ is called the *tail field*. When Λ is a subset of I^∞, by the expression $T\Lambda$ we mean $T\Lambda = \{T\mathbf{j} \mid \mathbf{j} \varepsilon \Lambda\}$.

THEOREM 1. *T maps $\mathfrak{F}_\infty$ one-to-one onto itself, and* $\mathcal{G} = \{\Lambda \varepsilon \mathfrak{F}_\infty \mid T\Lambda = \Lambda\}$.

This theorem states that if we regard $\mathfrak{F}_\infty$ as a set of "points," then T acts as a permutation on $\mathfrak{F}_\infty$, and $\mathcal{G}$ is the set of fixed points. Blackwell has shown that modulo equivalence relations, there is an isomorphism between $\mathcal{G}$ and the class of almost closed sets. We will show that the class of almost closed sets can be embedded in a class of objects which is isomorphic in the same way to $\mathfrak{F}_\infty$. Within this class, the almost closed sets correspond to objects which are invariant under the action of a shift function. Furthermore, the isomorphism commutes with this shift.

PROOF OF THEOREM 1. T^{-1} is a countably additive map from $\mathfrak{F}$ into $\mathfrak{F}$, so it is easy to show that $T^{-1}\mathfrak{F}_m = \mathfrak{F}_{m+1}$. For any set $\Lambda \subseteq I^\infty$, $T(T^{-1}\Lambda) = \Lambda$, and so it follows that $T\mathfrak{F}_{m+1} = \mathfrak{F}_m$ $(m \geqq 0)$ and these observations imply that $T\mathfrak{F}_\infty = \mathfrak{F}_\infty$. To show that T is one-to-one, suppose for $\Lambda_1, \Lambda_2 \varepsilon \mathfrak{F}_\infty$, $T\Lambda_1 = T\Lambda_2$. This means

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that there is a $\mathbf{j}_1 \varepsilon \Lambda_1$, $\mathbf{j}_2 \varepsilon \Lambda_2$ such that $T\mathbf{j}_1 = T\mathbf{j}_2$. Because Λ_1 and Λ_2 are in $\mathfrak{F}_\infty$, we can assume without loss of generality that $x_0(\mathbf{j}_1) = x_0(\mathbf{j}_2)$. But since $T\mathbf{j}_1 = T\mathbf{j}_2$, it follows that $\mathbf{j}_1 = \mathbf{j}_2$, hence $\Lambda_1 = \Lambda_2$. Finally, if $\Lambda \varepsilon \mathcal{G}$, then $\Lambda \varepsilon \mathfrak{F}_0$, so for any integer $n \geqq 0$, $\Lambda = T^{-n}\Lambda \varepsilon \mathfrak{F}_n$, hence $\Lambda \varepsilon \mathfrak{F}_\infty$. Furthermore, $\Lambda = T(T^{-1}\Lambda) = T\Lambda$. Hence, $\mathcal{G} \subseteq \{\Lambda \varepsilon \mathfrak{F}_\infty \mid T\Lambda = \Lambda\}$. The reverse containment follows from the fact that T is one-to-one over $\mathfrak{F}_\infty$, and this completes the proof.

2. The structure of the tail field. Let $p(i, j)$ be a stochastic matrix over $I \times I$. For each $b \varepsilon I$, there is a unique probability measure P_b over $\mathfrak{F}$ such that $P_b(x_0 = b) = 1$, and for $j, i, i_{n-1}, \cdots \varepsilon I$,

$$P_b(x_{n+1} = j \mid x_n = i, x_{n-1} = i_{n-1}, \cdots) = p(i, j).$$

With respect to the probability space $(I^\infty, \mathfrak{F}, P_b)$, the sequence $x_0, x_1, \cdots$ is a temporally homogeneous Markov chain, with transition function p. For some fixed element of I, say 0, we assume that for each $j \varepsilon I$, there is an $n \geqq 0$ such that $p^n(0, j) > 0$. We let $P = P_0$.

We wish to consider briefly the space-time chain corresponding to $x_0, x_1, \cdots$. Informally, this is the sequence $(0, x_0)$, $(1, x_1)$, $\cdots$ in the state space

$$J = \{(n, i) \mid n = 0, 1, 2, \cdots; i \varepsilon I\}.$$

However, we wish to consider the invariant field for the space-time chain, and to do this a more careful definition is necessary. Our purpose in considering the space-time chain is to motivate subsequent definitions, and the main results of this paper will not be based on what follows.

