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NOTE ON A 'MULTIVARIATE' FORM OF BONFERRONI'S INEQUALITIES[1]

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The well-known Bonferroni inequalities (see Feller [1]) provide a sequence of upper and lower bounds on the probability that exactly (or at least) k among n events occur. This note presents an analogous result in the case where one deals with r (finite) classes of events.

For notational convenience derivations are restricted to the case $r = 2$. Let $\{A_1, \cdots, A_M\}$, $\{B_1, \cdots, B_N\}$ be two classes of events. For integers m and n $0 \leqq m \leqq M$, $0 \leqq n \leqq N$ define $P_{[m,n]} = \Pr$ (exactly m A_i's and exactly n B_j's occur). $P_{(m,n)}$ is defined analogously with 'at least' replacing 'exactly.' Let $S_{m,n} = \sum' \Pr(A_{i_1} \cdots A_{i_m} B_{j_1} \cdots B_{j_1})$, where $\sum'$ denotes summation over the indices $1 \leqq i_1 < \cdots < i_m \leqq M$; $1 \leqq j_1 < \cdots < j_n \leqq N$. It is known (see Fréchet [2]) that

$$P_{[m,n]} = \sum_{t=m+n}^{M+N} \sum_{i+j=t} (-1)^{t-(m+n)} \binom{i}{m} \binom{j}{n} S_{i,j} \tag{1}$$

and hence, solving the linear system (1),

$$S_{m,n} = \sum_{t=m+n}^{M+N} \sum_{i+j=t} \binom{i}{m} \binom{j}{n} P_{[i,j]} . \tag{2}$$

A 'bivariate' form of Bonferroni's inequalities is given in the following:

THEOREM 1. *For any non-negative integer k,*

$$\sum_{t=m+n}^{m+n+2k+1} \sum_{i+j=t} f(i, j; t) \leqq P_{[m,n]} \leqq \sum_{t=m+n}^{m+n+2k} \sum_{i+j=t} f(i, j; t), \tag{3}$$

where $f(i, j; t) = (-1)^{t-(m+n)} \binom{i}{m} \binom{j}{n} S_{i,j}$.

PROOF. It suffices to show that $R(r) = \sum_{t=r}^{M+N} \sum_{i+j=t} (-1)^{t-r} \binom{i}{m} \binom{j}{n} S_{i,j} \geqq 0$ for $r \geqq m + n$. Using (2) we have

$$\begin{aligned} R(r) &= \sum_{t=r}^{M+N} \sum_{i+j=t} (-1)^{t-r} \binom{i}{m} \binom{j}{n} \sum_{y=i}^{M} \sum_{z=j}^{N} \binom{y}{i} \binom{z}{j} P_{[y,z]} \\ &= \sum_{i=m}^{M} \sum_{j=r-i}^{N} \sum_{y=i}^{M} \sum_{z=j}^{N} (-1)^{i+j-r} \binom{i}{m} \binom{j}{n} \binom{y}{i} \binom{z}{j} P_{[y,z]} \\ &= \sum_{i=m}^{M} \sum_{y=i}^{M} \sum_{z=r-i}^{N} \binom{z-n-i}{r-i-n-1} \binom{i}{m} \binom{y}{i} \binom{z}{n} P_{[y,z]} \geqq 0. \end{aligned}$$

An analogous result holds for $P_{(m,n)}$. First note that using (1) and an elementary combinatorial identity (for example, 12.8 on page 62 of [1]),

$$P_{(m,n)} = \sum_{y=m}^{M} \sum_{z=n}^{N} P_{[y,z]} = \sum_{i=m}^{M} \sum_{j=n}^{N} (-1)^{i+j-(m+n)} \binom{i-1}{m-1} \binom{j-1}{n-1} S_{i,j} ; \tag{4}$$

hence, by solving the linear system (4) (using 12.13 on page 62 of [1]),

$$S_{i,j} = \sum_{i=m}^{M} \sum_{j=n}^{N} \binom{i-1}{m-1} \binom{j-1}{n-1} P_{(i,j)} . \tag{5}$$

An argument analogous to that used in Theorem 1 yields

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THEOREM 2. *For any non-negative integer* k,

(6) $\sum_{t=m+n}^{m+n+2k+1} \sum_{i+j=t} g(i, j; t) \leqq P_{(m,n)} \leqq \sum_{t=m+n}^{m+n+2k} \sum_{i+j=t} g(i, j; t)$,

where $g(i, j; t) = (-1)^{t-(m+n)} \binom{i-1}{m-1} \binom{j-1}{n-1} S_{i,j}$.

A useful application of these inequalities has been made. Suppose the two classes of events are $\{A_i^N \ (i = 1, \cdots, N)\}$ and $\{B_j^N \ (j = 1, \cdots, N)\}$ where as $N \to \infty$, $S_{i,j} \to \xi_1^i \xi_2^j / i! j!$. In this case it is easily seen that $P_{[m,n]} \to (e^{-\xi_1} \xi_1^m / m!) \cdot (e^{-\xi_2} \xi_2^n / n!)$ as $N \to \infty$. This situation arises in consideration of certain simultaneous sequences of 'rare' events [3].

The general (multivariate) form of inequalities (3) and (6) is easily established. Let $\{A_{ij}(i = 1, \cdots, M_j)\}$ $(j = 1, \cdots, r)$ be r (finite) classes of events. For non-negative integers m_j $(j = 1, \cdots, r)$, $0 \leqq m_j \leqq M_j$, define $P_{[m_1, \cdots, m_r]}$, $P_{(m_1, \cdots, m_r)}$ and $S_{m_1, \cdots, m_r}$ in the obvious manner. Then for any non-negative integer k,

(7) $$\sum_{t=\Sigma m_j}^{\Sigma m_j + 2k+1} \sum_{\Sigma i_j = t} f(i_1, \cdots, i_r; t) \leqq P_{[m_1, \cdots, m_r]} \leqq \sum_{t=\Sigma m_j}^{\Sigma m_j + 2k} \sum_{\Sigma i_j = t} f(i_1, \cdots, i_r; t)$$

where $f(i_1, \cdots, i_r; t) = (-1)^{t - \Sigma m_j} \prod_{j=1}^{r} \binom{i_j}{m_j} S_{i_1, \cdots, i_r}$, and the indices satisfy $m_j \leqq i_j \leqq M_j$ $(j = 1, \cdots, r)$. Clearly similar bounds exist for $P_{(m_1, \cdots, m_r)}$. They are obtained from (7) by replacing $f(i_1, \cdots, i_r; t)$ by

$$g(i_1, \cdots, i_r; t) = (-1)^{t - \Sigma m_j} \prod_{j=1}^{r} \binom{i_j - 1}{m_j - 1} S_{i_1, \cdots, i_r}.$$

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