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MARTINGALE EXTENSIONS OF A THEOREM OF MARCINKIEWICZ AND ZYGMUND[1]

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1. Introduction. Suppose that $(d_n, n \geqq 1)$ is an orthonormal sequence of independent random variables and $(a_n, n \geqq 1)$ is a sequence of real numbers. Kac and Steinhaus [4] prove that if d_n^2 are uniformly integrable and $\sum a_n d_n$ converges a.s., then $\sum a_n^2 < \infty$. Marcinkiewicz and Zygmund [5] improve this result by replacing the uniform integrability of d_n^2 by $E|d_n| \geqq \delta > 0$ for every n. Recently Gundy [3] has extended the latter to martingales as follows: Let $(x_n = d_1 + \cdots + d_n, \mathfrak{F}_n, n \geqq 1)$ be a martingale with $E(d_n^2 \mid \mathfrak{F}_{n-1}) = 1$ a.s. and $E(|d_n| \mid \mathfrak{F}_{n-1}) \geqq \delta$ a.s. for some number $\delta \geqq 0$, and let $(\varphi_n, \mathfrak{F}_{n-1}, n \geqq 1)$ be a stochastic sequence, i.e., φ_n are $\mathfrak{F}_{n-1}$ measurable random variables. Then except on a null set, $\sum \varphi_n^2 < \infty$, $\sum \varphi_n^2 d_n^2 < \infty$ and $\sum \varphi_n d_n$ converges are equivalent.

In [6] (also in [7], p. 123), Zygmund proves the following summability result: Let d_n be independent, identically distributed random variables with $P[d_1 = \pm 1] = \frac{1}{2}$ and let $(a_{m,n}, m \geqq 1, n \geqq 1)$ be a double sequence of real numbers such that $\lim_{m\to\infty} a_{m,n} = a_n$, finite, for each n. If $\sum_{k=1}^{\infty} a_{m,k} d_k = T_m$ a.s. and $P[T_m \text{ converges}] > 0$, then $\sum a_n^2 < \infty$.

In Section 3, we shall give a new proof of Gundy's theorem and improve it slightly. In Section 4, Marcinkiewicz and Zygmund's theorem is extended to a summability result, which includes Zygmund's theorem as a special case.

2. Notation and lemmas. Let $(\Omega, \mathfrak{F}, P)$ be a probability space, $(\mathfrak{F}_n, n \geqq 1)$ be a sequence of σ-fields with $\mathfrak{F}_n \subset \mathfrak{F}_{n+1} \subset \mathfrak{F}$ and $(x_n, n \geqq 1)$ be a sequence of random variables. If $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \cdots$ and x_n is $\mathfrak{F}_n$-measurable for each n, the sequence $(x_n, \mathfrak{F}_n, n \geqq 1)$ is said to be a stochastic sequence. We always put $\mathfrak{F}_0 = \{\emptyset, \Omega\}$. For a set A, the indicator function of A is denoted by $I(A)$, and the integral $\int_A x\, dP$ is shortened as $\int_A x$. If $(x_n = d_1 + \cdots + d_n, \mathfrak{F}_n, n \geqq 1)$ is a martingale, the sequence $(d_n, \mathfrak{F}_n, n \geqq 1)$ is called a martingale difference sequence.

LEMMA 1. *Let $d \geqq 0$ be a random variable and $\mathcal{G} \subset \mathfrak{F}$ be a σ-field. Put $m = E(d \mid \mathcal{G})$ and $v = E^{\frac{1}{2}}(d^2 \mid \mathcal{G})$. If $\lambda \geqq 0$ is a $\mathcal{G}$-measurable random variable and $P[m < \infty] = 1$, then*

$$vP(d > \lambda v \mid \mathcal{G}) \geqq \lambda(m - 2\lambda v), \tag{1}$$

$$E(d^2 I[\lambda d < 1] \mid \mathcal{G}) \geqq (m - \lambda v^2)^2 \quad \text{on} \quad [m \geqq \lambda v^2]. \tag{2}$$