Let J^∞ denote the countable cross product space for J, whose elements are of the form $[(n_0, j_0), (n_1, j_1), \cdots]$, where each n_k is a non-negative integer, and $j_k \varepsilon I$. Let $Y_n : J^\infty \rightarrow J$ denote the usual coordinate function ($n \geqq 0$), and let $\mathfrak{F}(Y) = \mathcal{B}(Y_0, Y_1, \cdots)$. We define a measure Q on $\mathfrak{F}(Y)$, such that $Y_0, Y_1, \cdots$ is a Markov chain over $(J^\infty, \mathfrak{F}(Y), Q)$, and $Q(Y_0 = (0, 0)) = 1$,

$$Q(Y_{n+1} = (n + 1, j) \mid Y_n = (n, i)) = p(i, j).$$

The original chain $x_0, x_1, \cdots$ can be "recovered" by defining a projection $\pi: J \rightarrow I$, such that $\pi(n, i) = i$, and setting $z_n = \pi(Y_n)$ ($n \geqq 0$). Let $\mathfrak{F}(z) = \mathcal{B}(z_0, z_1, \cdots)$. Then clearly the chain $z_0, z_1, \cdots$ over the probability space $(J^\infty, \mathfrak{F}(z), Q)$ is equivalent to the chain $x_0, x_1, \cdots$ over $(I^\infty, \mathfrak{F}, P)$.

Let

$$\mathfrak{F}_\infty(Y) = \bigcap\nolimits_{n \geqq 0} \mathcal{B}(Y_n, Y_{n+1}, \cdots),$$

$$\mathfrak{F}_\infty(z) = \bigcap\nolimits_{n \geqq 0} \mathcal{B}(z_n, z_{n+1}, \cdots),$$

and let $\mathcal{G}(Y)$ denote the invariant field for the space-time chain. In the following, we write a.s. (almost surely) to mean with respect to the measure Q.

THEOREM 2. *The fields* $\mathfrak{F}_\infty(Y)$, $\mathfrak{F}_\infty(z)$ *and* $\mathfrak{F}(Y)$ *are* a.s. *equivalent.*

PROOF. Theorem 1 implies that $\mathcal{G}(Y) \subseteq \mathfrak{F}_\infty(Y)$.

For each integer $k \geqq 0$, let $\pi_k: I \rightarrow J$ by $\pi_k(i) = (k, i)$. Then $Y_k = \pi_k(z_k)$ a.s. and so $\mathcal{B}(Y_k, Y_{k+1}, \cdots) \subseteq \mathcal{B}(z_k, z_{k+1}, \cdots)$ a.s., hence $\mathcal{F}_\infty(Y) \subseteq \mathcal{F}_\infty(z)$ a.s.

Now select $\Lambda \,\varepsilon\, \mathcal{F}_\infty(z)$. Let

$$f(n, i) = Q(\Lambda \mid z_n = i)$$

defined whenever $Q(z_n = i) > 0$. Now $Y_n = (n, z_n)$ a.s., hence

$$\begin{aligned} f(Y_n) &= Q(\Lambda \mid z_n) \quad \text{a.s.} \\ &= Q(\Lambda \mid z_n, z_{n-1}, \cdots) \quad \text{a.s.} \end{aligned}$$

and so by a theorem in martingale theory ([4], p. 332), $\lim f(Y_n)$ exists a.s., and is equal to

$$\lim_{n\rightarrow\infty} f(Y_n) = \chi_\Lambda \quad \text{a.s.}$$

where χ_Λ is the indicator function of Λ. Clearly, then, Λ is a.s. an invariant set, hence $\mathcal{F}_\infty(z) \subseteq \mathcal{G}(Y)$ a.s., and the proof is complete.

From this point on, we write a.s. to mean with respect to the measure P.

Blackwell has shown that every invariant set $\Lambda \,\varepsilon\, \mathcal{G}$ is a.s. equivalent to a set of the form $\{x_n \,\varepsilon\, A \text{ i.o.}\}$ (i.o. means "infinitely often") where A is an almost closed subset of I, that is, a subset such that

$$\{x_n \,\varepsilon\, A \text{ i.o.}\} = \{x_n \,\varepsilon\, A \text{ all large } n\} \quad \text{a.s.}$$

If we apply this characterization to the invariant field $\mathcal{G}(Y)$ of the space-time chain, and use the equivalence between $\mathcal{G}(Y)$ and $\mathcal{F}_\infty$ implied by Theorem 2, we find that every set in the tail field $\mathcal{F}_\infty$ is P-a.s. equivalent to a set of the form

$$\{x_n \,\varepsilon\, A_n \text{ i.o.}\}$$

where $A_0, A_1, \cdots$ is a sequence of subsets of I, such that

$$(1) \qquad \{x_n \,\varepsilon\, A_n \text{ i.o.}\} = \{x_n \,\varepsilon\, A_n \text{ all large n}\} \quad \text{a.s.}$$

Thus, we obtain a characterization of $\mathcal{F}_\infty$ in terms of sequences of subsets of I. We now consider this characterization with respect to the shift T, and in light of Theorem 1. Certain desirable properties are suggested.

