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NOTE ON SHIFT-INVARIANT SETS

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In this note we prove a theorem which implies that shift-invariant sets in a bilateral product space with infinite invariant measure are contained in the remote σ-algebra (also called tail σ-algebra), if the shift is conservative. After completing the paper we noticed that, in a paper as yet unpublished, K. Dugdale obtained the theorem in the case where the remote σ-algebra is trivial; it seems that his method, based on induced transformations, does not yield our result. W. Parry [6] asserts the same special case of the theorem under some assumptions on the measure space. We further show that in the dissipative case it may happen that the remote σ-algebra is trivial and some invariant sets are not; and that it may also happen that all invariant sets are trivial and the remote σ-algebra is not. Our examples involve transient random walks.

1. Let $(E_k, \mathfrak{F}_k)$, $k = 0, \pm 1, \cdots$, be countably many copies of a measurable space $(E_0, \mathfrak{F}_0)$ and let

$$(\Omega, \mathfrak{A}) = \prod_{k=-\infty}^{+\infty} (E_k, \mathfrak{F}_k).$$

Let X_k be the mapping assigning to the point $\omega = (\cdots, \omega_{-1}, \omega_0, \omega_1, \cdots) \varepsilon \Omega$ its kth coordinate $\omega_k \varepsilon E_k$. The *shift* T on Ω is the transformation defined by $X_k(T\omega) = X_{k+1}(\omega)$. The σ-algebra generated by $X_m, X_{m+1}, \cdots$ is denoted by $\mathfrak{A}_m$. The (right) *remote σ-algebra* $\mathfrak{A}_\infty$ is by definition $\bigcap_{m=0}^{\infty} \mathfrak{A}_m$. A transformation T on Ω is called *invertible* iff T is one-to-one, onto and $A \varepsilon \mathfrak{A}$ implies $T^{-1}A \varepsilon \mathfrak{A}$, $TA \varepsilon \mathfrak{A}$. Clearly, the shift is invertible. Let μ be a fixed measure on $\mathfrak{A}$; henceforth all relations are modulo sets of μ measure zero. An invertible transformation T is called *measure-preserving* iff $A \varepsilon \mathfrak{A}$ implies $\mu(T^{-1}A) = \mu(A) = \mu(TA)$. The σ-algebra of *invariant sets* is defined by: $A \varepsilon \mathfrak{I}$ iff $A \varepsilon \mathfrak{A}$ and $T^{-1}A = A = TA$. A set $A \varepsilon \mathfrak{A}$ is called *wandering* iff the sets $\cdots, T^{-1}A, A, TA, \cdots$ are mutually disjoint. T is called *conservative* iff every wandering set has measure zero.

We state our theorem somewhat abstractly, without reference to the shift. To apply the theorem to sequences $(X_n)_{n=-\infty}^{n=+\infty}$, assume that $\mathfrak{A}_0$ is generated by $X_0, X_1, \cdots$.

THEOREM. *Let T be an invertible conservative measure-preserving transformation on a measure space $(\Omega, \mathfrak{A}, \mu)$ and let $\mathfrak{A}_0$ be a σ-subalgebra such that $T^{-1}\mathfrak{A}_0 \subset \mathfrak{A}_0$ and $\bigcup_{k=0}^{\infty} T^k\mathfrak{A}_0$ generates $\mathfrak{A}$. Assume that μ restricted to $\mathfrak{A}_0$ is σ-finite. Then the σ-algebra $\mathfrak{I}$ of invariant sets is contained in the remote σ-algebra $\mathfrak{A}_\infty = \bigcap_{k=0}^{\infty} T^{-k}\mathfrak{A}_0$, modulo μ-null sets.*

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PROOF. The σ-algebra $T^{-k}\mathfrak{A}_0$ is denoted by $\mathfrak{A}_k$ ($k = 0, \pm 1, \cdots$). Let $A \in \mathfrak{I}$ and let $h \in L_1^+$ (the class of integrable non-negative functions) be such that $\{h > 0\} = A$. Let $\delta > 0$. Since μ is σ-finite on $\mathfrak{A}_0$, there exists a set $B \in \mathfrak{A}_0$ with $\mu(B) < \infty$ and $\|h1_{B^c}\|_1 < \delta/2$. For sufficiently large n, $h1_B$ differs in L_1 norm by less than $\delta/2$ from some $\mathfrak{A}_{-n}$ measurable function h_δ; hence $\|h - h_\delta\|_1 < \delta$. Therefore one can obtain a sequence of positive numbers k_p and a sequence of functions h_p, each h_p measurable on $\mathfrak{A}_{-k_p}$, $p = 1, 2, \cdots$, such that

$$\sum_{p=1}^{\infty} \|h_p - h\|_1 < \infty. \tag{1.1}$$

Let $f \in L_1^+$ be measurable on $\mathfrak{A}_0$ and strictly positive on Ω. By the ratio ergodic theorem of Stepanoff-Hopf (see [4], p. 49)

$$D_n(h_p, f) =_{\text{def}} \sum_{k=0}^{n-1} h_p \cdot T^k / \sum_{k=0}^{n-1} f \cdot T^k$$