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PROOF. To prove (1), we can assume that $\lambda > 0$.

$$m = E(d \mid \mathcal{G}) \leqq \lambda v + E(dI[\lambda v < d \leqq v/\lambda] \mid \mathcal{G}) + E(dI[d > v/\lambda] \mid \mathcal{G})$$
$$\leqq 2\lambda v + vP(d > \lambda v \mid \mathcal{G})/\lambda,$$

which yields (1). Since $E(dI[\lambda d \geqq 1] \mid \mathcal{G}) \leqq \lambda E(d^2I[\lambda d \geqq 1] \mid \mathcal{G}) \leqq \lambda v^2$,

$$E(d^2I[\lambda d < 1] \mid \mathcal{G}) \geqq E^2(dI[\lambda d < 1] \mid \mathcal{G}) \geqq (m - \lambda v^2)^2 \quad \text{on} \quad [m \geqq \lambda v^2].$$

From Lemma 1, immediately follows:

LEMMA 1$'$. *Under the conditions of Lemma* 1, *if* $P[3\lambda v \leqq m < \infty] = 1$ *for some constant* $\lambda > 0$, *then*

$$(1) \qquad P(d > \lambda v \mid \mathcal{G}) \geqq \lambda^2,$$

and if $\varphi \geqq 0$ *is a* $\mathcal{G}$*-measurable random variable*,

$$(2') \qquad E(d^2I[\varphi d < 1] \mid \mathcal{G}) \geqq (3\lambda - \varphi v)^2 v^2 \quad on \quad [3\lambda \geqq \varphi v].$$

LEMMA 2. *Let* $(d_n, \mathfrak{F}_n, n \geqq 1)$ *be a non-negative stochastic sequence such that there exists a constant* $\lambda > 0$ *satisfying*

$$(3) \qquad \infty > E(d_n \mid \mathfrak{F}_{n-1}) \geqq \lambda v_n, \quad where \quad v_n = E^{\frac{1}{2}}(d_n^2 \mid \mathfrak{F}_{n-1}),$$

for $n \geqq 1$, *then*

$$(4) \qquad P[\sup v_n = \infty, \sup d_n < \infty] = 0.$$

PROOF. Put $A = [\sup d_n < K]$ for $K > 0$. By Lévy's martingale version (see [2], p. 324) of Borel-Cantelli lemma, $\sum P(d_n \geqq K \mid \mathfrak{F}_{n-1}) < \infty$ on A. By (1$'$), for almost all $\omega \in A$, $\lambda v_n(\omega) \leqq K$ for all large n. Hence $\sup v_n < \infty$ on A. Letting $K \to \infty$, we obtain (4).

LEMMA 3. *Let* $(\varphi_n, \mathfrak{F}_{n-1}, n \geqq 1)$ *be a stochastic sequence and let* $(e_n, \mathfrak{F}_n, n \geqq 1)$ *be a martingale difference sequence with* $Ee_n^2 < \infty$. *Put* $d_n = \varphi_n e_n$, $v_n = \varphi_n E^{\frac{1}{2}}(e_n^2 \mid \mathfrak{F}_{n-1})$ *and* $x_n = d_1 + \cdots + d_n$. *For constants* $K > 0$ *and* $M > 0$, *let* $t = \inf\{n \mid |x_n| > K \text{ or } |v_{n+1}| > M\}$. *Then for* $j \geqq 1$,

$$(5) \qquad (K^2 + K)^2 \geqq \textstyle\sum_1^j \int_{[t \geqq k]} d_k^2(I[|d_k| < K^2] - 2K^{-1}I[|d_k| \geqq K^2]).$$

PROOF. Since $E\, d_k^2 I[t \geqq k] = Ev_k^2 I[t \geqq k] \leqq M^2$, $\sum_1^j d_k I[t \geqq k]$ is a martingale and

$$E \textstyle\sum_1^j d_k^2 I[t \geqq k] = E(\sum_1^j d_k I[t \geqq k])^2 = Ex^2_{\min(t,j)} \leqq K^2 + \int_{[t \leqq j]} (2x_{t-1}\, d_t + d_t^2)$$
$$\leqq (K^2 + K)^2 + \int_{[t \leqq j, |d_t| \geqq K^2]} (2x_{t-1}\, d_t + d_t^2)$$
$$\leqq (K^2 + K)^2 + (1 + 2K^{-1}) \textstyle\sum_1^j \int_{[t \geqq k, |d_k| \geqq K^2]} d_k^2,$$

which yields (5).

