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B. RAMANCHANDRAN. *Advanced Theory of Characteristic Functions*. Statistical Publishing Society, Calcutta, 1967. vii + 208 pp. $7.00.

Review by R. CUPPENS

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This book gives an account of the present state of the theory of analytic characteristic functions. Therefore, a better title would be "Theory of analytic characteristic functions". It can be decomposed in three parts: general theory (Chapters 1–3), decomposition theory (Chapters 4–7), applications (Chapter 8).

After a Chapter 0 devoted to some results of the theory of functions of real and complex variables, Chapter 1 recalls (mostly without proof) the elementary theory of characteristic functions: distribution functions, moments, fundamental properties of characteristic functions, infinitely divisible and lattice distributions. [It must be noted that the definition of "negative binomial-type distribution function" (p. 25) is not the classical one.] In the Chapter 2, the author studies analytic characteristic functions and their extensions (defined by J. Marcinkiewicz and studied recently by C. G. Esseen and the author): the characteristic functions which are boundary values of analytic functions. This chapter contains the Raikov's theorem on analyticity strip, its extension to boundary characteristic functions, the relation between boundary characteristic functions and bounded distributions, the Raikov's theorem on decompositions of analytic characteristic functions and the validity in the analyticity strip of the classical (Lévy, Lévy-Khintchine, Kolmogorov) representations for characteristic functions which are infinitely divisible and analytic. [The definition of boundary characteristic functions given here is not the simplest possible: indeed, it is sufficient to say that a characteristic function is boundary value of an analytic function g, that is $\lim_{y\to 0+} g(t + iy) = f(t)$: the continuity of g in the closed strip follows immediately from the uniform continuity of f. Moreover, the proof of Theorem 2.1.2 (the purely imaginary points of the boundary of the strip are singular points) is unusefully intricate: the characteristic function f is analytic in the strip $\{-\alpha < \operatorname{Im} z < \beta\}$ if and only if $\int_{-\infty}^{+\infty} e^{-yx}\, dF(x)$ converges for $-\alpha < y < \beta$. If β is not a singular point, then this integral converges for $-\alpha < y < \beta'$ with $\beta' > \beta$ and f is analytic in the strip $\{-\alpha < \operatorname{Im} z < \beta'\}$; this is in contradiction with the hypothesis and proves the theorem]. In Chapter 3, a detailed account of some results on order and type of entire characteristic functions due to P. Lévy, Pólya and the author is given.

With Chapter 4, begins the study of decomposition theorems. It gives the classical theorems on the decomposition of normal, Poisson and binomial laws and especially the Linnik theorem (with the simple proof due to I. V. Ostrovskiy) on the decomposition of the convolution of normal and Poisson laws. Chapter 5 studies the unsolved problem of the characterization of the class I_0 of infinitely divisible characteristic functions without indecomposable factors.

For the statement of the results, we introduce the class $\mathcal{L}$ of characteristic functions of the kind

$$\ln f(t) = i\beta t - \gamma t^2 + \sum_{r=1}^{2} \sum_{m=-\infty}^{\infty} \lambda_{m,r}(e^{i\mu_{m,r}t} - 1 - i\mu_{m,r}t(1 + \mu_{m,r}^2)^{-1})$$

where β is real, $\gamma \geqq 0$, $\lambda_{m,r} \geqq 0$, $\mu_{m,1} > 0$, $\mu_{m,2} < 0$ and $\mu_{m+1,r}/\mu_{m,r}$ is an integer. Section 5.1 states the Linnik theorem: If $\gamma > 0$ and if f belongs to I_0, then f belongs to $\mathcal{L}$. The (very difficult) proof of this theorem is only sketched. Section 5.2 gives the Ostrovskiy's reciprocal:

If $f \varepsilon \mathcal{L}$ and if

$$(*) \qquad \lambda_{m,r} = O[\exp(-k\mu_{m,r}^2)] \qquad (m \to +\infty)$$

for some $k > 0$, then f belongs to I_0.

[In this direction, a result by Ostrovskiy (*Vestnik Leningrad Univ* **19** (1964) n° **19** 51–60) showing that for lattice distributions (*) can be replaced by $\lambda_{m,r} = o[\exp(-2|\mu_{m,r}| \log |\mu_{m,r}|)]$ and a result by Goldberg and Ostrovskiy (*Ukrain. Mat. Ž.* **19** (1967) n° **3**, 104–106) showing that (*) cannot be replaced by $\lambda_{m,r} = O[\exp(-k|\mu_{m,r}|)]$ must be noted]. Then, some results by Cramer and Ostrovskiy on the decomposition of infinitely divisible characteristic functions without normal factor are given. [Recently, the reviewer has proved that the sufficient condition that f belongs to I_0 given by Theorem 5.3.3 is also necessary provided the measures in Lévy's representation of f have almost everywhere continuous densities] Chapter 6 gives some other decomposition theorems due to Ostrovskiy with application to products of two and three Poisson-type characteristic functions. [But the example (p. 124) of products of four Poisson-type charateristic functions (which is not the original one of Ostrovskiy) is not interesting, since it can be deduced directly from Theorem 5.2. On the other hand, the historical note (p. 108) is inaccurate for a point: the fact that the product of two Poisson-type characteristic functions belongs always to I_0 has been completely proven by P. Lévy (*Comptes Rendus du Colloque de Genève. Actualités Sci. Indust.* n° **736**, 25–59. Paris, Hermann, (1938)).] An appendix to this chapter gives some other results of Linnik. Chapter 7 considers α-decompositions of characteristic functions and gives a rigorized proof of Mamay's theorem.

Chapter 8 gives some applications of α-decompositions to characterization problems (for example, the Skitovitch-Darmois theorem). A more complete study of these problems can be found in the book of Lukacs and Laha "Applications of characteristic functions". Finally, in a first appendix a proof of some classical theorems (uniqueness, continuity, Bochner's theorem), using the product of characteristic function by a normal characteristic function is given. This method found again by P. K. Pathak is due to A. Wintner and was used by E. K. Haviland (Amer. J. Math. 57 (1935) 382–388) for proving the continuity theorem for characteristic functions of several variables. A second appendix extends some results of the book to functions which are not characteristic functions. In this direction, Theorem A II 2 attributed to Pólya (1949) is far older: it has been proved by Plancherel and Pólya (*Comment. Math. Helv.* **9** (1936/37) 224–248).

It is possible to regret for this book many references to the books "Characteristic functions" by Lukacs and "Decomposition of probability laws" by Linnik, whereas less emphasis for some minor subjects would allow to make it independent. Nevertheless, in its state, it is a good complement of Lukac's book and it is indispensable for anybody who studies characteristic functions and is not intimate with Russian literature.