converges as $n \to \infty$ to a finite limit $D(h_p, f)$, measurable on $\mathfrak{A}_{-k_p}$ and invariant under T, therefore measurable on $\mathfrak{A}_\infty$. We now apply the corollary on p. 195 of Neveu [5], which remains valid if the operator is induced by a measure-preserving point-transformation on a σ-finite measure space, to the functions $g_p = h - h_p$. It follows that the limit $D(h, f)$ of $D_n(h, f)$ is measurable on $\mathfrak{A}_\infty$. Since A is invariant, $D(h, f) = 0$ on A^c. On the other hand, applying the Stepanoff-Hopf theorem with Ω replaced by A and h and f interchanged, we obtain that $D(f, h)$ is finite on A, hence $D(h, f)$ is positive on A. It follows that $A \in \mathfrak{A}_\infty$.

2. We now give an example such that $\mathfrak{A}_\infty$ is trivial and $\mathfrak{I}$ is not, and an example where the converse is true.

Let $(\Omega^+, \mathfrak{A}^+)$ be the unilateral product space $\prod_{k=0}^{\infty} (E_k, \mathfrak{F}_k)$, where every E_k is the set of all integers and $\mathfrak{F}_k$ is the σ-algebra of its subsets. Let $X_0, X_1, \cdots$ again be the coordinate mappings and define $\mathfrak{A}_\infty^+$ by analogy. Let π be a probability measure on $\mathfrak{A}^+$.

The following lemma is taken from Blackwell and Freedman [1], though not explicitly stated there. For the convenience of the reader we present a short proof here.

LEMMA. *If X_0 is an integer-valued random variable and $X_n = X_0 + Y_1 + \cdots + Y_n$ is a sum of X_0 and of n independent identically distributed integer-valued random variables $Y_1, \cdots, Y_n$ such that $X_0, X_1, \cdots$ is an aperiodic irreducible random walk, then $\mathfrak{A}_\infty^+$ is trivial.*

PROOF. At first assume $\pi(X_0 = 0) = 1$. The events in $\mathfrak{A}_\infty^+$ are invariant under any finite permutation of the Y_n's and hence by the Hewitt-Savage zero-one law (cf. [2], p. 122), $\mathfrak{A}_\infty^+$ is trivial. Now proceed as in [1]: Let J be the set of all integers k such that $\pi(X_0 = k) > 0$. Let $p^{(n)}(i, j)$ be the n-step transition probability from i to j. For $i, j \in J$ there is, by aperiodicity, some integer n such that $p^{(n)}(i, j) > 0$ and $p^{(n)}(j, j) > 0$. If $A \in \mathfrak{A}_\infty^+$, then

$$\pi(A \mid X_0 = i) = p^{(n)}(i, j)\pi(A \mid X_n = j) + (1 - p^{(n)}(i, j))\pi(A \mid X_n \neq j). \tag{2.1}$$

If $\pi(A \mid X_0 = i) > 0$, then $\pi(A \mid X_0 = i) = 1$ by the first argument and hence

$\pi(A \mid X_n = j) = 1$ by (2.1). Since $p^{(n)}(j, j) > 0$, it follows that $\pi(A \mid X_0 = j) > 0$ and hence again by the first argument, $\pi(A \mid X_0 - j) = 1$. Thus $\pi(A \mid X_0 = i) > 0$ for a state i implies that $\pi(A \mid X_0 = j) = 1$ for all states $j \varepsilon J$. Therefore $\mathcal{A}_\infty^+$ is trivial.

We now assume that $J = E_0$ and we replace π by an equivalent infinite invariant measure μ. Such a measure may be obtained by dividing π on each set $\{X_0 = k\}$ by $\pi(X_0 = k)$. μ is now extended to the bilateral product space $(\Omega, \mathcal{A})$ by stationarity, yielding the Markov measure corresponding to the invariant marginal distribution $\mu(X_0 = j)$ for $j \varepsilon E_0$, and the transition probabilities of the considered aperiodic random walk. By a theorem of Harris and Robbins [3], the shift T on $(\Omega, \mathcal{A}, \mu)$ is dissipative iff the random walk is transient, which we assume. The (right) remote σ-algebra $\mathcal{A}_\infty$ clearly is trivial iff $\mathcal{A}_\infty^+$ is trivial.

If $(\Omega, \mathcal{A}, \mu)$ is thus constructed, then the events $A_k = \{\omega : X_i(\omega) = 0$ for exactly k indices $i\}$ constitute a non-trivial partition of the space Ω. Hence $\mathcal{I}$ is non-trivial, but by the lemma $\mathcal{A}_\infty$ is trivial.

To give the second announced example, consider the deterministic random walk $X_n = X_0 + n$, with $\mu(X_0 = k) = 1$ for all k. μ assigns measure one to every point $\omega = (\cdots, \omega_{-1}, \omega_0, \omega_1, \cdots)$ such that $\omega_k = \omega_0 + k$ for all k. $\mathcal{A}_\infty = \mathcal{A}$, but $\mathcal{I}$ in this case is trivial. (There are related examples at the end of [1].)

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