LEMMA 4. *Let* $d_n \geqq 0$ *be a sequence of random variables and for some constant* $\lambda > 0$,

$$(6) \qquad \infty > E\, d_n \geqq 3\lambda u_n, \quad where \quad u_n = E^{\frac{1}{2}}\, d_n^2.$$

for $n \geqq 1$. *If* $\sup u_n = \infty$, *then* $\sup P[d_n > K] \geqq \lambda^2$ *for every constant* $K > 0$.

PROOF. Putting $\mathcal{G} = \{\varnothing, \Omega\}$ in Lemma 1′, we have $P[d_n > \lambda u_n] \geqq \lambda^2$ for $n \geqq 1$. For $k = 1, 2, \cdots$, choose n_k such that $u_{n_k} \geqq k$. Then for any constant $K > 0$ and $\lambda k \geqq K$,

$$\sup P[d_n > K] \geqq P[d_{n_k} > K] \geqq P[d_{n_k} > \lambda k] \geqq P[d_{n_k} > \lambda u_{n_k}] \geqq \lambda^2.$$

LEMMA 5. (Burkholder [1], Lemma 4). *To each $\delta > 0$ corresponds an $\alpha > 0$ with the following property: If $(x_n = d_1 + \cdots + d_n, \mathfrak{F}_n, n \geqq 1)$ is a martingale and $\infty > E|d_k| \geqq \delta E^{\frac{1}{2}} d_k^2$ for $n \geqq 1$, then $E|x_n| \geqq \alpha E^{\frac{1}{2}} x_n^2$.*

3. A new proof of Gundy's theorem.

THEOREM 1. *Let $(\varphi_n, \mathfrak{F}_{n-1}, n \geqq 1)$ be a stochastic sequence and $(e_n, \mathfrak{F}_n, n \geqq 1)$ be a martingale difference sequence with $Ee_n^2 < \infty$. Put $d_n = \varphi_n e_n$, $v_n = |\varphi_n| E^{\frac{1}{2}}(e_n^2 \mid \mathfrak{F}_{n-1})$, and $x_n = d_1 + \cdots + d_n$. If there exists a constant $\lambda > 0$ such that for $n \geqq 1$ and all large $K > 0$,*

$$E(d_n^2 I[|d_n| < K] \mid \mathfrak{F}_{n-1}) \geqq (3\lambda - v_n K^{-1})^2 v_n^2 \quad \text{on} \quad [3\lambda \geqq v_n K^{-1}], \tag{7}$$

then except on a null set, the following statements are equivalent:

$$\sup |x_n| < \infty, \qquad \sup v_n < \infty, \tag{8}$$

$$\sum (d_n^2 + v_n^2) < \infty, \tag{9}$$

$$\sum d_n^2 < \infty, \qquad \sup v_n < \infty, \tag{10}$$

$$x_n \ \text{converges}, \qquad \sup v_n < \infty, \tag{11}$$

$$\sum v_n^2 < \infty. \tag{12}$$

PROOF. We shall prove that (i)(8) implies (9) and that (ii)(10) implies (11). That (9) implies (10) and (12) and that (11) implies (8) are obvious. In ([2], p. 323), Doob proved that (12) implies (11).

