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EPSILON ENTROPY OF GAUSSIAN PROCESSES[1]

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0. Summary. This paper shows that the epsilon entropy of any mean-continuous Gaussian process on $L_2[0, 1]$ is finite for all positive ϵ. The epsilon entropy of such a process is defined as the infimum of the entropies of all partitions of $L_2[0, 1]$ by measurable sets of diameter at most ϵ, where the probability measure on L_2 is the one induced by the process. Fairly tight upper and lower bounds are found as $\epsilon \rightarrow 0$ for the epsilon entropy in terms of the eigenvalues of the process.

1. Introduction. Let $x(t)$ be a mean-continuous Gaussian process with mean zero on the unit interval. Then its covariance function $R(s, t)$ is a continuous function on the unit square and its eigenfunction expansion

$$R(s, t) = \sum_{n=1}^{\infty} \lambda_n \varphi_n(s) \varphi_n(t)$$

converges uniformly [1, p. 478]. The eigenvalues $\lambda_n = \sigma_n^2$ are non-negative numbers with $\sum \lambda_n < \infty$. The eigenfunctions $\{\phi_n(t)\}$ form an orthonormal system in $L_2[0, 1]$, and are continuous.

If we assume the process is measurable [1, p. 502], then the paths are functions in $L_2[0, 1]$, and we take $L_2[0, 1]$ as the probability space. This gives a measure on the Borel sets of $L_2[0, 1]$, which is uniquely determined by the covariance function.

One way of determing this measure is to take our process to be the sum of the Karkunen-Loeve series

$$x(t) = \sum_{n=1}^{\infty} x_n \varphi_n(t),$$

where the $\{x_n\}$ are independent Gaussian random variables, with $Ex_n = 0$, $Ex_n^2 = \lambda_n$. If we take Ω_0 to be the product space of the x_n, this series converges in $L_2[[0, 1] \times \Omega_0]$. The subset Ω of Ω_0 on which $\sum x_n^2 < \infty$ has probability 1, and is a Hilbert space under the norm $\|\{x_n\}\|^2 = \sum x_n^2$. The map $\{x_n\} \rightarrow x(t)$ is an isometry of Ω onto the subspace Ω^* of $L_2[0, 1]$ generated by the eigenfunctions. This mapping induces a measure in L_2 which is concentrated on the subspace Ω^*.

For $\epsilon > 0$, we define an ϵ-*partition* of $\bar{X} = L_2[0, 1]$ (with the given probability measure) to be a finite or denumerable collection of disjoint ϵ-sets (Borel sets of diameter $\leqq \epsilon$) which cover a subset of L_2 of measure 1. More generally, an ϵ; δ-*partition* is such a collection of sets which omits a subset of L_2 with measure no greater than δ. Let such a partition U consists of sets U_i of measures $p_i = \mu(U_i)$,

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$\sum p_i = 1$. Then the *entropy* of U is defined as the entropy of the discrete distribution $p_1, p_2, \cdots$:

$$H(U) = \sum p_i \log (1/p_i).$$

(we use logarithms with base e for convenience).

The *ϵ-entropy of $\bar{X}$*, $H_\epsilon(\bar{X})$, is the infimum of $H(U)$ over all ϵ-partitions U of $\bar{X}$. The ϵ; *δ-entropy* $H_{\epsilon;\delta}(\bar{X})$ is defined similarly as the infimum over all ϵ; δ-partitions. (If $U = \{U_i\}$ is an ϵ; δ-partition with $\mu(U_i) = p_i$, $\sum p_i = m \geqq 1 - \delta$,

$$H(U) = \sum p_i/m \log (m/p_i).$$

These concepts were introduced in a more general setting in [2]. It was shown there that $H_{\epsilon;\delta}(\bar{X})$ is finite for $\delta > 0$.

Note that any partition U can be restricted to the subspace Ω^* of $L_2[0,1]$ on which the measure is concentrated. This subspace can be identified with the Hilbert space Ω of sequences $\{x_n\}$, where the coordinates are independent Gaussian random variables. Thus the ϵ-entropy of the process depends only on the measure on Ω, and not on how Ω is embedded in $L_2[0, 1]$. It is a function only of the eigenvalues $\{\lambda_n\}$.

The purpose of these definitions is to make precise the notion of "Data Compression". Thus, $H_\epsilon(\bar{X})$ is, in a sense made precise in [3], the channel capacity needed to describe sample functions of $\bar{X}$ to within ϵ in L_2-norm with probability 1.

Reference [2] showed that for mean-continuous, but not necessarily Gaussian processes $\bar{X}$ on the unit interval, the following holds:

(1) $H_\epsilon(\bar{X})$ is finite for every $\epsilon > 0$ provided the eigenvalues λ_n of $\bar{X}$ (written as usual in non-increasing order) satisfy

$$\sum n\lambda_n < \infty.$$

(2) If, on the other hand,

$$\sum n\lambda_n = \infty,$$

then there exists a mean-continuous process $\bar{X}$ on the unit interval such that, for every $\epsilon > 0$ no matter how large, $H_\epsilon(\bar{X})$ is infinite.

One of the main results of this paper is that, if $\bar{X}$ is a Gaussian process, $H_\epsilon(\bar{X})$ is finite for every positive ϵ no matter how small, and no matter how slowly the eigenvalues λ_n approach 0 (as long, of course, as $\sum \lambda_n < \infty$). Another is that $H_\epsilon(\bar{X})$ is a continuous function of ϵ for a fixed mean-continuous Gaussian process $\bar{X}$ on the unit interval. We also find upper and lower bounds for $H_\epsilon(\bar{X})$ which are reasonably tight as $\epsilon \to 0$. These bounds are given in terms of the eigenvalues of the process.

In [4], it is shown that if the only partitions of $L_2[0, 1]$ that are allowed are products of partitions of each eigenfunction axis, then the resulting entropy, called *product epsilon entropy* need not be finite. In fact, a necessary and sufficient condition that product epsilon entropy be finite for one (or all) positive epsilon is that the "entropy of the eigenvalues"

$$\sum \lambda_n \log 1/\lambda_n$$

be finite. The reason that $H_\epsilon(\bar{X})$ is always finite for a Gaussian process when $\epsilon > 0$ is that the partitions that are used to show finiteness of $H_\epsilon(\bar{X})$ involve finite-dimensional subspaces of $L_2[0, 1]$ generated by an arbitrarily large finite number of eigenfunctions. The partitions used on these subspaces differ in an essential way from products of one-dimensional partitions, as we shall see.

We shall also need the concept of epsilon entropy for spaces other than mean-continuous Gaussian processes. The definition readily suggests itself: the epsilon entropy of a separable metric space under a probability distribution under which every open set is measurable is the infimum of the entropies of all epsilon partitions of the space, where the entropy is defined using the given probability measure.

2. Continuity of $H\epsilon(X)$. In this section, we show that if $\bar{X}$ is a mean-continuous Gaussian process and $\epsilon > 0$, then $H_\epsilon(\bar{X})$ is continuous in ϵ; we shall assume the result, to be proved later in the paper, that $H_\epsilon(\bar{X})$ is finite for every positive ϵ. Since the continuity of H_ϵ in ϵ is not used subsequently, there is no loss in the assumption.

In [2], it was shown that if the measure μ on $\bar{X}$ has no atoms, then

$$H_\epsilon(\bar{X}) \to \infty, \qquad \text{as} \quad \epsilon \to 0.$$

Since $\bar{X}$ has at least one positive eigenvalue (because we assumed that $R(s, t)$is not identically 0), μ is non-atomic. Thus, if $H_0(\bar{X})$ is interpreted as $+\infty$, $H_\epsilon(\bar{X})$ is continuous even at 0.

Continuity from above in ϵ was proved in [2]. Thus the only thing which remains to be shown here is that $H_\epsilon(\bar{X})$ is continuous from below, for $\epsilon > 0$. This is proved in Theorem 1, in a more general context: the ϵ; δ-entropy $H_{\epsilon;\delta}(\bar{X})$ is continuous from below in ϵ, for $\delta \geqq 0$. The following necessary lemma is of interest in its own right.

LEMMA 1. *If $\bar{X}$ is the Hilbert space of a mean continuous Gaussian process on the unit interval, and C a closed convex set in $\bar{X}$, any measurable subset of the set E of extreme points of C has measure zero. (That is, E has inner measure zero. Actually, it can be shown that the set E is itself measurable, but this will not be needed.)*

PROOF. Let S be a measurable subset of E, and $\chi(x)$ the characteristic function of S. The space $\bar{X}$ is the product of the one-dimensional space of its first coordinate with a Hilbert space $\bar{Y}$, with product measure. If ν_1, ν_2 denote the measures in these spaces, then by Fubini's theorem

$$\mu(S) = \int_{\bar{X}} \chi(x)\, d\mu(x) = \int_{\bar{Y}} \left[\int_{-\infty}^{\infty} \chi(x_1, y)\, d\nu_1(x_1)\right] d\nu_2(y).$$

The distribution of x_1 is continuous, and for fixed y the function $\chi(x_1, y)$ is non-zero for at most two values of x_1. Hence the inner integral is zero, and $\mu(S) = 0$. We can now state and prove Theorem 1.

THEOREM 1. *The ϵ; δ entropy of a Gaussian process on $L_2[0, 1]$ is continuous from below in ϵ.*

PROOF. Let $\bar{X}$ be the Hilbert space of the process. It is sufficient to show that for any $\alpha > 0$ there is an $\epsilon' < \epsilon$ with

$$H_{\epsilon';\delta}(\bar{X}) \leqq H_{\epsilon;\delta}(\bar{X}) + \alpha.$$

More directly, let U be an ϵ; δ-partition with entropy $H_{\epsilon;\delta}(\bar{X})$ which exists by [2]. Then it is sufficient to show that there is an ϵ'; δ-partition U' with

$$H(U') \leqq H(U) + \alpha.$$

The set covered by U has measure $1 - \delta$. The partition U' will be constructed so that it also has this property. To construct U', we will first form an ϵ'; δ'-partition V ($\delta' > \delta$) by reducing the sets of U. This partition will then be augmented by intersecting the part of $\bar{X}$ covered by U but not by V with an ϵ'-partition of $\bar{X}$.

