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# AN OPTIONAL STOPPING THEOREM

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Let $Y = (Y_n, \mathfrak{F}_n, n \geqq 0)$ denote a stochastic sequence of integrable random variables such that each $Y_n$ is measurable with respect to $\mathfrak{F}_n$, $(\mathfrak{F}_n)$ being an increasing sequence of sub-$\sigma$-fields of the $\sigma$-field of the underlying probability space. The notion of *stopping time* will be relative to $(\mathfrak{F}_n)$. It may or may not be true that there exists a constant $M$ such that

$$|E[Y_T]| \leqq M \tag{1}$$

for every bounded stopping time $T$. In Theorem 1 it is shown that for a certain very special class of sequences a suitable $M$ does exist. This fact is used in Theorem 2 to obtain a result in the ergodic theory of Markov chains. Related results have been observed before, see for instance [1], but the present result seems new and the proof is short and intuitive.

Let $X = (X_n, \mathfrak{F}_n, n = 0)$ denote a Markov chain with state space $(S, \mathfrak{B})$ and stationary transition probability function $P(x, B)$. Write $P^k(x, A)$ for the $k$-step transition probability function.

THEOREM 1. *Let $g$ be a real valued measurable function on $(S, \mathfrak{B})$ which is bounded in absolute value. Suppose*

$$Gg(x) = \lim_{n\to\infty} \textstyle\sum_{k=1}^{n} (P^k g)(x)$$

*exists for all $x \varepsilon S$, and that $Gg$ is bounded in absolute value. Let $Y = (Y_n, \mathfrak{F}_n, n \geqq 0)$, $Y_n = \sum_{k=0}^{n} g(X_k)$. Then there exists an $M$ such that* (1) *holds for every bounded stopping time $T$.*

PROOF. Write

$$Z_n = \textstyle\sum_{k=0}^{n} g(X_k) + Gg(X_n)$$

and observe that $Z = (Z_n, \mathfrak{F}_n, n \geqq 0)$ is a martingale. For bounded stopping times $T$ the optional sampling theorem for martingales implies $E[Z_T] = E[Z_0]$, hence

$$|E[Y_T] - E[Z_0]| \leqq \sup_x |Gg(x)|.$$

The Markov chain $X$ is *recurrent in the sense of Harris* if there exists a $\sigma$-finite measure $\pi$ on $(S, \mathfrak{B})$ such that $\pi = \pi P$ and $B \varepsilon \mathfrak{B}$, $\pi(B) > 0$ implies

$$\lim_{n\to\infty} P_x[\textstyle\bigcup_{k=0}^{n} [X_k \varepsilon B]] = 1, \qquad x \varepsilon S. \tag{2}$$

If the convergence in (2) is uniform in $x$, $X$ is *uniformly recurrent*. The measure $\pi$ is unique up to a constant factor, and in the uniformly recurrent case it is necessarily finite and will be assumed to be normalized to be a probability measure.

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($X$ is uniformly recurrent iff it satisfies hypothesis ($D$) on page 192 of [2] with a single ergodic set.)

COROLLARY. *If $X$ is uniformly recurrent and $g$ is a real valued measurable function on $(S, \mathcal{B})$ which is bounded in absolute value and satisfies $\int_S g(x)\pi(dx) = 0$, the conclusion of Theorem 1 holds.*

PROOF. It follows easily from known properties of uniformly recurrent chains that $Gg$ exists and is bounded in absolute value. For the general case can easily be reduced to that in which there are no cycles (aperiodic case), and then $P^n(x, \cdot)$ converges to $\pi$ in variation, uniformly in $x$ and geometrically fast [2].

Let $X$ be recurrent in the sense of Harris, $D \varepsilon \mathcal{B}$, $\pi(D) > 0$. Let $T_1, T_2, \cdots$ be the first, second, $\cdots$ entrance times of $(X_n)$ into $D$. If the chain $X_{T_1}, X_{T_2}, \cdots$ is uniformly recurrent, call $D$ a *Doblin set.*

THEOREM 2. *Let $D$ be a Doblin set, $A \varepsilon \mathcal{B}$, $B \varepsilon \mathcal{B}$, $A \subseteq D$, $B \subseteq D$. Then $|\sum_{k=0}^{N} P[X_k \varepsilon A]\pi(B) - P[X_k \varepsilon B]\pi(A)|$ is bounded uniformly in $N$.*

PROOF. It suffices to prove the result under the assumption that $A$ and $B$ are disjoint. Let $V_0 = (X_0, X_1 \cdots, X_{T_1})$, $V_1 = (X_{T_1+1}, X_{T_1+2}, \cdots, X_{T_2}), \cdots$. Observe that $V = (V_k, \mathfrak{F}_{T_k}, k \geqq 0)$ is a uniformly recurrent chain. A typical state for this chain is $v = (x_0, x_1, \cdots, x_n)$, where $x_n \varepsilon D$, $x_m \varepsilon S - D$ for $0 \leqq m < n$. Let

$$\begin{aligned} g(v) &= \pi(B), \quad x_n \varepsilon A \\ &= -\pi(A), \quad x_n \varepsilon B \\ &= 0, \quad x_n \varepsilon D - (A \cup B). \end{aligned}$$

Furthermore let $q(v) = n + 1$, so that $q(v)$ equals the "length of $v$". One easily verifies that if $\tilde{\pi}$ is the invariant measure for $V$, $\int g(v)\, \tilde{\pi}\, (dv) = 0$. For given $N$ let $T_N = \sup\{n: \sum_{k=0}^{n} q(V_k) \leqq N\}$. Evidently $T_N \leqq N$ so the corollary applies to give a uniform bound on $|E[\sum_{k=0}^{T_N} g(V_k)]|$ and this implies the truth of the Theorem.

## REFERENCES

[1] METIVIER, M. Existence of an invariant measure and an Ornstein's ergodic theorem. *Ann. Math. Statist.* **40** 79–96.

[2] DOOB, J. L. (1953). *Stochastic Processes.* Wiley, New York.