1) Suppose $\Lambda \,\varepsilon\, \mathcal{F}_\infty$, and $\{A_n; n \geqq 0\}$ is a sequence of subsets of I satisfying (1), such that

$$\Lambda = \{x_n \,\varepsilon\, A_n \text{ i.o.}\} \quad \text{a.s.}$$

Now $T\{x_n \,\varepsilon\, A_n \text{ i.o.}\} = \{x_n \,\varepsilon\, A_{n+1} \text{ i.o.}\}$ and so it would be natural to expect that

$$(2) \qquad T\Lambda = \{x_n \,\varepsilon\, A_{n+1} \text{ i.o.}\} \quad \text{a.s.}$$

2) Let $\{B_n; n \geqq 0\}$ be a sequence satisfying (1), such that indeed

$$T\Lambda = \{x_n \,\varepsilon\, B_n \text{ i.o.}\} \quad \text{a.s.}$$

Then comparing this identity with (2), it would be desirable to have $A_{n+1} = B_n$ for all n, or some similar correspondence.

However, neither of the above properties hold. The basic problem is that it is not sufficient that $\Lambda_1 = \Lambda_2$ a.s. in order to have $T\Lambda_1 = T\Lambda_2$ a.s. Therefore, a sequence $\{A_n; n \geqq 0\}$ may satisfy (1), but the "shifted" sequence $\{A_{n+1}; n \geqq 0\}$ may not.

A characterization similar to the one above which achieves these desirable properties can be obtained by strengthening (1). It appears to be more straightforward to begin again from scratch rather than to build upon the preceeding discussion regarding space-time chains. This is the object of this section. Some of the proofs will be seen to be very similar to those in [3].

An event $\Lambda \varepsilon \mathfrak{F}$ is said to be a *null set* if $P(\Lambda) = 0$. We will call a set $\Lambda \varepsilon \mathfrak{F}_\infty$ a *small set* if $P(T^k\Lambda) = 0$ for every integer k (positive or non-positive). Theorem 1 shows that $T^k\Lambda$ is measurable for all such integers. Any small set is a null set, and any null set which is also in $\mathcal{G}$ is small. We call two sets Λ_1 and Λ_2 in $\mathfrak{F}_\infty$ *equivalent* if the symmetric difference $\Lambda_1 \triangle \Lambda_2 = (\Lambda_1 - \Lambda_2) \cup (\Lambda_2 - \Lambda_1)$ is a small set. When this equivalence relation is restricted to $\mathcal{G}$, it coincides with the one Chung places on $\mathcal{G}$ in his discussion of Blackwell's results. Following Chung's notation, we will write $\Lambda_1 \doteq \Lambda_2$ to mean that Λ_1 and Λ_2 are equivalent. When equivalence classes of sets are considered, we will describe the situation by the phrase "modulo small sets."

Let $A = \{A_n; n \geqq 0\}$ be a sequence of subsets of I. We define subsets $\mathfrak{L}^*(A)$ and $\mathfrak{L}_*(A)$ of I^∞ by the following relations:

$$\mathfrak{L}^*(A) = \operatorname{limsup}_{n\to\infty} \{\mathbf{j} \mid x_n(\mathbf{j}) \,\varepsilon\, A_n\},$$

$$\mathfrak{L}_*(A) = \operatorname{liminf}_{n\to\infty} \{\mathbf{j} \mid x_n(\mathbf{j}) \,\varepsilon\, A_n\}.$$

Equivalent expressions are the following:

$$\mathfrak{L}^*(A) = \{\mathbf{j} \mid x_n(\mathbf{j}) \,\varepsilon\, A_n \text{ i.o.}\},$$

$$\mathfrak{L}_*(A) = \{\mathbf{j} \mid x_n(\mathbf{j}) \,\varepsilon\, A_n \text{ all large } n\}.$$