Let η be a number between 0 and ϵ, and $W = \{W_j\}$ be an η-partition of $\bar{X}$ of finite entropy. We claim that there is a number $\beta > 0$ such that if Y is any set of measure less than β,

$$\sum \mu(Y \cap W_j)/(1 - \delta) \log ((1 - \delta)/\mu(Y \cap W_j)) < \alpha/2. \tag{1}$$

To see this, consider

$$e_j = 1/\{(1 - \delta) \min [\mu(Y), \mu(W_j)]\}.$$

If $\mu(Y)$ is sufficiently small,

$$e_j \log (1/e_j) \leqq \mu(W_j)/(1 - \delta) \log ((1 - \delta)/(\mu(W_j)),$$

hence,

$$\sum e_j \log (1/(e_j)) \to 0 \qquad \text{as} \quad \mu(Y) \to 0,$$

by dominated convergence. On the other hand,

$$1/(1 - \delta)\mu(Y \cap W_j) \leqq e_j,$$

from which the result follows. For the given β, the partition V will be made to have measure greater than $1 - \delta - \beta$.

Let $U = \{U_j\}$. Pick J so that

$$\textstyle\sum_{j=1}^{J} \mu(U_j) > 1 - \delta - \beta/2. \tag{2}$$

Let C_j be the closed convex hull of U_j, and

$$S_j = \{x \mid x \,\varepsilon\, C_j, \sup_{y \varepsilon C_j} d(x, y) = \epsilon\}.$$

S_j is clearly a closed, hence measurable, set. Furthermore, it follows easily from the parallelogram law for the metric in Hilbert space that $S_j \subset E_j$, the set of extreme points of C_j. Hence, by Lemma 1, $\mu(S_j) = 0$.

For $\rho > 0$, define

$$C_j^\rho = \{x \mid x \,\varepsilon\, C_j, \inf_{j \varepsilon S_j} d(x, y) \geqq \rho\},$$

also a closed set; $C_j^\rho = C_j$ if S_j is empty. As ρ decreases to zero, C_j^ρ increases, and

$$\lim_{\rho \to 0^+} C_j^\rho = C_j - S_j.$$

Hence,

$$\lim_{\rho\to 0+} \mu(C_j^\rho) = \mu(C_j - S_j) = \mu(C_j).$$

There is a family $\{K_\rho\}$ of compact sets with measure approaching 1 as $\rho \to 0$ ([6], page 64). If we take

$$D_i^\rho = C_i^\rho \cap K_\rho ,$$

then

$$\lim_{\rho\to 0} \mu(D_j^\rho) = \mu(C_j).$$

Since $C_j^\rho \subset C_j - S_j$, every point pair in C_j^ρ has separation less than ϵ, and D_j^ρ, a compact subset, has diameter strictly less than ϵ. Define

$$U_j^\rho = U_j \cap D_j^\rho.$$

Then U_j^ρ has diameter less than ϵ, and since $U_j - U_j^\rho \subset C_j - D_j^\rho$,

$$\lim_{\rho\to 0} \mu(U_j^\rho) = \mu(U_j).$$

Choose $\rho > 0$ so that for $j = 1, \cdots, J$

$$\mu(U_j^\rho) > \mu(U_j) - 2^{-j-1}\beta,$$

and

$$(3) \quad \mu(U_j^\rho)/(1-\delta)\log((1-\delta)/\mu(U_j^\rho)) < \mu(U_j)/(1-\delta)\log((1-\delta)/\mu(U_j)) + 2^{-j-1}\alpha.$$

Then if we define

$$\epsilon' = \max[\eta, \operatorname{diam}(U_1^\rho), \cdots, \operatorname{diam}(U_J^\rho)],$$

ϵ' is less than ϵ and $V = \{U_j^\rho, j = 1, \cdots, J\}$ is an ϵ'-partition of a part of $\bar{X}$ of probability

$$\sum_{j=1}^J \mu(U_j^\rho) > \sum_{j=1}^J \mu(U_j) - \beta/2 > 1 - \delta - \beta,$$

by (2). Let Y be the set covered by U but not by V. It has measure less than β. Hence (1) applies, and if we take

$$U' = V \cup \{Y \cap W_j\},$$

U' is an ϵ'; δ-partition covering the same set as U, with

$$\begin{aligned} H(U') &= \sum_{j=1}^J \mu(U_j^\rho)/(1-\delta)\log((1-\delta)/\mu(U_j^\rho)) \\ &\quad + \sum \mu(Y\cap W_j)/(1-\delta)\log((1-\delta)/\mu(Y\cap W_j)) \\ &< \sum_{j=1}^J \mu(U_j)/(1-\delta)\log((1-\delta)/\mu(U_j)) + \tfrac{1}{2}\alpha + \tfrac{1}{2}\alpha \\ &\leqq H(U) + \alpha, \end{aligned}$$

by (1) and (3). This completes the proof of Theorem 1.

3. Lower bounds for $H_\epsilon(\underset{\sim}{X})$. In this section, we derive some lower bounds for the ϵ-entropy of a mean-continuous Gaussian process on the unit interval.

First, note that for any ϵ-partition $U = \{U_j\}$ of $\underset{\sim}{X}$, if $U(x)$ denotes the set U_j which contains x, we have

$$H(U) = E \log \{1/\mu[U(x)]\}. \tag{4}$$

This expression is decreased if we replace $U(x)$ by the sphere of radius ϵ about x. It follows that

$$H_\epsilon(\underset{\sim}{X}) \geqq E_y \log [1/\mu\{x \mid d(x, y) \leqq \epsilon\}], \tag{5}$$

where d denotes the metric in $\underset{\sim}{X}$, and E_y indicates that the expectation is to be taken with respect to y. The first lower bound to be derived is a lower bound for the right side of (5).

First, we need an upper bound for $\mu\{x \mid d(x, y) \leqq \epsilon\}$. We obtain this upper bound from the following lemma:

LEMMA 2. *If Z is a non-negative random variable with characteristic function f, then for a and $b \geqq 0$,*

$$\Pr\{Z \leqq a\} \leqq e^{ba} f(ib).$$

PROOF. Let $F(x)$ be the distribution function of Z, and let W be a random variable with the distribution function

$$\begin{aligned}\Pr\{W \leqq x\} &= \textstyle\int_0^x e^{-by}\, dF(y) / \int_0^\infty e^{-by}\, dF(y) \\ &= 1/f(ib) \textstyle\int_0^x e^{-by}\, dF(y).\end{aligned}$$

If E is an exponential random variable independent of W, with distribution function $1 - e^{-bx}$, then

$$\begin{aligned}\Pr\{W + E \leqq a\} &= \textstyle\int_0^a [1 - e^{-b(a-x)}] 1/f(ib) e^{-bx}\, dF(x) \\ &= \Pr\{W \leqq a\} - e^{-ba}/f(ib) F(a).\end{aligned}$$

Hence,

$$\begin{aligned}F(a) &= e^{ba} f(ib)[\Pr\{W \leqq a\} - \Pr\{W + E \leqq a\}] \\ &\leqq e^{ba} f(ib),\end{aligned}$$

which proves the lemma.

The next lemma gives an upper bound for the probability of the ϵ-sphere about a fixed point y.

LEMMA 3. *Let a mean-continuous Gaussian process $\underset{\sim}{X}$ have eigenvalues $\{\lambda_n\}$. Then in the L_2 norm d, for any fixed $y \in \underset{\sim}{X}$, we have*

$$\mu\{x \mid d(x, y) \leqq \epsilon\} \leqq \inf_{b \geqq 0} e^{b\epsilon^2} / \left(\prod (1 + 2b\lambda_n)\right)^{\frac{1}{2}} \exp\left[-\sum b y_n^2/(1 + 2b\lambda_n)\right].$$

PROOF. We apply Lemma 2 with $Z = d(x, y)^2$, $a = \epsilon^2$. This gives

$$\Pr\{x \mid d(x, y) \leq \epsilon\} \leqq \inf_{b \geq 0} e^{b\epsilon^2} f(ib),$$

where

$$f(s) = E \exp [is\, d(x, y)^2] = E \exp [is \sum (x_n - y_n)^2]$$
$$= \prod E \exp [is (x_n - y_n)^2] = \prod \{(1 - 2is\lambda_n)^{-\frac{1}{2}} \exp (isy_n^2/(1 - 2is\lambda_n))\}.$$

The lemma follows by putting $s = ib$ in this expression.

Using the estimate of Lemma 3 in (5), we arrive at the lower bound

$$(6) \quad H_\epsilon(\bar{X}) \geqq E_y \sup_{b \geqq 0} \{-b\epsilon^2 + \tfrac{1}{2} \sum \log (1 + 2b\lambda_n) + \sum by_n^2/(1 + 2b\lambda_n)\}.$$

The disadvantage of this estimate is that a set of diameter ϵ containing y has been replaced by a sphere of diameter 2ϵ. Another lower bound will be derived which does not have this disadvantage. We first prove that the sphere of radius $\epsilon/2$ about the origin has at least as much probability as any set of diameter ϵ in $\bar{X}$, a result of independent interest. Actually, strict inequality can be proved but is not needed.