(i) For $M > 0$, choose $K > 0$ such that $K\lambda^2 \geqq 4$ and $2K^2\lambda \geqq M$. Define $t = \inf\{n \mid |x_n| > K \text{ or } v_{n+1} > M\}$. Then by Lemma 3,

$$(K^2 + K)^2 \geqq \textstyle\sum_1^j \int_{[t \geqq k]} (I[|d_k| < K^2] - 2K^{-1} I[|d_k| \geqq K^2])\, d_k^2.$$

On the set $[t \geqq k]$, $3\lambda - v_k K^{-2} \geqq 3\lambda - MK^{-2} \geqq \lambda$. Hence by (7),

$$\begin{aligned} (K^2 + K)^2 & \\ \geqq \textstyle\sum_1^j \int_{[t \geqq k]} (\lambda^2 - 2K^{-1}) v_k^2 &\geqq \lambda^2 \textstyle\sum_1^j \int_{[t \geqq k]} v_k^2/2 = \lambda^2 \textstyle\sum_1^j \int_{[t \geqq k]} d_k^2/2. \end{aligned} \tag{13}$$

Therefore $\sum (v_n^2 + d_n^2) < \infty$ on $[t = \infty] = [\sup |x_n| \leqq K, \sup v_n \leqq M]$. Letting $K \to \infty$ and then $M \to \infty$, we obtain that $\sum (v_n^2 + d_n^2) < \infty$ on $[\sup |x_n| < \infty, \sup v_n < \infty]$.

(ii) For $M > 0$, choose $K > 0$ such that $2K\lambda \geqq M$. Define

$$t = \inf\{n \mid \textstyle\sum_1^n d_k^2 > K \quad \text{or} \quad v_{n+1} > M\}.$$

Then for $j \geqq 1$,

$$\begin{aligned} \textstyle\sum_1^j E\, d_k^2 I[t \geqq k] &= \textstyle\sum_1^j E\, d_k^2 I[t > k] + \textstyle\sum_1^j E\, d_k^2 I[t = k] \\ &\leqq K + K^2 + \textstyle\sum_1^j E\, d_k^2 I[t = k, |d_k| \geqq K]. \end{aligned}$$

Hence by (7),

$$K^2 + K \geqq \sum_1^j E\, d_k^2 I[t \geqq k,\ |d_k| < K] \geqq \sum_1^j E(I[t \geqq k] v_k^2 (3\lambda - v_k K^{-1})^2.$$

On the set $[t \geqq k]$, $3\lambda - v_k K^{-1} \geqq 3\lambda - MK^{-1} \geqq \lambda$. Therefore

$$K^2 + K \geqq \lambda^2 \sum_1^j E(I[t \geqq k] v_k^2),$$

$$\sum E(I[t \geqq k]\, d_k^2) = \sum E(I[t \geqq k] v_k^2) < \infty.$$

Since $\sum_1^j I[t \geqq k]\, d_k$ is a martingale, $\sum I[t \geqq k]\, d_k$ converges a.s. Thus x_n converges on $[t = \infty] = [\sum d_k^2 \leqq K, \sup v_n \leqq M]$. Letting $K \to \infty$ and then $M \to \infty$, we have that x_n converges on $[\sum d_n^2 < \infty, \sup v_n < \infty]$. The proof is completed.

THEOREM 1′. *Let $(\varphi_n, \mathfrak{F}_{n-1}, n \geqq 1)$ be a stochastic sequence and $(e_n, \mathfrak{F}_n, n \geqq 1)$ be a martingale difference sequence with $Ee_n^2 < \infty$ and $u_n = E^{\frac{1}{2}}(e_n^2 \mid \mathfrak{F}_{n-1})$. If there exists a constant $\lambda > 0$ such that for $n \geqq 1$,*

$$(7') \qquad E(|e_n| \mid \mathfrak{F}_{n-1}) \geqq 3\lambda u_n,$$

then except on a null set, $\sup |\sum_1^n \varphi_k e_k| < \infty$, $\sum \varphi_n^2 e_n^2 < \infty$, $\sum \varphi_n e_n$ *converges and* $\sum \varphi_n^2 u_n^2 < \infty$ *are all equivalent.*

PROOF. Put $d_n = \varphi_n e_n$ and $v_n = |\varphi_n| E^{\frac{1}{2}}(e_n^2 \mid \mathfrak{F}_{n-1})$. Then (7′) implies that $E(|d_n| \mid \mathfrak{F}_{n-1}) \geqq 3\lambda v_n$. Since $P[v_n < \infty] = 1$, from Lemma 1′, we have that (7) holds and from Lemma 2, $\sup v_n < \infty$ if $\sup |d_n| < \infty$. Thus Theorem 1′ follows from Theorem 1.

