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THE SAMPLING DISTRIBUTION OF AN ESTIMATOR ARISING IN CONNECTION WITH THE TRUNCATED EXPONENTIAL DISTRIBUTION

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1. Introduction. Let $T_1, T_2, \cdots, T_N$ be independent exponentially distributed random variables with $P\{T_j \leqq t\} = 1 - e^{-\lambda t}$ for $t \geqq 0$, and let $T_{(1)}, \cdots, T_{(N)}$ be the corresponding order statistics. For ease of exposition we shall speak of the T_j as failure times of N parallel test items. We consider a situation where observation is truncated at time τ. $D(\tau) = \max\{n: T_{(n)} \leqq \tau\}$ is the number of failures observed at or prior to time τ.

$$M(\tau) = \sum_{j=1}^{D(\tau)} T_{(j)}$$

is the total time on test until time τ for items failing at or prior to time τ, and $L(\tau) = M(\tau) + \tau\{N - D(\tau)\}$ is the total time on test until time τ. Then $\lambda^*(\tau) = D(\tau)/L(\tau)$ is a common estimator for λ. The purpose of the present note is to establish the sampling distribution for $\lambda^*(\tau)$.

2. Previous results. Sverdrup (1961) gives large sample properties for $\lambda^*(\tau)$. Bartholomew (1963) studies (small sample) properties of $1/\lambda^*(\tau)$ as an estimator for $\theta = 1/\lambda$, conditional on $D(\tau) > 0$. An interesting application of Bartholomew's result is given by Barlow et al. (1968).

A result closely related to our Theorem 1 is mentioned by Epstein [(1960), pp. 85–86]. Our result (2) was obtained by different methods by Bain and Weeks (1964).

3. Preliminaries. (i) If $V_1, \cdots, V_m$ are independent and uniformly distributed over $[0, a]$, and if $Z = \sum_{j=1}^m V_j$, then Z has a probability density

$$\phi_m(z, a) = (a^m\Gamma(m))^{-1} \sum_{\nu=0}^m (-1)^\nu \binom{m}{\nu} (z - \nu a)_+^{m-1}$$

for all z. Here $x_+^0 = 1$ for $x > 0$, $x_+^0 = 0$ for $x \leqq 0$, and $x_+^n = \{\max(x, 0)\}^n$ for $n = 1, 2, \cdots$.

(ii) Let T be distributed as the T_j and let $U = T$ for $T < \tau$, $U = \tau$ for $T \geqq \tau$. Then $P\{U \leqq u \mid T \leqq t\} = b_t(1 - e^{-\lambda u})$ for $0 \leqq u \leqq t$, with $b_t = (1 - e^{-\lambda t})^{-1}$, and the corresponding (conditional) density is

$$f_t(u) = b_t\lambda e^{-\lambda u} \qquad \text{for} \quad 0 < u < t.$$

(iii) Let f_t^{n*} be the nth convolution of f_t with itself, and let F_t^{n*} be the corresponding distribution function. A simple induction then shows that

$$f_t^{n*}(u) = b_t^n(\lambda t)^n e^{-\lambda u}\phi_n(u, t). \tag{1}$$

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(iv) Finally, for $m = 0, 1, \cdots, N$,

$$P\{D(\tau) = m\} = \binom{N}{m}(1 - e^{-\lambda\tau})^m e^{-\lambda\tau(N-m)}.$$

4. The sampling distribution.

THEOREM 1. *Given that $D(\tau) = m$, $T_{(1)}, \cdots, T_{(m)}$ may be regarded as the order statistics of m independent variables each of which has the probability density f_τ.*

The proof is straightforward. An immediate result is

$$(2) \qquad P\{M(\tau) \leqq u \mid D(\tau) = m\} = F_\tau^{m*}(u) \qquad \text{for} \quad m = 1, 2, \cdots, N.$$

If $F_t^{0*}(u) = 1$ for $u \geqq 0$, $F_t^{0*}(u) = 0$ otherwise, (2) holds also for $m = 0$. Thus $P\{L(\tau) \leqq x \mid D(\tau) = m\} = F_\tau^{m*}\{x - (N - m)\tau\}$, and

$$(3) \qquad \begin{aligned} &P\{\lambda^*(\tau) \leqq v \mid D(\tau) = m\} \\ &= F_\tau^{0*}(v) && \text{when} \quad m = 0, \\ &= 1 - F_\tau^{m*}\{mv^{-1} - (N - m)\tau\} && \text{when} \quad m = 1, 2, \cdots, N \quad \text{and} \quad v > 0, \\ &\quad \text{and} \\ &= 0 && \text{otherwise.} \end{aligned}$$

The distribution of $\lambda^*(\tau)$ is then found to be

$$(4) \qquad \begin{aligned} P\{\lambda^*(\tau) \leqq v\} &= \textstyle\sum_{m=0}^{N} \{1 - F_\tau^{m*}(mv^{-1} - (N - m)\tau)\}\binom{N}{m} \\ &\quad \cdot (1 - e^{-\lambda\tau})^m e^{-\lambda\tau(N-m)} && \text{for} \quad v > 0, \\ &= e^{-\lambda\tau N} && \text{for} \quad v = 0, \quad \text{and} \\ &= 0 && \text{for} \quad v < 0. \end{aligned}$$

We may also easily find expressions for such quantities as $E(\lambda^*)^k$ by (3), but the formulae are so messy that they probably are of little practical value.

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