LEMMA 4. *Let $\bar{X}$ be the Hilbert space of a Gaussian process, and V any measurable set in $\bar{X}$ with* diam $(V) \leqq \epsilon$. *Then* $\mu(V) \leqq \mu[S_{\epsilon/2}(0)]$, *where* $S_{\epsilon/2}(0)$ *is the sphere of radius* $\epsilon/2$ *about the origin.*

PROOF. We construct a sequence of sets $V_0 = V, V_1, V_2, \cdots$ by symmetrization as follows: given V_{j-1}, we consider $\bar{X}$ as the product of the one dimensional space of the coordinate x_j and the space $Y = \{y\}$ of the other coordinates. The measure on this product space is product measure. For given y, let $\nu(y)$ be the one-dimensional Lebesgue measure of $\{x_j \mid (x_j, y) \,\varepsilon\, V_{j-1}\}$. Define

$$V_j = \{(x_j, y) \mid |x_j| < \tfrac{1}{2}\nu(y)\}.$$

It is easily shown that diam $(V_j) \leqq$ diam (V_{j-1}). Furthermore, if we write $\mu(V_j)$ as an iterated integral, integrating over x_j first, it is clear that

$$\mu(V_j) \geqq \mu(V_{j-1}).$$

Thus for all j we have

$$\text{diam } (V_j) \leqq \epsilon, \qquad \mu(V_j) \geqq \mu(V).$$

The set V_j, $j \geqq 1$, is symmetric in $x_1, \cdots, x_j$: if $(x_1, \cdots, x_j, x_{j+1}, \cdots) \,\varepsilon\, V_j$, then $\pm x_1, \cdots, \pm x_j, x_{j+1}, \cdots) \,\varepsilon\, V_j$.

Let W_j and Z_j be the cylinder sets

$$W_j = \{x \mid x_1 = y_1, \cdots, x_j = y_j \quad \text{for some} \quad y \,\varepsilon\, V_j\},$$
$$Z_j = \{x \mid x_1^2 + \cdots + x_j^2 \leqq \epsilon^2/4\}.$$

For any $x \,\varepsilon\, W_j$, there is a point of the form $(x_1, \cdots, x_j, y_{j+1}, y_{j+2}, \cdots)$ in V_j. By the symmetry of V_j,

$$(-x_1, \cdots, -x_j, y_{j+1}, y_{j+2}, \cdots) \,\varepsilon\, V_j.$$

Since diam $(V_j) \leqq \epsilon$, the distance between these two points is at most ϵ, which implies

$$x_1^2 + \cdots + x_j^2 \leqq \epsilon^2/4.$$

Hence $W_j \subset Z_j$. Also, we have $V_j \subset W_j$. Hence

$$\mu(Z_j) \geqq \mu(V_j) \geqq \mu(V).$$

On the other hand, $\mu[S_{\epsilon/2}(0)] = \lim_{j\to\infty} \mu(Z_j)$. Thus the conclusion of the lemma follows.

Applying Lemma 4 to (4), we get

$$H_\epsilon(\bar{X}) \geqq \log \{1/\mu[S_{\epsilon/2}(0)]\}. \tag{7}$$

The following theorem presents two lower bounds: $L_\epsilon(\bar{X})$, derived from (6), and $M_\epsilon(\bar{X})$, derived from (7). Note that $L_\epsilon(\bar{X})$ is always weaker. It is of interest mainly because of Theorem 4 of the next section, which bounds $H_\epsilon(\bar{X})$ from above in terms of $L_\epsilon(\bar{X})$.

THEOREM 2. *Let $\bar{X}$ be a mean-continuous Gaussian process wth eigenvalues $\{\lambda_n\}$. Define $b = b(\epsilon) \geqq 0$ by*

$$\begin{aligned} \sum \lambda_n/(1 + b\lambda_n) &= \epsilon^2, \qquad \sum \lambda_n > \epsilon^2, \\ b &= 0, \qquad \sum \lambda_n \leqq \epsilon^2. \end{aligned} \tag{8}$$

Put

$$L_\epsilon(\bar{X}) = \tfrac{1}{2} \sum \log [1 + \lambda_n b(\epsilon)] \tag{9}$$

and

$$M_\epsilon(\bar{X}) = \tfrac{1}{2} \sum \log [1 + \lambda_n b(\epsilon/2)] - \tfrac{1}{8}\epsilon^2 b(\epsilon/2). \tag{10}$$

Then

$$H_\epsilon(\bar{X}) \geqq M_\epsilon(\bar{X}) \geqq L_\epsilon(\bar{X}).$$

PROOF. From (6), we have

$$H_\epsilon(\bar{X}) \geqq E_y\{-b\epsilon^2 + \tfrac{1}{2} \sum \log (1 + 2b\lambda_n) + \sum b y_n^2/(1 + 2b\lambda_n)\},$$

for any $b \geqq 0$. Replacing b by $b/2$,

$$H_\epsilon(\bar{X}) \geqq -\tfrac{1}{2}b\epsilon^2 + \tfrac{1}{2} \sum \log (Hb\lambda_n) + \tfrac{1}{2} \sum b\lambda_n/(1 + b\lambda_n). \tag{11}$$

Take $b = b(\epsilon)$. Then this inequality reduces to $H_\epsilon(\bar{X}) \geqq L_\epsilon(\bar{X})$. This inequality also follows from the remainder of the proof.

To obtain the inequality $H_\epsilon(\bar{X}) \geqq M_\epsilon(\bar{X})$, we have by Lemma 3 (with b replaced by $\beta/2$)

$$\mu[S_{\epsilon/2}(0)] \leqq \inf_{\beta\geqq 0} \exp (\tfrac{1}{8}\beta\epsilon^2)/(\prod (1 + \beta\lambda_n))^{\frac{1}{2}}.$$

Hence by (7)

$$H_\epsilon(\bar{X}) \geqq \sup_{\beta\geqq 0} [-\tfrac{1}{8}\beta\epsilon^2 + \tfrac{1}{2} \sum \log (1 + \beta\lambda_n)].$$

This expression is maximized by setting $\beta = b(\epsilon/2)$, which proves $H_\epsilon(X) \geqq M_\epsilon(X) \geqq 0$.

Finally, to show $M_\epsilon(\bar{X}) \geqq L_\epsilon(\bar{X})$, it is sufficient to treat the case $\sum \lambda_n > \epsilon^2$, since $L_\epsilon(\bar{X}) = 0 \leqq M_\epsilon(\bar{X})$ otherwise. Then the difference is

$$\begin{aligned} D &= M_\epsilon(\bar{X}) - L_\epsilon(\bar{X}) \\ &= \tfrac{1}{2} \sum \log [(1 + \lambda_n b(\epsilon/2)]/[1 + \lambda_n b(\epsilon)] - \tfrac{1}{8}\epsilon^2 b(\epsilon/2). \end{aligned}$$

If we define $f(t) = \sum \lambda_n/(1 + t\,\lambda_n)$, and $\beta = b(\epsilon/2)$, $b = b(\epsilon)$, then

$$\begin{aligned} D &= \tfrac{1}{2} \textstyle\int_1^\beta f(t)\, dt - \tfrac{1}{2}\beta f(\beta) \\ &= -\tfrac{1}{2} \textstyle\int_b^\beta t\, df(t) - \tfrac{1}{2} b f(b) \\ &= \tfrac{1}{2} \textstyle\int_{\epsilon^2/4}^{\epsilon^2} f^{-1}(u)\, du - \tfrac{1}{2} b f(b), \end{aligned}$$

by integration by parts. From the form of $f(t)$, $tf(t)$ is a non-decreasing function. Thus for $u \leqq \epsilon^2$,

$$f^{-1}(u) \geqq \epsilon^2 f^{-1}(\epsilon^2)/u = bf(b)/u,$$

and

$$D \geqq \tfrac{1}{2} bf(b)[\textstyle\int_{\epsilon^2/4}^{\epsilon^2} du/u - 1] = \tfrac{1}{2} bf(b) \log(4/\epsilon) > 0.$$

This completes the proof of Theorem 2.

It can also be shown that $M_\epsilon(\bar{X})$ is greater than the right side of (11), for all $b > 0$, not merely for $b = b(\epsilon)$.

Next we give an improvement on the lower bound $M_\epsilon(\bar{X})$, which is difficult to use in general, but will be evaluated for special processes in Section 4. This is based on the following lemma:

LEMMA 5. *Let $x_1, \cdots, x_n$ be independent Gaussian random variables with $Ex_j = 0$, $Ex_j^2 = \lambda_j > 0$, $j = 1, \cdots, n$. Consider the n-dimensional probability space $\bar{X}$ of $x_1, \cdots, x_n$ under the Euclidian metric d. Let $a = (a_1, \cdots, a_n)$ be a fixed point of $\bar{X}$ with $d(a, 0) > \epsilon$ and $S_\epsilon(a)$ the set of points x with $d(x, a) \leqq \epsilon$. There is a translation $x \to x' = x + b$ such that for any x in $S_\epsilon(a)$ the probability density $p(x)$ satisfies the inequality*

$$p(x')/p(x) \geqq \exp\left(\tfrac{1}{2} \textstyle\sum_{k=1}^n \lambda_k a_k^2 q^2/(\epsilon + \lambda_k q)^2\right], \tag{12}$$

where q is the unique positive solution of

$$\textstyle\sum_{k=1}^n a_k^2/(\epsilon + \lambda_k q)^2 = 1. \tag{13}$$

PROOF. The density $p(x)$ has the form

$$p(x) = \exp[-\tfrac{1}{2} Q(x)],$$

where

$$Q(x) = \textstyle\sum_{k=1}^n x_k^2/\lambda_k .$$

We need to show that

$$Q(x') - Q(x) \leqq -\textstyle\sum_{k=1}^n \lambda_k a_k^2 q^2/(\epsilon + \lambda_k q)^2 \tag{14}$$

for all points of $S_\epsilon(a)$, when b is suitably chosen.