Clearly for any sequence A, $\mathfrak{L}_*(A)$ and $\mathfrak{L}^*(A)$ are elements of $\mathfrak{F}_\infty$, and $\mathfrak{L}_*(A) \subseteq \mathfrak{L}^*(A)$. A sequence A is a *transient sequence* if $\mathfrak{L}^*(A)$ is a small set. It follows that A is a transient sequence iff

$$P(x_n \,\varepsilon\, A_{n+k} \text{ i.o. for some } k) = 0.$$

If A is a transient sequence, then $\mathfrak{L}_*(A) \doteq \mathfrak{L}^*(A)$. If this relation holds for a sequence A which is not transient, we will call A a *tail sequence*. Equivalently, A is a tail sequence if for every integer k such that $P(x_n \,\varepsilon\, A_{n+k} \text{ i.o.}) > 0$, we have the relation

$$P(x_n \,\varepsilon\, A_{n+k} \text{ all large } n \mid x_n \,\varepsilon\, A_{n+k} \text{ i.o.}) = 1.$$

Let $\mathfrak{I}$ denote the class of sequences which are either tail sequences or transient sequences. If $A \,\varepsilon\, \mathfrak{I}$, let $\mathfrak{L}(A) = \mathfrak{L}^*(A)$.

When $A = \{A_n; n \geqq 0\}, B = \{B_n; n \geqq 0\}$ are sequences of subsets of I, natural meanings are assigned to such set-theoretic symbols as A^c, $A \cup B$, $A \subseteq B$ etc. For example

$$A^c = \{A_n^c; n \geqq 0\}, \qquad A \cup B = \{A_n \cup B_n; n \geqq 0\},$$
$$A \subseteq B \text{ if } A_n \subseteq B_n \text{ for all } n.$$

We also define a shift function T operating on such sequences, by setting $TA = \{A_{n+1}; n \geqq 0\}$. We will call A^c the *complement of* A, $A \cup B$ the *union* of A and B, and TA the *shift* of A.

THEOREM 3. (a) $\mathfrak{J}$ *is closed under the action of* T, *and under the operations of complementation and finite unions (hence under all finite set-theoretic operations).*

(b) *For* $A, B \in \mathfrak{J}$, $\mathfrak{L}(A)^c \doteq \mathfrak{L}(A^c)$ *and* $\mathfrak{L}(A \cup B) \doteq \mathfrak{L}(A) \cup \mathfrak{L}(B)$.

(c) $\mathfrak{L}(A) \doteq \mathfrak{L}(B)$ iff $A \triangle B$ *is a transient sequence, for* $A, B \in \mathfrak{J}$.

(d) $\mathfrak{L}(TA) = T\mathfrak{L}(A)$ if $A \in \mathfrak{J}$.

PROOF. The assertions in (a) and (b) regarding the union are implied by the following observation regarding $A, B \in \mathfrak{J}$:

$$\begin{aligned} \mathfrak{L}^*(A \cup B) &\subseteq \mathfrak{L}^*(A) \cup \mathfrak{L}^*(B) \\ &\doteq \mathfrak{L}_*(A) \cup \mathfrak{L}_*(B) \\ &\subseteq \mathfrak{L}_*(A \cup B). \end{aligned}$$

A similar relation implies the assertions regarding complementation. Now if $\mathfrak{L}(A) \doteq \mathfrak{L}(B)$, by definition the set $\mathfrak{L}(A) \triangle \mathfrak{L}(B)$ is small, so for $A, B \in \mathfrak{J}$, $\mathfrak{L}(A \triangle B)$ is small, i.e., $A \triangle B$ is transient. This proves (c). The assertions in (a) and (d) regarding T follow directly from the definitions, and the proof of these assertions is left to the reader. This completes our proof.

If A and B are elements of $\mathfrak{J}$, we will say they are *equivalent* if $A \triangle B$ is transient. When it is useful, we will write in this case $A \doteq B$, and use the phrase "modulo transient sequences" to mean the obvious thing. Theorem 2, part (c) shows that $\mathfrak{L}$ maps $\mathfrak{J}$ one-to-one into $\mathfrak{F}_\infty$, modulo transient sequences in $\mathfrak{J}$ and small sets in $\mathfrak{F}_\infty$. We now show that this map is onto

THEOREM 4. (a) *Let* $\Lambda \in \mathfrak{F}_\infty$, *and let* $f_n(i) = P_i(T^n\Lambda)$, *where* n *is any integer, and* $i \in I$. *For any integer* k, *the limit of the sequence* $f_k(x_0), f_{k+1}(x_1), \cdots$ *exists* a.s., *and is given by*

$$\lim_{n\to\infty} f_{n+k}(x_n) = \chi_{T^k\Lambda} \quad \text{a.s.}$$

(b) *For each* $n \geqq 0$, *let* $A_n = \{i \mid f_n(i) > \frac{1}{2}\}$ *and let* $A = \{A_n ; n \geqq 0\}$. *Then* $A \in \mathfrak{J}$ *and* $\mathfrak{L}(A) \doteq \Lambda$.