When $u_n = 1$ a.s. for $n \geqq 1$, the equivalence of $\sum \varphi_n^2 e_n^2 < \infty$, $\sum \varphi_n e_n$ converges, and $\sum \varphi_n^2 < \infty$ under the conditions of Theorem 1′ has been established by Gundy [3] by a different method.

THEOREM 2. *Let $(d_n, \mathfrak{F}_n, n \geqq 1)$ be a martingale difference sequence with $E\, d_n^2 < \infty$ and $v_n = E^{\frac{1}{2}}(d_n^2 \mid \mathfrak{F}_{n-1})$. For $m \geqq 1$, let $(\varphi_{m,n}, \mathfrak{F}_{n-1}, n \geqq 1)$ be a stochastic sequence such that for each $n \geqq 1$,*

$$P[\lim_m \varphi_{m,n} = \varphi_n \text{ finite}] = 1.$$

Put $s_{m,n} = \sum_{k=1}^n \varphi_{m,k} d_k$. If there exists a constant $1 > \lambda > 0$ such that for $n \geqq 1$,

$$(14) \qquad E(|d_n| \mid \mathfrak{F}_{n-1}) \geqq 3\lambda v_n,$$

then $\sum \varphi_n^2 v_n^2 < \infty$ on the set $[\sup_{m,n} |s_{m,n}| < \infty]$.

PROOF. For $M > 0$, choose $K \geqq M$ such that $K^2\lambda \geqq \max(4, M)$. Define $t = t_m = \inf\{n \mid |s_{m,n}| > K \text{ or } |\varphi_{m,n+1} v_{n+1}| > M\}$. By Lemma 1′ and (13), for $j = 1, 2, \cdots$,

$$(15) \qquad 2(K^2 + K)^2 \geqq \lambda^2 \sum_{k=1}^j \int_{[t_m \geqq k]} \varphi_{m,k}^2 v_k^2.$$

Put $A = [\sup_{m,n} |s_{m,n}| < M, \sup_{m,n} |\varphi_{m,n} v_n| < M]$. Then

$$2(K^2 + K)^2 \geqq \lambda^2 \sum_{k=1}^j \int_A \varphi_{m,k}^2 v_k^2.$$

By Fatou lemma, $2(K^2 + K)^2 \geqq \lambda^2 \sum_{k=1}^j \int_A \varphi_k^2 v_k^2$. Hence $\sum \int_A \varphi_n^2 v_n^2 < \infty$ and $\sum \varphi_n^2 v_n^2 < \infty$ on A. Since M is arbitrary, $\sum \varphi_n^2 v_n^2 < \infty$ on $[\sup_{m,n} |s_{m,n}| < \infty,$

$\sup_{m,n} |\varphi_{m,n} v_n| < \infty]$. Now let $\eta_n = \sup_m |\varphi_{m,n}|$. Since $P[|\varphi_n| < \infty] = 1$, $(\eta_n, \mathfrak{F}_{n-1}, n \geqq 1)$ is a stochastic sequence. By (14), $E(\eta_n |d_n| \mid \mathfrak{F}_{n-1}) \geqq 3\lambda \eta_n v_n$ for $n \geqq 1$. By Lemma 2, $P[\sup_n \eta_n v_n = \infty, \sup_n \eta_n |d_n| < \infty] = 0$. Hence $\sup_{m,n} |\varphi_{m,n} v_n| = \sup_n \eta_n v_n < \infty$ on $[\sup_n \eta_n |d_n| < \infty]$. Therefore $\sum \varphi_n^2 v_n^2 < \infty$ on $[\sup_{m,n} |s_{m,n}| < \infty]$.