Putting $x = a + \epsilon\sigma$, $\|\sigma\| \leqq 1$, we have

$$Q(x') - Q(x) = Q(b) + 2 \sum b_k \lambda_k^{-1}(a_k + \epsilon\sigma_k).$$

Taking the maximum over all σ with $\|\sigma\| \leqq 1$

$$M(b) = \max_\sigma [Q(x') - Q(x)] = Q(b) + 2 \sum b_k a_k/\lambda_k + 2\epsilon q,$$

where $q = (\sum b_k^2/\lambda_k^2)^{\frac{1}{2}}$.
To obtain (14), choose the value of b which minimizes $M(b)$. Setting the partial derivatives of $M(b)$ equal to zero, we get

$$b_j = -q\lambda_j a_j/(\epsilon + q\lambda_j), \qquad j = 1, \cdots, n,$$

where q satisfies (13). For this choice of b,

$$M(b) = -\sum \lambda_k a_k^2 q^2/(\epsilon + \lambda_k q)^2,$$

which verifies (14) and proves Lemma 5.

The improvement to the lower bound $M_\epsilon(\bar{X})$ can now be given.

THEOREM 3. *Let $\bar{X}$ be the Hilbert space of a mean-continuous Gaussian process on* [0, 1]. *Define the non-negative random variable $q = q(x)$ by*

$$q = 0, \qquad \|x\| \leqq \epsilon,$$

and for $\|x\| > \epsilon$, by

$$\sum x_k^2/(\epsilon + \lambda_k q)^2 = 1. \tag{15}$$

where $\{\lambda_k\}$ are the eigenvalues of the process. Then

$$H_\epsilon(\bar{X}) \geqq M_\epsilon(\bar{X}) + \tfrac{1}{2} \sum E\lambda_k x_k^2 q^2/(\epsilon + \lambda_k q)^2. \tag{16}$$

PROOF. Let n be a positive integer, and for

$$\textstyle\sum_{k=1}^n x_k^2 > \epsilon^2, \tag{17}$$

define $q_n = q_n(x)$ as the positive solution of

$$\textstyle\sum_{k=1}^n x_k^2/(\epsilon + \lambda_k q_n)^2 = 1, \tag{18}$$

while $q_n = 0$ if (17) is violated. Consider any set U of diameter $\leqq \epsilon$ containing the point $x^{(0)}$. Its projection into the space of coordinates $x_1, \cdots, x_n$ lies in $S_\epsilon(x_1^{(0)}, \cdots, x_n^{(0)})$. It follows that if we translate U by applying the translation b of Lemma 5 to the first n coordinates, we get a new set V with

$$\mu(V) \geqq \mu(U) \exp \{\tfrac{1}{2} \textstyle\sum_{k=1}^n \lambda_k x_k^{(0)2} q_n(x^{(0)})^2/[\epsilon + \lambda_k q_n(x^{(0)})]^2\}.$$

By Lemma 4, $\mu(V) \leqq \mu[S_{\epsilon/2}(0)]$. Hence

$$\mu(U) \leqq \mu[S_{\epsilon/2}(0)] \exp\{-\tfrac{1}{2} \textstyle\sum_{k=1}^n \lambda_k x_k^{(0)2} q_n(x^{(0)})^2/[\epsilon + \lambda_k q_n(x^{(0)})]^2\}. \tag{19}$$

Now we shall show that $q_n \to q$ as $n \to \infty$. If $\|x\| \leqq \epsilon$, there is nothing to show, since $q_n = q = 0$. Suppose $\|x\| > \epsilon$, and let x_m be the first non-zero component of x. For any positive integer n, we have

$$\textstyle\sum_{k=1}^n x_k^2/(\epsilon + \lambda_k q)^2 = 1 - \sum_{k>n} x_k^2/(\epsilon + \lambda_k q)^2 = 1 - \delta_n, \tag{20}$$

where $\delta_n \geqq 0$, $\lim_{n\to\infty} \delta_n = 0$. For n sufficiently large, $\sum_{k=1}^{n} x_k^2 > \epsilon^2$. Then (18) holds. Comparing with (20), we see that $q_n \leqq q$, and

$$0 \leqq x_k^2/(\epsilon + \lambda_k q_n)^2 - x_k^2/(\epsilon + \lambda_k q)^2 \leqq \delta_n, \qquad k = 1, \cdots, n.$$

In particular, if $n > m$,

$$x_m^2 |(\epsilon + \lambda_m q_n)^{-2} - (\epsilon + \lambda_m q)^{-2}| \leqq \delta_n .$$

Taking the limit as $n \to \infty$, we see that $q_n \to q$. An equation similar to (20) shows that if $n' > n$, $q_{n'} \geqq q_n$, so the convergence is monotone.

For any fixed integer n_1, if $n > n_1$ we can terminate the series in (19) at n_1 (increasing the right side of the inequality):

$$\mu(U) \leqq \mu[S_{\epsilon/2}(0)] \exp\{-\tfrac{1}{2} \textstyle\sum_{k=1}^{n_1} \lambda_k x_k^{(0)2} q_n(x^{(0)})^2/[\epsilon + \lambda_k q_n(x^{(0)})]^2\}.$$

Applying this inequality to (4),

$$H_\epsilon(\underset{\sim}{X}) \geqq \log \{1/\mu[S_{\epsilon/2}(0)]\} + \tfrac{1}{2} \textstyle\sum_{k=1}^{n_1} E\lambda_k x_k^2 q_n/(\epsilon + \lambda_k q_n)^2.$$

Let $n \to \infty$. By monotone convergence,

$$H_\epsilon(\underset{\sim}{X}) \geqq \log \{1/\mu[S_{\epsilon/2}(0)]\} + \tfrac{1}{2} \textstyle\sum_{k=1}^{n_1} E\lambda_k x_k^2 q^2/(\epsilon + \lambda_k q)^2.$$

Now let $n_1 \to \infty$. As shown in the proof of Theorem 2, the first term is bounded below by $M_\epsilon(\underset{\sim}{X})$. This completes the proof of Theorem 3.

A result of Kolmogorov's [5, equation 12] implies that the ϵ-entropy has a lower bound

$$H_\epsilon(\underset{\sim}{X}) \geqq Y_\epsilon(\underset{\sim}{X}) = \tfrac{1}{2} \textstyle\sum_{n=1}^{N} \log \lambda_n \theta^{-2},$$

where N and θ are defined (for $\epsilon^2 \leqq \sum \lambda_n$) by the equation $\epsilon^2 = \sum \min(\theta^2, \lambda_n) \equiv N\theta^2 + \sum_{n \geqq N+1} \lambda_n$. A simple, but lengthy, variational argument shows that

$$L_\epsilon(\underset{\sim}{X}) \geqq Y_\epsilon(\underset{\sim}{X})$$

with equality only in the case where $\lambda_1 = \lambda_2 = \cdots \lambda_n$ and $\lambda_n = 0$ for $n > N$. (Kolmogorov's bound is actually a bound for the problem of communicating $\underset{\sim}{X}$ holding the expected square error to within ϵ^2.) In the finite-dimensional case, a result in [3] gives an even more precise lower bound for $H_\epsilon(\underset{\sim}{X})$. Hence we do not have to use Kolmogorov's bound.

4. An upper bound for $H_\epsilon(\underset{\sim}{X})$. In Theorem 4 below, we bound the ϵ-entropy of a Gaussian process from above asymptotically in terms of the quantity $L_\epsilon(\underset{\sim}{X})$ introduced in Theorem 2. The method of proof uses a special partition of $\underset{\sim}{X}$. To estimate its entropy, we need some preliminary lemmas which give bounds on the entropy of a finite dimensional Gaussian distribution. The first of these lemmas bounds the probability of being outside a spherical shell centered on the sphere of radius $n^{\frac{1}{2}}$, for the joint distribution of n independent unit normal variables.

LEMMA 6. *Let $\underset{\sim}{X}$ be the n-dimensional Euclidian space of n independent normal* *random variables of mean zero, variance* 1. *Let S be the spherical shell*

$$|n^{\frac{1}{2}} - (\sum x_j^2)^{\frac{1}{2}}| < d,$$

where $0 < d < n^{\frac{1}{2}}$, *and*

$$\nu(n, d) = 1 - \mu(S).$$

Then there is a universal constant C_1 *such that*

$$\nu(n, d) < C_1 e^{-d^2}/d.$$

PROOF. The probability density of $r = (\sum x_j^2)^{\frac{1}{2}}$ is

$$\rho(r) = 2^{1-n/2}/\Gamma(n/2) r^{n-1} e^{-\frac{1}{2}r^2}.$$

Put $r = n^{\frac{1}{2}} + s$. Applying Stirling's formula to the Gamma function shows that

$$\rho(r) = \pi^{-\frac{1}{2}} \exp(-\tfrac{1}{2}s^2 - sn^{\frac{1}{2}})(1 + sn^{-\frac{1}{2}})^{n-1}[1 + O(n^{-1})].$$

Now use the inequality $1 + x \leqq e^x$.
We have

$$\exp(-sn^{\frac{1}{2}})(1 + sn^{-\frac{1}{2}})^{n-1} \leqq \exp[-sn^{\frac{1}{2}} + sn^{-\frac{1}{2}}(n - 1)] \leqq e,$$

since $s \geqq -\sqrt{n}$. Thus $\rho(r) < \frac{1}{2}C_1 e^{-\frac{1}{2}s^2}$ for some constant C_1, and

$$\nu(n, d) = \int_{|s|>d} \rho(r)\, dr < \int_{|s|>d} \tfrac{1}{2}C_1 e^{-\frac{1}{2}s^2}\, ds < C_1 e^{-d^2}/d.$$

This proves Lemma 6.

The next lemma bounds the ϵ-entropy of the unit $(n - 1)$-sphere with the uniform probability distribution.

LEMMA 7. *Let* $\underset{\sim}{X}$ *be the unit sphere in n-dimensional Euclidian space, with a uniform probability distribution. If* β *and* γ *are positive numbers, then for* $\epsilon > 0$

$$H_\epsilon(\underset{\sim}{X}) < (1 + \beta)n \log[(2 + \gamma)/\epsilon] + C_4(\beta, \gamma),$$

where C_4 *depends only on* β *and* γ.