PROOF. The chain is temporally homogeneous, so for any event $\Lambda' \in \mathfrak{F}$, if $P(x_n = i) > 0$, then

$$P_i(\Lambda') = P(T^{-n}\Lambda' \mid x_n = i).$$

Suppose $\Lambda \in \mathfrak{F}_\infty$. From Theorem 1, $T^k\Lambda \in \mathfrak{F}_\infty$. Setting $\Lambda' = T^{n+k}\Lambda$, it follows from the definition of f that

$$f_{n+k}(i) = P(T^k\Lambda \mid x_n = i)$$

hence

$$f_{n+k}(x_n) = P(T^k\Lambda \mid x_n) \quad \text{a.s.}$$

The Markov property implies that

$$f_{n+k}(x_n) = P(T^k\Lambda \mid x_n, x_{n-1}, \cdots)$$

and so assertion (a) follows from a theorem in martingale theory in [4], page 332.

Let $A = \{A_n ; n \geqq 0\}$ be constructed as in (b). For any integer k

$$\begin{aligned} T^k\mathfrak{L}^*(A) &= \{x_n \in A_{n+k} \quad \text{i.o.}\} \\ &= \{f_{n+k}(x_n) > \tfrac{1}{2} \quad \text{i.o.}\}. \end{aligned}$$

It follows from (a) that

$$T^k\mathfrak{L}^*(A) = \{\chi_{T^k\Lambda} > \tfrac{1}{2}\} = T^k\Lambda \quad \text{a.s.}$$

and hence,

$$\begin{aligned} T^k\mathfrak{L}^*(A) &= \{f_{n+k}(x_n) > \tfrac{1}{2} \quad \text{all large} \quad n\} \quad \text{a.s.} \\ &= T^k\mathfrak{L}_*(A) \quad \text{a.s.} \end{aligned}$$

Since k is arbitrary, $\mathfrak{L}_*(A) \doteq \mathfrak{L}^*(A)$, which proves the theorem.

We summarize the results of this section in the following theorem, which is analogous to Chung's summary of Blackwell's results (Theorem 1 [3], Part I, Section 17). The proof follows directly from Theorems 3 and 4.

THEOREM 5. *For each $\Lambda \in \mathfrak{F}_\infty$ there is an $A \in \mathfrak{I}$, unique modulo transient sequences, such that $\mathfrak{L}(A) \doteq \Lambda$. This correspondence is an isomorphism with respect to complementation and finite unions and intersections, and commutes with T (modulo small sets in $\mathfrak{F}_\infty$, transient sequences in $\mathfrak{I}$).*

3. The structure of the invariant field. We define a subclass $\mathfrak{I}^*$ of $\mathfrak{I}$ by

$$\mathfrak{I}^* = \{A \in \mathfrak{I} \mid TA = A\}.$$

Clearly, the definitions imply that $A = \{A_n ; n \geqq 0\}$ is in $\mathfrak{I}^*$ iff there is an almost closed or a transient set D such that $A_k = D (k \geqq 0)$. Theorem 1, and the results of the last section, show, once we have cleared away a technicality discussed below, that $\mathfrak{I}^*$ and $\mathfrak{G}$ are isomorphic, hence, we obtain Blackwell's result.

The technicality concerns the nature of the equivalence relation which we have defined on $\mathfrak{F}_\infty$, which is stronger than the one defined by Blackwell over $\mathfrak{G}$. Theorem 6 implies that the class of sets $\Lambda \in \mathfrak{F}_\infty$ such that $T\Lambda \doteq \Lambda$, and the class of sequences $A \in \mathfrak{I}$, such that $TA \doteq A$ are isomorphic. The following result, however, shows that this is, in effect, enough to obtain the assertion in the first paragraph.

THEOREM 6. (a) *For each* $\Lambda \varepsilon \mathfrak{F}_\infty$, *such that* $T\Lambda \doteq \Lambda$ *there is a set* $\Lambda_0 \varepsilon \mathcal{G}$ *such that* $\Lambda_0 \doteq \Lambda$.

(b) *For each* $B \varepsilon \mathfrak{J}$ *such that* $TB \doteq B$, *there is an almost closed or transient set* D *such that* $B \doteq \{D, D, D, \cdots\}$.

