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ZEROES OF INFINITELY DIVISIBLE DENSITIES[1]

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In this note, we prove the following theorem, which grew out of a question posed by R. K. Getoor.

Theorem. *Suppose $\{p_t\,; t > 0\}$ is a convolution semigroup of probability density functions; i.e., for all s, $t > 0$ and for all $x \varepsilon (-\infty, \infty)$*

$$p_{s+t}(x) = \int_{-\infty}^{\infty} p_s(x-y)p_t(y)\,dy. \tag{1}$$

Suppose further that $p_t(x)$ is jointly continuous in $t \varepsilon (0, \infty)$ and $x \varepsilon (-\infty, \infty)$. Then $\{x : p_t(x) = 0\}$ is empty for all $t > 0$, or $\{x : p_t(x) = 0\}$ is a half-line $(-\infty, ct]$ or $[ct, \infty)$, where c is some constant.

Corollary. *If p is an infinitely divisible density function whose characteristic function φ has the property that all positive powers $|\varphi|^t$ are integrable, then the set of zeroes of p is either empty or a closed half-line.*

Proof of Corollary. If $|\varphi|^t$ is integrable, the probability measure μ_t corresponding to φ^t has a continuous density p_t, and the p_t form a convolution semigroup. Since

$$p_t(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-ixy}\varphi^t(y)\,dy,$$

and since

$$|\varphi(y)| \leqq 1,$$

the dominated convergence theorem shows that if $t_n \to t > 0$ and $x_n \to x$, then $p_{t_n}(x_n) \to p_t(x)$; in other words, $p_t(x)$ is jointly continuous in t and x.

Proof of Theorem. Let M be the Levy measure for the process

$$\{p_t(x)\,dx; t > 0\},$$

so that the characteristic function φ^t of p_t is given by

$$\varphi^t(u) = \exp\{itbu - t\delta^2u^2/2 + t\int [e^{iux} - 1 - iu \sin x]M(dx)\}.$$

Clearly, if $\delta^2 > 0$, then $p_t(x) > 0$ for all t and x, so it may be assumed that $\delta^2 = 0$. We then have, for every bounded continuous function f such that $f(x) = O(x^2)$ near 0,

$$\int f(x)t^{-1}p_t(x)\,dx \to' \int f(x)M(dx) \quad \text{as } t \to 0. \tag{2}$$

Since the distributions of the process are continuous, a well-known theorem of Hartman and Wintner [1] tells us that M must have infinite mass near 0. Without loss of generality, it may be assumed that M has infinite mass on the positive

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side of 0. We now define $G \subset (0, \infty) \times (-\infty, \infty)$ by

$$G = \{(t, x) : p_t(x) > 0\}.$$

The equation (1) together with the continuity of the $p_t(\cdot)$ shows that G is an open subsemigroup of $(0, \infty) \times (-\infty, \infty)$. Let H denote the closure of G in $[0, \infty) \times (-\infty, \infty)$. Then H is a closed subsemigroup of $[0, \infty) \times (-\infty, \infty)$.

I claim that $(0, b) \varepsilon H$ for all $b \geqq 0$. Firstly, if $b = 0$, the claim follows at once from the fact that $p_t(x)\, dx \to \delta_0(dx)$ weakly as $t \to 0$. Secondly, if $b > 0$, using (2) and recalling that M must have mass arbitrarily close to 0 on the positive side, it follows that G must possess members in each wedge $\{(t, x) : x > \beta t\}$ arbitrarily close to 0. We use this remark to construct a sequence (t_n, x_n) in G such that

$$x_n \to 0 \quad \text{as} \quad n \to \infty, \quad \text{and} \tag{3}$$

$$t_n/x_n \to 0 \quad \text{as} \quad n \to \infty. \tag{4}$$

Let $\epsilon > 0$ be given; we shall construct a member (t, x) of G such that $t < \epsilon$ and $|x - b| < \epsilon$, thus justifying the claim. The construction proceeds as follows: we may assume $\epsilon < b$; choose n so large that $x_n < \epsilon$ and $t_n/x_n \leqq \epsilon/(2b)$, using (3) and (4). We may plainly then choose a positive integer m such that $|mx_n - b| <$ and $mt_n < \epsilon_n$, and we take $(t, x) = (mt_n, mx_n) \varepsilon G$.