PROOF. For $\epsilon \geqq 2$, $H_\epsilon(\underset{\sim}{X}) = 0$. Hence we can assume $\epsilon < 2$. First consider the measure $p_n = p_n(\epsilon)$ of a cap of diameter ϵ on the sphere. By a suitable choice of coordinates, if σ_n is the area of the unit sphere in n-space,

$$p_n = \sigma_{n-1}/\sigma_n \int_0^{\epsilon/2} r^{n-2}\, dr (1 - r^2)^{-\frac{1}{2}}.$$

Set $\epsilon_0 = 2/(1 + \frac{1}{2}\gamma)$. Then, using Stirling's formula to estimate

$$\sigma_{n-1}/\sigma_n = \pi^{-\frac{1}{2}}\Gamma(n/2)/\Gamma((n - 1)/2),$$

we have

$$C_1' n^{-\frac{1}{2}}(\epsilon/2)^{n-1} < p_n < C_2' n^{-\frac{1}{2}}(\epsilon/2)^{n-1}/(1 - \epsilon^2/4)^{\frac{1}{2}},$$

or, for $\epsilon \leqq \epsilon_0$,

$$C_1' n^{-\frac{1}{2}}(\epsilon/2)^{n-1} < p_n < C_3' n^{-\frac{1}{2}}(\epsilon/2)^{n-1}, \tag{21}$$

where C_3' depends on γ.

To get a fairly efficient partition of $\bar{X}$ we can proceed as follows: Take a maximal set of points with distances $\geqq \epsilon/2$. Let there be N of these points. If we take the neighborhood of radius $\epsilon/2$ about each point, we have an ϵ-covering of $\bar{X}$, from which an ϵ-partition can be extracted. To estimate N, the number of sets, note that the caps of diameter $\epsilon/2$ centered at the N points have disjoint interiors, so that $Np_n(\epsilon/2) < 1$. Using the estimate (21) with $\epsilon/2$ for ϵ, we find

$$N < C_4' n^{\frac{1}{2}} (4/\epsilon)^{n-1}. \tag{22}$$

Unfortunately, the inequality

$$H_\epsilon(\bar{X}) \leqq \log N$$

is too weak for the conclusion of the lemma. This partition will be used in combination with another partition of part of the space.

The other partition is obtained from probabilistic considerations. If k points are independently chosen at random on the sphere, and the cap of diameter ϵ is taken about each, the measure of the set omitted by each cap is $1 - p_n$, and by the independence, $q = (1 - p_n)^k$ is the expected value of the measure of the set not covered by any of the caps. The k caps cover a set whose measure depends continuously on the positions of the k points, hence there is a way to pick these caps so that they cover a set of measure exactly $1 - q$.

Now take

$$k = \log p_n / \log (1 - p_n) + \delta,$$

with $0 \leqq \delta \leqq 1$. Then $q = p_n(1 - p_n)^\delta$. Consider the partition of the sphere into two parts Y and Z, where Y is the part of the sphere covered by the k ϵ-caps and Z the complement of Y. The part Y has an ϵ-partition consisting of k sets, while the whole sphere, hence also Z, has an ϵ-partition with N sets, where N satisfies (22). Hence, considering Z and Y as metric spaces in their own right, we have

$$H_\epsilon(Z) \leqq \log N,$$
$$H_\epsilon(Y) \leqq \log k,$$

and

$$\begin{aligned} H_\epsilon(\bar{X}) &\leqq q \log q^{-1} + (1 - q) \log (1 - q)^{-1} + (1 - q) H_\epsilon(Y) + q H_\epsilon(Z) \\ &\leqq \log 2 + \log k + p_n \log N. \end{aligned} \tag{23}$$

For $\epsilon \leqq \epsilon_0 = 2/(1 + \frac{1}{2}\gamma)$, the term

$$p_n \log N < C_3' n^{-\frac{1}{2}} (\epsilon/2)^{n-1} \{ n \log (4/\epsilon) + \tfrac{1}{2} \log n + \log C_4' \}$$

has a bound depending only on γ. To estimate the term $\log k$, note that p_n has a bound $B(\gamma) < 1$ for $\epsilon < \epsilon_0$, hence

$$-\log (1 - p_n) > C_5' p_n,$$

and

$$k < C_6' p_n^{-1} \log p_n^{-1} + \delta < C_7' p_n^{-1} \log p_n^{-1}.$$

Thus

$$\log k < \log p_n^{-1} + \log\log p_n^{-1} + \log C_7'. \tag{24}$$

For $\beta > 0$ and $p_n < B(\gamma)$, the inequality

$$\log\log p_n^{-1} < \tfrac{1}{2}\beta \log p_n^{-1} + C_8'(\beta, \gamma) \tag{25}$$

is satisfied. Furthermore,

$$\begin{aligned}\log p_n^{-1} &< (n-1)\log(2/\epsilon) + \tfrac{1}{2}\log n - \log C_1' \\ &< n\log(2/\epsilon) + \beta/(2+\beta)n\log(1+\gamma/2) + C_9'.\end{aligned}$$

Combining this inequality with (24) and (25),

$$\begin{aligned}\log k &< (1+\beta/2)\log p_n^{-1} + C_{10}' \\ &< (1+\beta/2)n\log(2/\epsilon) + \tfrac{1}{2}\beta n\log(1+\gamma/2) + C_{11}'.\end{aligned}$$

For $\epsilon \leqq \epsilon_0$, $2/\epsilon > 1 + \gamma/2$. Hence

$$\log k < (1+\beta)n\log(2/\epsilon) + C_{11}',$$

and using this inequality in (23),

$$H_\epsilon(\bar{X}) < (1+\beta)n\log(2/\epsilon) + C_4(\beta, \gamma). \tag{26}$$

The inequality (26) is only valid for $\epsilon < 2/(1+\gamma/2)$. To get an inequality which holds for $\epsilon < 2$, put

$$\epsilon' = \epsilon/(1+\gamma/2)$$

and apply (26) to $H_{\epsilon'}(\bar{X})$. Then

$$H_\epsilon(\bar{X}) \leqq H_{\epsilon'}(\bar{X}) < (1+\beta)n\log[(2+\gamma)/\epsilon] + C_4(\beta, \gamma).$$

This completes the proof of Lemma 7.

The next lemma bounds the ϵ-entropy of Euclidian n-space under the joint distribution of n independent Gaussian random variables.

Lemma 8. *Let $\bar{X}$ be the n-dimensional Euclidean space of n independent normal random variables of mean zero, and variance $\lambda_1, \cdots, \lambda_n$. Let α be a number between 0 and 1, and for $0 < (1-\alpha)\epsilon < 2(n\lambda)^{\frac{1}{2}}$ set*

$$\nu = \nu(n, (1-\alpha)\epsilon/(2\lambda^{\frac{1}{2}})),$$

where λ is the maximum of $\lambda_1, \cdots, \lambda_n$. Then there is a universal constant C_2 such that

$$H_\epsilon(\bar{X}) < (1+\beta)n\log[(2+\gamma)(n\lambda)^{\frac{1}{2}}/(\alpha\epsilon)] + n\nu\log^+[(n\lambda)^{\frac{1}{2}}/\epsilon] + C_4(\beta, \gamma) + C_2(1+n\nu),$$

if β, γ are any positive numbers, and $C_4(\beta, \gamma)$ is the constant of Lemma 7.

Proof. Replacing $\lambda_1, \cdots, \lambda_n$ by λ can only increase $H_\epsilon(\bar{X})$. Hence we can assume $\lambda_1 = \cdots = \lambda_n = \lambda$. Next, we can assume $\lambda = 1$, for the result is invariant under the change of scale $\lambda \to t\lambda$, $\epsilon^2 \to t\epsilon^2$. Let S be the shell of Lemma 6 with

$d = (1 - \alpha)\epsilon/2$, and $Y = X - S$. Then $\mu(Y) = \nu(n, d) = \nu$, by definition, and

$$(27) \quad H_\epsilon(\bar{X}) \leqq \nu \log \nu^{-1} + (1 - \nu)\log(1 - \nu)^{-1} + \nu H_\epsilon(Y) + (1 - \nu)H_\epsilon(S) \leqq H_\epsilon(S) + \nu H_\epsilon(Y) + \log 2.$$

First we estimate $H_\epsilon(Y)$. If $E_Y[\sum x_j^2] = nq$, then each coordinate has $E[x_j^2] = q$, by symmetry, and if we partition Y with a product partition formed from $\epsilon n^{-\frac{1}{2}}$-partitions of each coordinate, we see [2, Theorem 8] that

$$(28) \quad H_\epsilon(Y) \leqq n[\log[(nq)^{\frac{1}{2}}/\epsilon] + C],$$

where C is a universal constant. Now we need to estimate

$$q = E_Y(\sum x_j^2/n).$$

First, if all the points of Y within a certain distance of the origin 0 are removed the value of this expectation is increased. Leaving out all points within distance $2n^{\frac{1}{2}}$ of 0, Y becomes the region $\sum x_j^2 > 4n$ in $\bar{X}$. Hence

$$q < 1/n \int_{2n^{\frac{1}{2}}}^{\infty} r^{n+1}e^{-\frac{1}{2}r^2}\,dr \Big/ \int_{2n^{\frac{1}{2}}}^{\infty} r^{n-1}e^{-\frac{1}{2}r^2}\,dr.$$

Replacing r by $tn^{\frac{1}{2}}$

$$(29) \quad q < \int_2^\infty f(t)^n t\,dt \Big/ \int_2^\infty f(t)^n t^{-1}\,dt,$$

where

$$f(t) = te^{-\frac{1}{2}t^2}.$$

Since $f(t)$ is a decreasing function on $(2, \infty)$, the ratio in (29) is a decreasing function of n, hence bounded by its value when $n = 1$. Thus q has a bound independent of n, and (28) implies

$$(30) \quad H_\epsilon(Y) \leqq n[\log(n^{\frac{1}{2}}/\epsilon) + C'].$$

Next we estimate $H_\epsilon(S)$. Let $h_n(\eta)$ be the η-entropy of a uniform distribution on the unit sphere in n-dimensions. Then there is an η-partition of the sphere which has this entropy. If each set of the partition is extended radially between spheres $1 - \eta' < r < 1 + \eta'$, we obtain an $\eta + 2\eta'$-partition of the shell between these spheres. It follows that $h_n(\eta)$ bounds the $\eta + 2\eta'$-entropy of any spherically symmetric distribution in the shell. Set $\eta = \alpha\epsilon n^{-\frac{1}{2}}$, $\eta' = \frac{1}{2}(1 - \alpha)\epsilon n^{-\frac{1}{2}}$. After a change of scale, we get

$$H_\epsilon(S) \leqq h_n(\alpha\epsilon n^{-\frac{1}{2}}).$$

Now $h_n(\eta)$ was estimated in Lemma 7, according to which

$$H_\epsilon(S) < (1 + \beta)n \log[(2 + \gamma)n^{\frac{1}{2}}/(\alpha\epsilon)] + C_4(\beta, \gamma).$$

Combining this inequality with (30) and (27) completes the proof of Lemma 8.