PROOF. Suppose $\Lambda \doteq T\Lambda$. Let

$$\Lambda_0 = \bigcup_{k=-\infty}^{\infty} T^k\Lambda.$$

Clearly $\Lambda_0 \varepsilon \mathcal{G}$. Now $T^k\Lambda \triangle \Lambda$ is small for each k, hence

$$\Lambda \triangle \Lambda_0 = \bigcup_{k=-\infty}^{\infty} (T^k\Lambda \triangle \Lambda)$$

is small, i.e., $\Lambda_0 \doteq \Lambda$.

Now assume $B \varepsilon \mathfrak{J}$, and $TB \doteq B$. Let $\Lambda = \mathfrak{L}(B)$.

Let $A = \{A_n ; n \geqq 0\}$ be the element of $\mathfrak{J}$ constructed from Λ as in Theorem 4. Then $\mathfrak{L}(A) \doteq \Lambda \doteq \mathfrak{L}(B)$, so $A \doteq B$. Since $T^k\Lambda \doteq \Lambda$ for each k, it follows from the construction that $A_k = A_0$ for each $k \geqq 0$, and since $A \varepsilon \mathfrak{J}$, it follows that A_0 is either almost closed or transient. This completes the proof.

4. Atomic tail sequences. An almost closed set is *atomic* if it does not contain two disjoint almost closed sets. A corollary to Blackwell's theory is that corresponding to each atom of the invariant field is a unique (modulo transient sets) atomic set. We now consider a similar correspondence for the tail field.

A set $\Lambda \varepsilon \mathfrak{F}_\infty$ which is not small is called an *atom* if for any element $\Lambda_0 \varepsilon \mathfrak{F}_\infty$ such that $\Lambda_0 \subseteq \Lambda$, Λ_0 is small or $\Lambda - \Lambda_0$ is small. The tail sequences which correspond to atoms will be called *atomic tail sequences*; such a sequence cannot be expressed as the union of two disjoint tail sequences. If A is an atomic tail sequence, then for any integer $k > 0$, either $T^kA \cap A$ is transient, or $T^kA \doteq A$. We set $N = \infty$ if the latter never happens, otherwise, we set

$$N = \inf\{k > 0 \mid T^kA \doteq A\}.$$

N is called the *asymptotic period* of the tail sequence A. We now show that atomic tail sequences are cyclic, in the sense of the following theorem.

THEOREM 7. *Let* A *be an atomic tail sequence with asymptotic period* N.

(a) *If* $N = \infty$, *then there exists a sequence* $B = \{B_n ; n \geqq 0\}$ *of disjoint sets, such that* $A \doteq B$.

(b) *If* $N < \infty$, *then there exists a sequence* $B_0, B_1, \cdots, B_{N-1}$ *of disjoint sets, such that* $A \doteq \{B_0, B_1, \cdots, B_{N-1}, B_0, \cdots\}$.

(c) *In either case,* $\cup B_k$ *is an atomic almost closed set.*

PROOF. We will prove (a), and leave (b) to the reader. Let A be an atomic tail sequence with infinite asymptotic period, and let $\Lambda_0 = \mathfrak{L}(A)$. When $i \neq j$, $T^i\Lambda_0 \cap T^j\Lambda_0$ is a small set. For any point $b \varepsilon I$, we have assumed that there is an $n \geqq 0$ such that $P(x_n = b) > 0$. Now, we have the relation

$$P(\Lambda \mid x_n = b) = P_b(T^n\Lambda)$$

for any set $\Lambda \varepsilon \mathfrak{F}_\infty$. Let Λ be the small set $T^{i-n}\Lambda_0 \cap T^{j-n}\Lambda_0$, and we obtain

$$P_b(T^i\Lambda_0 \cap T^j\Lambda_0) = 0. \tag{3}$$

Now set $f_n(i) = P_i(T^n\Lambda_0)$, and let $B_n = \{i \mid f_n(i) > \frac{1}{2}\}$. By Theorems 3 and 4, $B = \{B_n ; n \geqq 0\}$ is in $\mathfrak{J}$ and is equivalent to A. By (3), $B_0, B_1, \cdots$ are mutually disjoint. This establishes part (a).