Since H is a subsemigroup of $[0, \infty) \times (-\infty, \infty)$, we notice now that if $(t, x) \varepsilon H$, then $(t, x + y) \varepsilon H$ for all $y \geqq 0$.

Observe also that H contains points on every vertical line in $[0, \infty) \times (-\infty, \infty)$. Define $\xi(t) = \inf \{x : (t, x) \varepsilon H\}$ and $\eta(t) = \inf \{x : p_t(x) > 0\}$. We have $-\infty \leqq \xi(t) < \infty$ and $-\infty \leqq \eta(t) < \infty$. Our next task is to prove that $\xi(t) = \eta(t) = ct$ for some constant c, $-\infty \leqq c < \infty$. Firstly, since $G \subset H$, $\xi(t) \leqq \eta(t)$. If it is the case that $\eta(t) = -\infty$ for some t, then the semigroup property of G implies that $\eta(s) = -\infty$ for $s > t$, and also $\eta(t/2) = -\infty$. Repetition of the argument gives $\eta(t) \equiv -\infty$, and therefore $\xi(t) \equiv \eta(t) \equiv ct$ with $c = -\infty$. The only other possibility is that $\eta(t)$ is finite for all $t > 0$. Using once again the semigroup property of G, an easy argument leads to the result that $\eta(t + s) = \eta(t) + \eta(s)$ for all $t, s > 0$. Since G is open, there is an interval I in $(0, \infty)$ on which η is uniformly bounded above, and therefore we must have $\eta(t) = ct$ for some finite constant c. Now, G is contained in the wedge $\{(t, x) : x > ct\}$ and so H is contained in the wedge $W = \{(t, x) : x \geqq ct\}$. Thus $\xi(t) \geqq ct$, hence $\xi(t) = \eta(t) = ct$ for all t. Consequently, $H = W$.

What remains to be proven is that $p_t(x)$ can have no zeroes in the interior of the wedge W. In case $c = -\infty$, if $p_t(x) = 0$ for some (t, x), we choose (r, y) so that $r < t$ and $p_r(y) > 0$. Then, in fact, $p_s(z) > 0$ for (s, z) in a neighborhood of (r, y), and the semigroup property of G implies that $p_s(z) = 0$ in a neighborhood of $(t - r, x - y)$, contradicting the fact that $H = [0, \infty) \times (-\infty, \infty)$. In case c is finite, we have to modify the argument slightly. If $p_t(x) = 0$ and $x > ct$, the fact that $\eta(t) = ct$ implies that $p_s(z) > 0$ in some open set between the rays with slope c and x/t, and with $s < t$. The semigroup property of G then implies that $p_s(z) = 0$ in an open set lying above the ray with slope x/t and hence within the wedge W. As before, this contradicts the fact that $H = W$. Hence, $p_t(x)$ can have no zeroes in the interior of W, and the proof of the theorem is complete.

REMARKS. The theorem and its corollary are open to extension in at least two ways. Firstly, a multidimensional analogue would presumably be that the densities would be zero except on a certain family of cones. Secondly, it would be nice if one could prove that the set of zeroes of a continuous infinitely divisible density is either empty, or a closed half-line, without having to make any other assumptions. Our method of proof breaks down completely in such generality. It should be noted that another way of stating the theorem is that if $\{X_t ; t > 0\}$ is a process with stationary independent increments whose distributions have densities $p_t(x)$ jointly continuous in t and x, then if $p_t(x) = 0$ for some t and x, then there exists c such that $X_t - ct$ is either a subordinator, or the negative of a subordinator.

REFERENCE

[1] HARTMAN, P. and WINTNER, A. (1942). On the infinitesimal generators of integral convolutions. *Amer. J. Math.* **64** 273–298.