An alternate upper bound is obtained in Lemma 9. The bounds of both Lemma 8 and Lemma 9 are needed in Theorem 4.

LEMMA 9. *Let $\bar{X}$ be the n-dimensional Euclidian space of n independent normal random variables of mean zero with variances $\lambda_1, \cdots, \lambda_n$, and $\lambda = \max(\lambda_1, \cdots, \lambda_n)$. There is a universal constant C_3 such that if $\epsilon > 2(n\lambda)^{\frac{1}{2}}$,*

$$H_\epsilon(\bar{X}) < C_3 n^{\frac{3}{2}}[ge^{(1-\frac{1}{2}g^2)/2}]^n,$$

where $g = \epsilon/(2(n\lambda)^{\frac{1}{2}})$.

PROOF. As in the proof of the preceding lemma, we can assume $\lambda_1 = \cdots = \lambda_n = \lambda = 1$. Then $r = (\sum x_j^2)^{\frac{1}{2}}$ has density

$$\rho(r) = 2^{1-n/2}/\Gamma(n/2) r^{n-1} e^{-\frac{1}{2}r^2},$$

and the probability that $r > \epsilon/2$ is

$$p = 2^{1-n/2}/\Gamma(n/2) \int_{\epsilon/2}^{\infty} r^{n-1} e^{-\frac{1}{2}r^2}\, dr.$$

Put

$$\epsilon = 2g(n)^{\frac{1}{2}}$$

and assume $g > 1$. After the substitution $r = tn^{\frac{1}{2}}$, the coefficient of the integral can be estimated by Stirling's formula, and we have

$$p < Cn^{\frac{1}{2}} e^{n/2} \int_g^\infty t^{n-1} e^{-\frac{1}{2}nt^2}\, dt.$$

Since the function $te^{-\frac{1}{2}t^2}$ is decreasing for $t > g$,

$$p < Cn^{\frac{1}{2}} e^{n/2} (ge^{-\frac{1}{2}g^2})^{n-1} \int_g^\infty e^{-\frac{1}{2}t^2}\, dt < C' n^{\frac{1}{2}} e^{n/2} g^{n-2} e^{-\frac{1}{2}ng^2}. \tag{31}$$

Now we proceed as in the proof of Lemma 7, letting Z be the set $r \leqq \epsilon/2$ and Y the set $r > \epsilon/2$. The diameter of Z is ϵ. Hence its ϵ-entropy is zero, and

$$H_\epsilon(\bar{X}) \leqq p \log p^{-1} + (1-p)\log(1-p)^{-1} + pH_\epsilon(Y). \tag{32}$$

To estimate $H_\epsilon(Y)$ we again need to bound $nq = E_Y(r^2)$. This q is given by a formula similar to (29), in which we can replace n by 1 to get an upper bound:

$$q \leqq \int_g^\infty t^2 e^{-\frac{1}{2}t^2}\, dt / \int_g^\infty e^{-\frac{1}{2}t^2}\, dt < Cg^2.$$

Thus as in (28)

$$H_\epsilon(Y) \leqq n[\log[(nq)^{\frac{1}{2}}/\epsilon] + C] < nC''.$$

Combining this bound with (31) and (32),

$$\begin{aligned} H_\epsilon(\bar{X}) &\leqq p[1 + \log(1/p) + nC''] \\ &\leqq Cn^{\frac{1}{2}} e^{n/2} g^{n-2} e^{-\frac{1}{2}ng^2}[\tfrac{1}{2}ng^2 - (n-2)\log g - \tfrac{1}{2}\log n + nC''] \\ &< C_3 n^{\frac{3}{2}} g^n e^{\frac{1}{2}n - \frac{1}{2}ng^2}. \end{aligned}$$

Lemma 9 is proved.

We are at last ready to state and prove the upper bound of Theorem 4.

THEOREM 4. *Let m be any positive number less than $\frac{1}{2}$. Then*

$$H_\epsilon(\bar{X}) \leqq L_{m\epsilon}(\bar{X})[1 + o(1)]$$

as $\epsilon \to 0$. In particular, $H_\epsilon(\bar{X})$ is finite for $\bar{X}$ a mean-continuous Gaussian process on the unit interval and $\epsilon > 0$.

PROOF. The idea of the proof is as follows: For any $\delta > 0$, $\bar{X}$ will be broken up as the product of a sequence of finite-dimensional spaces $\{\bar{X}_k\}$ in a way which depends on δ as well as on ϵ, so that, for the optimum product partition U,

$$H(U) \leqq (1 + \delta)L_{m\epsilon}(\bar{X})[1 + o(1)].$$

The meshes $\{\epsilon_k\}$ of the component partitions are suggested by the series (8). The most natural product partitions to try are one-dimensional product partitions, where we take

$$\epsilon_k^2 = A^2\lambda_k/(1 + b\lambda_k) \tag{33}$$

for the partition of the kth coordinate. It turns out that this does not always work. In fact, if the eigenvalues decrease slowly enough, there are no one-dimensional product ϵ-partitions with finite entropy [4], even if $\sum \lambda_k$ is finite. However, for small ϵ, this is the best way to handle the large eigenvalues, and there is a first range of k in which one-dimensional subspaces are used. Beyond this point, the dimensions of the subspaces are consecutive integers beginning with 1. This sequence of subspaces is also split up into two ranges; up to a certain point, the entropy of the subspace is estimated by Lemma 8. Beyond this point, Lemma 9 is applied.

The ϵ-entropy of a 1-dimensional Gaussian distribution of variance λ is a function $h(\epsilon\lambda^{-1})$ with the property that

$$h(\epsilon) \sim \log (1/\epsilon)$$

as $\epsilon \to 0$ ([4], Lemma 7). Thus, there is a number $\delta_2 < 1$ such that

$$h(\epsilon\lambda^{-\frac{1}{2}}) < (1 + \delta) \log (\lambda^{\frac{1}{2}}/\epsilon)$$

if $\epsilon < \delta_2\lambda^{\frac{1}{2}}$.

Let A be a constant with

$$2 < A < 1/m, \tag{34}$$

and define M by

$$A^2/(1 + M) = \delta_2^2. \tag{35}$$

Then for k such that

$$b\lambda_k > M, \tag{36}$$

if we define ϵ_k by (33), the ϵ_k-entropy h_k of the kth coordinate satisfies the inequality

$$h_k < (1 + \delta) \log (\lambda_k^{\frac{1}{2}}/\epsilon_k) = \tfrac{1}{2}(1 + \delta)[\log (1 + b\lambda_k) - 2 \log A],$$

so

$$(37) \quad h_k < (1 + \delta)\tfrac{1}{2} \log (1 + b\lambda_k).$$

Let (36) be true for $k \leqq k_1$. Group the coordinates $x_{k_1+1}, x_{k_1+2}, \cdots$ into spaces $\bar{X}_{k_1+1}, \bar{X}_{k_1+2}, \cdots$, where for $j > k_1$, $\bar{X}_j$ is the space of coordinates x_k with

$$k_1 + \tfrac{1}{2}(j - k_1)(j - k_1 - 1) < k \leqq k_1 + \tfrac{1}{2}(j - k_1 + 1)(j - k_1).$$

The space $\bar{X}_j$ has dimension $n_j = j - k_1$.

For $j > k_1$, let $\bar{\lambda}_j$ be the maximum eigenvalue of the coordinates in $\bar{X}_j$. Define ϵ_j by

$$(38) \qquad \epsilon_j^2 = A^2 n_j \bar{\lambda}_j/(1 + b\bar{\lambda}_j).$$

Next, pick K, α, and γ with $\gamma > 0$, $0 < \alpha < 1$, such that

$$(39) \qquad 2 < 2K < (2 + \gamma)/\alpha < A.$$

Define j_2 to be the last value of $j\,(>k_1)$ for which

$$(40) \qquad (2 + \gamma)(n_j\bar{\lambda}_j)^{\frac{1}{2}}/(\alpha\epsilon_j) > K,$$

if such values exist. By (38) and (39),

$$\lim_{j\to\infty} (2 + \gamma)(n_j\bar{\lambda}_j)^{\frac{1}{2}}/(\alpha\epsilon_j) = (2 + \gamma)/(\alpha A) < 1 < K,$$

so that (40) is eventually violated. If (40) is never satisfied, put $j_2 = k_1$. Since the left side of (40) is monotonic in j, (40) is satisfied whenever $k_1 < j \leqq j_2$.