4. Some summability results. In this section, we shall assume that $a_{m,n}$ is a double sequence of real numbers with $\lim_m a_{m,n} = a_n$ for each n. For a stochastic sequence $(d_n, \mathfrak{F}_n, n \geqq 1)$, we put $s_{m,n} = \sum_{k=1}^{n} a_{m,k} d_k$.

THEOREM 3. *Let $(d_n, \mathfrak{F}_n, n \geqq 1)$ be a martingale difference sequence with $E d_n^2 = 1$ and for some constant $\lambda > 0$,*

$$E|d_n| \geqq 3\lambda, \qquad (n \geqq 1). \tag{16}$$

(i) *If for a fixed $m \geqq 1$,*

$$\lim_{K\to\infty} P[|s_{m,n}| > K] = 0 \quad \text{unif. } in \quad n, \tag{17}$$

then $\sum_{n=1}^{\infty} a_{m,n}^2 < \infty$, $\lim_n s_{m,n} = s_m$ a.s. and in L_2, and for some $\alpha > 0$, independent of m, $E|s_m| \geqq \alpha E^{\frac{1}{2}} s_m^2$.

(ii) *If (17) holds for every $m \geqq 1$, and*

$$\lim_{K\to\infty} P[|s_m| > K] = 0 \quad \text{unif. } in \quad m, \tag{18}$$

then $\sum a_n^2 < \infty$. In particular, $\sum a_n^2 < \infty$, if

$$\lim_{K\to\infty} P[|s_{m,n}| > K] = 0 \quad \text{unif. } in \quad m \quad and \quad n. \tag{19}$$

(iii) *If $E(d_n^2 \mid \mathfrak{F}_{n-1}) = 1$ for $n \geqq 1$ and if for some constants $\lambda > 0$, $\eta > 0$ and $K > 0$,*

$$E(|d_n| \mid \mathfrak{F}_{n-1}) \geqq 3\lambda, \tag{20}$$

$$\sup_{m,n} |a_{m,n}| < \infty, \qquad \inf_m P[\sup_n |s_{m,n}| \leqq K] \geqq \eta, \tag{21}$$

then $\sum a_n^2 < \infty$.

PROOF. (i) By Lemma 5 and (16), there exists $\alpha > 0$, independent of m, such that

$$E|s_{m,n}| \geqq \alpha E^{\frac{1}{2}} s_{m,n}^2. \tag{22}$$

By Lemma 4, if $\sup_n E s_{m,n}^2 = \infty$, then $\lim_{k\to\infty} \sup_n P[|s_{m,n}| \geqq K] \geqq \lambda^2$, which contradicts (17). Therefore $\sup_n E s_{m,n}^2 = \sum_{n=1}^{\infty} a_{m,n}^2 < \infty$, $\lim_n s_{m,n} = s_m$ a.s. and in L_2, and by (22), $E|s_m| \geqq \alpha E^{\frac{1}{2}} s_m^2$.

(ii) By (i), for each $m \geqq 1$, $\lim_n s_{m,n} = s_m$ a.s. and in L_2 and $E|s_m| \geqq \alpha E^{\frac{1}{2}} s_m^2$. By Lemma 4, if $\sup_m E s_m^2 = \infty$, then $\lim_{k\to\infty} \sup_m P[|s_m| \geqq K] > 0$, which contradicts (18). Hence $\sup_m E s_2^m = \sup_m \sum_{n=1}^{\infty} a_{m,n}^2 < \infty$. By Fatou lemma, $\sum a_n^2 < \infty$.