For part (c), if A is an atomic tail sequence, $\Lambda_0 = \mathfrak{L}(A)$ is an atom of $\mathfrak{F}_\infty$. Let

$$\Lambda = \bigcup_{k=-\infty}^{\infty} T^k \Lambda_0 .$$

Λ is in $\mathcal{G}$. Suppose Λ_1 is in $\mathcal{G}$, and $\Lambda_1 \subseteq \Lambda$. Then $\Lambda_0 \cap \Lambda_1 \subseteq \Lambda_0$, hence $\Lambda_0 \cap \Lambda_1$ is small, or is equivalent to Λ_0. In the first case, for every integer k, $T^k(\Lambda_0 \cap \Lambda_1) = (T^k\Lambda_0) \cap \Lambda_1$ is small, hence $\Lambda_1 = \Lambda \cap \Lambda_1 = \cup T^k \Lambda \cap \Lambda_1$ is small. In the second case, we have $T^k\Lambda_0 \cap \Lambda_1 \doteq T^k \Lambda_0$ for every k, hence $\cup T^k\Lambda_0 \cap \Lambda_1 = \Lambda$, i.e., $\Lambda \cap \Lambda_1 \doteq \Lambda$. Since $\Lambda_1 \subseteq \Lambda$, this implies that $\Lambda_1 \doteq \Lambda$. Thus we see that Λ is an atom of $\mathcal{G}$. It follows from Blackwell's theorem (see [3], page 114) that $D = \{i \mid P_i(\Lambda) > \frac{1}{2}\}$ is an atomic almost closed set. Now (3) implies

$$P_i(\Lambda) = P_i(\cup T^k \Lambda_0) = \sum P_i(T^k\Lambda_0)$$

and so

$$D = \{i \mid \sum f_k(i) > \tfrac{1}{2}\} \supseteq \cup B_k$$

where f and B_k are defined in the proof of the first part of the theorem. Furthermore

$$\begin{aligned} P(x_n \,\varepsilon\, D \quad \text{i.o.}) &= P(\Lambda) \\ &= P(\textstyle\bigcup_{k=-\infty}^{\infty} T^k\{x_n \,\varepsilon\, B_n \quad \text{all large} \quad n\}) \\ &= P(x_n \,\varepsilon\, B_{n+k} \quad \text{all large} \quad n, \quad \text{some} \quad k) \\ &\leqq P(x_n \,\varepsilon\, \textstyle\bigcup_{k=0}^{\infty} B_k \quad \text{all large} \quad n). \end{aligned}$$

Hence

$$\begin{aligned} P(x_n \,\varepsilon\, D - \cup B_k \text{ i.o.}) &\leqq P(x_n \,\varepsilon\, D \text{ i.o.}) - P(x_n \,\varepsilon \cup B_k \quad \text{all large} \quad n) \\ &\leqq 0 \end{aligned}$$

so $\cup B_k$ differs from the atomic set D by at most a transient set, hence $\cup B_k$ is itself an atomic set, and the proof is complete.

To simplify the interpretation of Theorem 7, we assume that $\mathcal{G}$ is a.s. trivial. Then I is an atomic set: every almost closed set is the complement of a transient set. An example in [2] shows that it is possible for $\mathcal{G}$ to be trivial, yet $\mathfrak{F}_\infty$ contain no atoms. But if $\mathfrak{F}_\infty$ contains one atom Λ, it contains no non-atomic sets, and every other atom is of the form $T^k\Lambda$ for some integer k. If $T^k\Lambda \neq \Lambda$ for any k, the asymptotic period of the chain is infinite. Otherwise, the asymptotic period is $N = \inf \{k > 0 \mid T^k\Lambda \doteq \Lambda\}$. In this case there will exist a cycle of subsets $A_1, A_2, \cdots, A_N$, through which, eventually, the chain x_n circulates.

This, of course, is trivially the situation for a periodic chain. We will give an example of an aperiodic chain, in which the asymptotic period is greater than one. We first obtain a sufficient condition for $\mathfrak{F}_\infty$ to be atomic.

LEMMA. *Suppose* $\mathcal{G}$ *is trivial and there is a* $k > 0$, *an* $\epsilon > 0$ *such that* $p^k(i, i) \geqq \epsilon$ *or all* $i \varepsilon I$. *Then* $\mathfrak{F}_\infty$ *is atomic, and has at most k atoms.*

PROOF. We show that the hypotheses imply that $T^k\Lambda \doteq \Lambda$ for any set $\Lambda \varepsilon \mathfrak{F}_\infty$. Let $\Lambda \varepsilon \mathfrak{F}_\infty$. For any $i \varepsilon I$,

$$P(\Lambda \mid x_n = i) = \sum_{j\varepsilon I} P(\Lambda \mid x_{n+k} = j, x_n = i)P(x_{n+k} = j \mid x_n = i)$$
$$\geqq \epsilon P(\Lambda \mid x_{n+k} = i, x_n = i).$$