First we treat the subspaces $\bar{X}_j$ with $j > j_2$. Here

$$\epsilon_j \geqq 2g(n_j\bar{\lambda}_j)^{\frac{1}{2}},$$

where

$$g = (2 + \gamma)/(2\alpha K) > 1,$$

by (39). Applying Lemma 9,

$$h_j = h_{\epsilon_j}(\bar{X}_j) < C_3 n_j^{\frac{3}{2}}(ge^{\frac{1}{2}-\frac{1}{2}g^2})^{n_j}.$$

We have

$$S_1 = \textstyle\sum_{n=1}^{\infty} n^{\frac{3}{2}}(ge^{\frac{1}{2}-\frac{1}{2}g^2})^n < \infty,$$

since $ge^{\frac{1}{2}-\frac{1}{2}g^2} < 1$. The n_j are distinct positive integers. Hence

$$(41) \qquad \textstyle\sum_{j=j_2+1}^{\infty} h_j < C_3 S_1.$$

For $k_1 < j \leqq j_2$, we apply Lemma 8 with $\beta = \delta/2$. For this we require that

$$(42) \qquad (1 - \alpha)(2 + \gamma) < \alpha K.$$

This inequality, together with (40), ensures that $(1 - \alpha)\epsilon_j < (n_j\bar{\lambda}_j)^{\frac{1}{2}}$ in this range. Then

$$(43) \quad h_j < (1 + \delta/2)n_j \log^+ [(2 + \gamma)(n_j\bar{\lambda}_j)^{\frac{1}{2}}/(\alpha\epsilon_j)] + n_j\nu_j \log^+ [(n_j\bar{\lambda}_j)^{\frac{1}{2}}/\epsilon_j] + C_4(\delta/2, \gamma) + C_2(1 + n_j\nu_j),$$

where

$$(44) \quad \nu_j = \nu(n_j, (1 - \alpha)\epsilon_j/(2\bar{\lambda}_j^{\frac{1}{2}})) < 2C_1\bar{\lambda}_j^{\frac{1}{2}}/((1 - \alpha)\epsilon_j) \exp [-(1 - \alpha)^2\epsilon_j^2/(4\bar{\lambda}_j)],$$

by Lemma 6.

The eigenvalues in this range violate (36). Hence $b\bar{\lambda}_j \leqq M$, and by (38)

$$(45) \quad (n_j\bar{\lambda}_j)^{\frac{1}{2}}/\epsilon_j = A^{-1}(1 + b\bar{\lambda}_j)^{\frac{1}{2}} \leqq A^{-1}(1 + M)^{\frac{1}{2}} = M'.$$

Applying this inequality in (44),

$$(46) \quad n_j\nu_j < 2C_1M'/(1 - \alpha)n_j^{\frac{1}{2}}.$$

From (45) and (46), the terms after the first on the right side of (43) have a bound of the form $C_6n_j^{\frac{1}{2}}$, where C_6 depends on the constants which have been introduced. Choose C_7 so that

$$C_6n^{\frac{1}{2}} < 2^{-n}C_7 + \tfrac{1}{2}\delta n \log K, \qquad n \geqq 1.$$

By (40), the first logarithm on the right in (43) is greater than $\log K$. Hence

$$\begin{aligned} h_j &< (1 + \delta)n_j \log^+ [(2 + \gamma)(n_j\bar{\lambda}_j)^{\frac{1}{2}}/(\alpha\epsilon_j)] + 2^{-n_j}C_7 \\ &= (1 + \delta)n_j[\tfrac{1}{2} \log (1 + b\bar{\lambda}_j) + \log ((2 + \gamma)/(\alpha A))] + 2^{-n_j}C_7 \\ &< (1 + \delta)\tfrac{1}{2}n_j \log (1 + b\bar{\lambda}_j) + 2^{-n_j}C_7 \end{aligned}$$

by (39). Summing,

$$(47) \quad \textstyle\sum_{j=k_1+1}^{j_2} h_j < (1 + \delta) \sum_{j=k_1+1}^{j_2} \tfrac{1}{2}n_j \log (1 + b\bar{\lambda}_j) + C_7 .$$

If $\epsilon'^2 = \sum_1^\infty \epsilon_j^2$, $H_{\epsilon'}(\bar{X}) \leqq \sum_{j=1}^\infty h_j$. Combining (37), (41), and (47),

$$(48) \quad H_{\epsilon'}(\bar{X}) \leqq (1 + \delta)\{\tfrac{1}{2} \textstyle\sum_{k=1}^{k_1} \log (1 + b\lambda_k) + \tfrac{1}{2} \sum_{j=k_1+1}^{j_2} n_j \log (1 + b\bar{\lambda}_j)\} + C_3S_1 + C_7 .$$

It remains to determine the relation between ϵ' and ϵ, and to compare the expression in braces in (48) with $L_\epsilon(\bar{X})$. In this it is crucial that $b\bar{\lambda}_j$ is bounded for $j > k_1$.

First we consider

$$L_\epsilon' = \tfrac{1}{2} \textstyle\sum_{k=1}^{k_1} (1 + b\lambda_k) + \tfrac{1}{2} \sum_{j=k_1+1}^{j_2} n_j \log (1 + b\bar{\lambda}_j).$$

Since (36) is violated for $j > k_1$, if r is any positive integer

$$(49) \quad \textstyle\sum_{j=k_1+1}^{\infty} n_j \log (1 + b\bar{\lambda}_j) \leqq r \log (1 + M) + (r + 1)/r \sum_{j=k_1+2}^{\infty} (n_j - 1) \log (1 + b\bar{\lambda}_j).$$

Here $\bar{\lambda}_j$ is the first eigenvalue in $\bar{X}_j$, hence it is no larger than any eigenvalue in $\bar{X}_{j-1}$. For $j \geqq k_1 + 2$, $\bar{X}_{j-1}$ has dimension $n_j - 1$, hence

$$(n_j - 1) \log (1 + b\bar{\lambda}_j) \leqq \sum \log (1 + b\lambda_k),$$

where the summation is taken over x_k in $\bar{X}_{j-1}$. Applying this inequality in (49),

$$L_\epsilon' \leqq (r + 1)/rL_\epsilon(\bar{X}) + r \log (1 + M),$$

and (48) reduces to

$$\begin{aligned} H_{\epsilon'}(\bar{X}) &\leqq (1 + \delta)(r + 1)/r[L_\epsilon(\bar{X}) + r \log (1 + M)] + C_3S_1 + C_7 \\ &= (1 + \delta)(r + 1)/rL_\epsilon(\bar{X})[1 + o(1)]. \end{aligned}$$

Letting $r \to \infty$,

(50) $$H_{\epsilon'}(\bar{X}) \leqq (1 + \delta)L_\epsilon(\bar{X})[1 + o(1)].$$

Now we consider the value of ϵ'. We have

$$\epsilon'^2 = A^2\{\textstyle\sum_{k=1}^{k_1} \lambda_k/(1 + b\lambda_k) + \sum_{j=k_1+1}^{\infty} n_j\bar{\lambda}_j/(1 + b\bar{\lambda}_j)\}.$$

This is to be compared with the series (8) for ϵ^2. For any positive integer r,

$$b\epsilon'^2 \leqq A^2\{\textstyle\sum_{k=1}^{k_1} b\lambda_k/(1 + b\lambda_k) + r + (r + 1)/r \sum_{j=k_1+2}^{\infty} (n_j - 1)b\bar{\lambda}_j(/1 + b\bar{\lambda}_j)\},$$

since $b\bar{\lambda}_j/(1 + b\bar{\lambda}_j) < 1$. Comparing the jth term in the second series with the terms for $\bar{X}_{j-1}$ in (8), we see as above that

$$b\epsilon'^2 \leqq A^2[(r + 1)/rb\epsilon^2 + r] = A^2(r + 1)/rb\epsilon^2[1 + o(1)],$$

since $b\epsilon^2 \to \infty$ as $\epsilon \to 0$. Letting $r \to \infty$, we have

$$\epsilon' \leqq A\epsilon[1 + o(1)].$$

The constant A was selected to satisfy (34). Hence if ϵ is sufficiently small $\epsilon' < \epsilon/m$, and by (50)

$$H_{\epsilon/m}(\bar{X}) \leqq H_{\epsilon'}(\bar{X}) \leqq (1 + \delta)L_\epsilon(\bar{X})[1 + o(1)].$$

Here ϵ can be any positive number less than $(\sum \lambda_k)^{\frac{1}{2}}$. Replacing ϵ by $m\epsilon$, we have

$$H_\epsilon(\bar{X}) \leqq (1 + \delta)L_{m\epsilon}(\bar{X})[1 + o(1)].$$

Letting $\delta \to 0$ completes the proof of desired inequality. In particular, we conclude that $H_\epsilon(\bar{X})$ is finite. Theorem 4 is proved.

5. Entropy of special processes; the Wiener process. By the Wiener process, we mean that Gaussian process on [0, 1] which has covariance function $R(s, t) = \min (s, t)$, and

(51) $$\lambda_n = \pi^{-2}(n - \tfrac{1}{2})^{-2}, \qquad n = 1, 2, \cdots.$$

This can be treated as a special case of a more general process, such as the solutions of finite-order stochastic differential equations, with

(52) $$\lambda_n \sim An^{-p}, \qquad p > 1.$$

We first estimate $L_\epsilon(\bar{X})$ and $M_\epsilon(\bar{X})$ for such processes, to get the upper and lower bounds of Theorem 2 and 4. Then the lower bound of Theorem 3 will be treated to obtain the best bounds we know of for this class of processes.