(iii) Put $\sup_{m,n} |a_{m,n}| = M$ and choose $K \geqq M$ such that $K^2 \lambda \geqq \max(4, M)$. Define $t = t_m = \inf\{n \mid |s_{m,n}| > K\}$. Since $|v_n a_{m,n}| = |a_{m,n}| \leqq M \leqq K$, by (15) and (21), we have

$$2(K^2 + K)^2\lambda^{-2} \geqq \sum_{k=1}^{j} \int_{[t_m \geqq k]} a_{m,k}^2 \geqq \sum_{k=1}^{j} a_{m,k}^2 P[t_m = \infty]$$
$$\geqq \sum_{k=1}^{h} a_{m,k}^2 P[\sup_n |s_{m,n}| \leqq K] \geqq \eta \sum_{k=1}^{j} a_{m,k}^2 .$$

By Fatou lemma,

$$2(K^2 + K)^2\lambda^{-2} \geqq \eta \sum_1^j a_n^2$$

Hence $\sum a_n^2 < \infty$, and the proof is completed.

From Theorem 3(ii), immediately follows:

COROLLARY 1. *Let a_n be a sequence of real numbers and $(x_n = a_1 d_1 + \cdots + a_n d_n, \mathfrak{F}_n, n \geqq 1)$ be a martingale such that for some constant $\lambda > 0$,*

(23) $$E\, d_n^2 = 1, \qquad E|d_n| \geqq \lambda.$$

(i) *If $P[\sup_n |x_n| < \infty] = 1$ or* (ii) *x_{n_k} converges in distribution for some sub-sequence, then $\sum a_n^2 < \infty$ (and x_n converges* a.s. *and in L_2).*

Corollary 1(i) reduces Gundy's local condition: $E(d_n^2 \mid \mathfrak{F}_{n-1}) = 1$ and $E(|d_n| \mid \mathfrak{F}_{n-1}) \geqq \lambda$ to the global condition (23), when the stochastic sequence $(\varphi_n, \mathfrak{F}_{n-1}, n \geqq 1)$ is replaced by a sequence of constants. When d_n are independent random variables, Corollary 1(i) reduces to a result of Marcinkiewicz and and Zygmund [5].

Corollary 1(ii) is interesting in comparison with the following well known result: Let d_n be independent random variables. If $\sum d_n$ converges in distribution, then $\sum d_n$ converges a.s.

THEOREM 4. *Let d_n be a sequence of independent random variables such that for some constant $\lambda > 0$ and for every $n \geqq 1$,*

(24) $$E\, d_n = 0, \qquad E\, d_n^2 = 1, \qquad E|d_n| \geqq \lambda,$$

(25) $$P[\lim_{n\to\infty} s_{m,n} = s_m] = 1,$$

(26) $$P[\sup_m |s_m| < \infty] > 0.$$

Then $\sum a_n^2 < \infty$, if $|a_n| < \infty$ for each n.

PROOF. Since $\lim_m a_{m,n} = a_n$ finite, $\sup_m |a_{m,n}| < \infty$ for $n \geqq 1$, and the set $[\sup_m |s_m| < \infty]$ is a tail event. By zero-one law, $P[\sup_m |s_m| < \infty] = 1$. Let $d_1^*, d_2^*, \cdots$ be random variables such that d_j and d_j^* have the same distribution and that $d_1, d_1^*, d_2, d_2^*, \cdots$ are independent. Put $e_n = (d_n - d_n^*)2^{-\frac{1}{2}}$. Then $Ee_n = 0, Ee_n^2 = 1, 2^{\frac{1}{2}}E|e_n| \geqq 2E(d_n - d_n^*)I[d_n \geqq 0, d_n^* < 0] \geqq E|d_n| \geqq \lambda$. Therefore, we can assume that d_n is symmetric for each n. By Lévy's inequality (see [2], p. 106),

$$P[\sup_n |s_{m,n}| > K] \leqq 2\, P[|s_m| \geqq K] \leqq 2\, P[\sup_m |s_m| \geqq K].$$

Since $P[\sup_m |s_m| < \infty] = 1$, $\lim_{k\to\infty} P[\sup_n |s_{m,n}| \geqq K] = 0$ uniformly in m. Therefore $\sum a_n^2 < \infty$ by Theorem 3(ii).

When $P[d_n = \pm 1] = 1$, Theorem 4 is due to Zygmund ([6], also [7] p. 123).

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