By the Markov property, this implies

$$P(\Lambda \mid x_n = i) \geqq \epsilon P(\Lambda \mid x_{n+k} = i) = \epsilon P(T^k\Lambda \mid x_n = i).$$

Hence,

$$P(\Lambda \mid x_n) \geqq \epsilon P(T^k\Lambda \mid x_n) \quad \text{a.s.}$$

Letting $n \to \infty$, it follows that

$$\chi_\Lambda \geqq \epsilon\chi_{T^k\Lambda} \quad \text{a.s.}$$

hence $\Lambda \supseteq T^k\Lambda$ a.s. Applying this result to the complement of Λ, we obtain $\Lambda^c \supseteq T^k\Lambda^c = (T^k\Lambda)^c$, hence $\Lambda = T^k\Lambda$ a.s. It follows from this that for any integer n, $T^n\Lambda = T^k(T^n\Lambda) = T^n(T^k\Lambda)$ a.s., hence $\Lambda \doteq T^k\Lambda$.

If $\mathfrak{F}_\infty$ is not atomic, then there is an event $\Lambda \varepsilon \mathfrak{F}_\infty$, such that $P(T^j\Lambda) < 1/k$, $j = 0, 1, \cdots, k - 1$, and $0 < P(\Lambda)$. We let

$$\Lambda_0 = \Lambda \cup T\Lambda \cup \cdots \cup T^{k-1}\Lambda$$

then the above result implies that $\Lambda_0 \varepsilon \mathcal{G}$, a.s., hence $P(\Lambda_0)$ is zero or one. But we have

$$0 < P(\Lambda) \leqq P(\Lambda_0) \leqq \sum_{j=0}^{k-1} P(T^j\Lambda) < 1$$

which is a contradiction. Therefore $\mathfrak{F}_\infty$ is atomic.

Let Λ be an atom of $\mathfrak{F}_\infty$. Then $\Lambda_0 = \Lambda \cup T\Lambda \cup \cdots \cup T^{k-1}\Lambda$ is in $\mathcal{G}$, and $P(\Lambda_0) = 1 = \sum_{j=0}^{k-1} P(T^j\Lambda)$, therefore every other atom of $\mathfrak{F}_\infty$ is of the form $T^j\Lambda$, $j = 0, \cdots, k - 1$.

Our example is the following Markov chain whose state space I is the integers. Let α_n $(n \geqq 0)$ be a sequence of probabilities, such that $\sum_{n=0}^{\infty} \alpha_n < \infty$, $0 < \alpha_j < \frac{1}{2}$. We set $\alpha_n = 0$ for $n < 0$. The stochastic matrix p is defined by

$$\begin{aligned} p(n, m) &= (1 - \alpha_n)p, & m &= n + 1, \\ &= \alpha_n, & m &= n, \\ &= (1 - \alpha_n)q, & m &= n - 1, \\ &= 0, & &\text{otherwise,} \end{aligned}$$

where $p > q$, $p = 1 - q$. The chain is aperiodic, and it is not difficult to show it is transient, with $x_n \to \infty$ a.s., and that the number of "pauses," i.e., occasions in which $x_n = x_{n+1}$, is a.s. finite.

The invariant field is trivial. For suppose $\Lambda \varepsilon \mathcal{G}$. Set $f(x) = P_x(\Lambda)$. Then f satisfies

$$f(n) = (1 - \alpha_n)pf(n + 1) + \alpha_n f(n) + (1 - \alpha_n)qf(n - 1).$$

The α's cancel, and

$$f(n) = pf(n + 1) + qf(n - 1).$$

The only bounded solutions to this equation are constants, from this it follows that $P(\Lambda) = 0$ or 1.

We also note that $p^2(i, i) \geqq \frac{1}{2}pq > 0$. Hence $\mathfrak{F}_\infty$ is atomic, with at most two atoms. But since the chain eventually stops pausing, $\mathfrak{F}_\infty$ contains the following two disjoint sets of positive probability:

$$\Lambda_1 = \{x_{2n} \quad \text{is even} \quad \text{i.o.}\}, \qquad \Lambda_2 = \{x_{2n} \quad \text{is odd} \quad \text{i.o.}\}.$$

Hence $\mathfrak{F}_\infty$ has these two atoms, and so this aperiodic chain has asymptotic period 2.

Let E denote the class of even integers, and 0 the class of odd integers. The two atomic tail sequences are $\{A_0, A_1, \cdots\} = \{E, 0, E, 0, \cdots\}$ and $\{B_0, B_1, \cdots\} = \{0, E, 0, E, \cdots\}$.

An example of an aperiodic chain with infinite asymptotic period can be provided in a similar way.

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