We need to find the asymptotic behavior of b as a function of ϵ, given (52) and

$$\sum \lambda_n/(1 + b\lambda_n) = \epsilon^2.$$

Note that $b \to \infty$ as $\epsilon \to 0$. If A_1 is any number greater than A, $\lambda_n \leqq A_1 n^{-p}$ except for a finite number of values of n. Hence

$$\epsilon^2 < \sum_{n=1}^{\infty} A_1 n^{-p}/(1 + bA_1 n^{-p}) + O(b^{-1}).$$

It is easily shown that as $b \to \infty$,

$$\sum_{n=1}^{\infty} A_1 n^{-p}/(1 + bA_1 n^{-p}) \sim \int_0^\infty A_1 t^{-p}\, dt/(1 + bA_1 t^{-p})$$
$$= A_1^{p^{-1}} b^{p^{-1}-1}\, \pi/(p \sin(\pi/p)).$$

Hence

$$\epsilon^2 < A_1^{p^{-1}} b^{p^{-1}-1} \pi/(p \sin(\pi/p))[1 + o(1)].$$

Similarly, if $A_1 < A$ the reverse inequality holds. It follows that

$$\epsilon^2 \sim A^{p^{-1}} b^{p^{-1}-1} \pi/(p \sin(\pi/p)),$$

or

$$b(\epsilon) \sim A^{(p-1)^{-1}} [\pi/(p \sin(\pi/p))]^{p/(p-1)} \epsilon^{-2p/(p-1)}. \tag{53}$$

The same type of reasoning applies to the series for $L_\epsilon(\bar{X})$. We have by (9)

$$L_\epsilon(\bar{X}) = \tfrac{1}{2} \sum \log(1 + b\lambda_n)$$
$$\sim \tfrac{1}{2} \int_0^\infty \log(1 + bAt^{-p})\, dt$$
$$= (bA)^{1/p} \pi/(2 \sin(\pi/p)).$$

Using (53),

$$L_\epsilon(\bar{X}) \sim B_1 \epsilon^{-2/(p-1)},$$

where

$$B_1 = \tfrac{1}{2} p A^{(p-1)^{-1}} (\pi/(p \sin(\pi/p)))^{p/(p-1)}. \tag{54}$$

In applying Theorem 4, the rate of growth of $L_\epsilon(\bar{X})$ is sufficiently small that we can put $m = \frac{1}{2}$. Thus Theorem 4 gives us

$$H_\epsilon(\bar{X}) \gtrsim 2^{2/(p-1)} B_1 \epsilon^{-2/(p-1)}. \tag{55}$$

Now $M_\epsilon(\bar{X})$ can be quickly evaluated. From (9), (10), and (53),

$$M_\epsilon(\bar{X}) = L_{\epsilon/2}(\bar{X}) - \tfrac{1}{8}\epsilon^2 b(\epsilon/2)$$
$$\sim L_{\epsilon/2}(\bar{X}) - 1/p 2^{2/(p-1)} B_1 \epsilon^{-2/(p-1)},$$

and

$$H_\epsilon(\bar{X}) \geqq M_\epsilon(\bar{X}) \sim (p-1)/p2^{2/(p-1)}B_1\epsilon^{-2/(p-1)}. \tag{56}$$

In examining the lower bound of Theorem 3, we first prove a general lemma which applies to any Gaussian process for which the eigenvalues do not decrease too rapidly. It says that in some sense the random variable q behaves like the deterministic function $r = r(\epsilon)$ which is the positive solution of

$$\sum \lambda_n/(\epsilon + \lambda_n r)^2 = 1 \tag{57}$$

when $\epsilon^2 < \sum \lambda_n$. This can be made precise when the eigenvalues satisfy (52).

LEMMA 10. *Let the eigenvalues* $\{\lambda_n\}$ *(in non-increasing order) of a mean-continuous Gaussian process* $\bar{X}$ *have the following property: There is a sequence* $n_1 < n_2 < \cdots$ *such that*

$$(n_{k+1} - n_k)/\log k \to \infty \tag{58}$$

and

$$\lambda_{n_{k+1}}/\lambda_{n_k} \to 1 \tag{59}$$

as $k \to \infty$. *Let* δ *be given with* $0 < \delta < 1$. *Then for* ϵ *sufficiently small, and* q *as defined in Theorem* 3, *equation* (15), *we have*

$$\left|\sum x_k^2/(\epsilon + \lambda_k q)^2 \Big/ \sum \lambda_k/(\epsilon + \lambda_k q)^2 - 1\right| < \delta \tag{60}$$

and

$$\left|\sum \lambda_k x_k^2 q^2/(\epsilon + \lambda_k q)^2 \Big/ \sum \lambda_k^2 q^2/(\epsilon + \lambda_k q)^2 - 1\right| < \delta \tag{61}$$

except on a set of x *of probability less than* δ.

PROOF. Consider

$$S_k = \sum_{n=n_k+1}^{n_{k+1}} x_n^2/(\epsilon + \lambda_n q)^2.$$

Let $x_n = y_n \lambda_n^{\frac{1}{2}}$. Then

$$S_k = \sum_{n=n_k+1}^{n_{k+1}} \lambda_n y_n^2/(\epsilon + \lambda_n q)^2 \leqq \lambda_{n_k}/(\epsilon + \lambda_{n_{k+1}} q)^2 \sum_{n=n_k+1}^{n_{k+1}} y_n^2. \tag{62}$$

Set $n_{k+1} - n_k = m_k$, and let d_k be a positive number less than $m_k^{\frac{1}{2}}$, to be determined later. By Lemma 6, the last sum in (62) is less than $(m_k^{\frac{1}{2}} + d_k)^2$, except on a set of probability less than $C_1 e^{-d_k^2}/d_k$. Then

$$\begin{aligned} S_k &\leqq \lambda_{n_k} m_k/(\epsilon + \lambda_{n_{k+1}} q)^2 (1 + d_k m_k^{-\frac{1}{2}})^2 \\ &\leqq (\lambda_{n_k}/\lambda_{n_{k+1}})^3 (1 + d_k m_k^{-\frac{1}{2}})^2 \sum_{n=n_k+1}^{n_{k+1}} \lambda_n/(\epsilon + \lambda_n q)^2. \end{aligned}$$

If δ_1 is any positive number, then, by (58) and (59), if j is a sufficiently large integer, we have

$$\log k < \tfrac{1}{2}\delta_1^2 m_k$$

and also

$$\lambda_{n_k} < (1 + \delta_1)\lambda_{n_{k+1}}$$

for all $k \geqq j$. Make $\delta_1 < 1$ and $j \geqq 2$. For $k \geqq j$, put $d_k = (2 \log k)^{\frac{1}{2}}$. Then

$$S_k \leqq (1 + \delta_1)^5 \sum_{n=n_k+1}^{n_{k+1}} \lambda_n/(\epsilon + \lambda_n q)^2$$

except on a set of probability less than $C_1 e^{-d_k^2}/d_k$, and, summing for $k \geqq j$, we have

$$(63) \qquad \sum_{n=n_j+1}^{\infty} x_n^2/(\epsilon + \lambda_n q)^2 \leqq (1 + \delta_1)^5 \sum_{n=n_j+1}^{\infty} \lambda_n/(\epsilon + \lambda_n q)^2,$$

except on a set of probability less than

$$p_j = C_1 \sum_{k=j}^{\infty} e^{-d_k^2}/d_k = C_1 \sum_{k=j}^{\infty} k^{-2}(2 \log k)^{-\frac{1}{2}}.$$

Similarly,

$$(64) \quad \sum_{n=n_j+1}^{\infty} x_n^2/(\epsilon + \lambda_n q)^2 \geqq (1 + \delta_1)^{-3}(1 - \delta_1)^2 \sum_{n=n_j+1}^{\infty} \lambda_n/(\epsilon + \lambda_n q)^2$$

except on the same set of probability less than p_j.

From (64) and (15),

$$(65) \qquad \sum_{n=n_j+1}^{\infty} \lambda_n/(\epsilon + \lambda_n q)^2 \leqq (1 + \delta_1)^3(1 - \delta_1)^{-2},$$

which shows that as $\epsilon \to 0$, $q \to \infty$ uniformly, off the exceptional set. For the terms with $n \leqq n_j$, we have

$$S_0 = \sum_{n=1}^{n_j} x_n^2/(\epsilon + \lambda_n q)^2 \leqq \lambda_1/(\lambda_{n_j}^2 q^2) \sum_{n=1}^{n_j} y_n^2.$$

Applying Lemma 6 with $d = j$,

$$(66) \qquad S_0 < \lambda_1/(\lambda_{n_j}^2 q^2)(n_j^{\frac{1}{2}} + j)^2,$$

except on a set of probability less than $q_j = C_1 e^{-j^2}/j$. For the other series,

$$(67) \qquad S_0' = \sum_{n=1}^{n_j} \lambda_n/(\epsilon + \lambda_n q)^2 < n_j \lambda_1/(\lambda_{n_j}^2 q^2).$$

Now we can show (60). Take δ_1 so small that

$$(1 + \delta_1)^5 < 1 + \delta/4,$$
$$(1 + \delta_1)^{-3}(1 - \delta_1)^2 > 1 - \delta/4.$$

Next, make j sufficiently large that $p_j + q_j < \delta/2$. Off the exceptional set of probability $<p_j + q_j$, $q \to \infty$ uniformly as $\epsilon \to 0$. By (66) and (67), if ϵ is sufficiently small

$$S_0 < \delta/4,$$
$$S_0' < \delta/4.$$

By (65),

$$\sum_{n=1}^{\infty} \lambda_n/(\epsilon + \lambda_n q)^2 < 1 + \delta/2 < (1 - \delta)^{-1}.$$

By (63) and (15),

$$\sum_{n=1}^{\infty} \lambda_n/(\epsilon + \lambda_n q)^2 > (1 + \delta/4)^{-1} \sum_{n=n_j+1}^{\infty} x_n^2/(\epsilon + \lambda_n q)^2$$
$$> (1 - \delta/4)/(1 + \delta/4) > (1 + \delta)^{-1}.$$

Hence, (60) is true except on a set of probability less than $\delta/2$.

Similarly we can prove (61). The difference in the treatment of the initial terms in the two series is that here we know the series become infinite as $\epsilon \to 0$, while the sum of the first n_j terms is bounded (off an exceptional set of probability less than $\delta/2$). Lemma 10 is proved.

Now we shall apply this lemma and Theorem 3 to processes satisfying (52).

THEOREM 5. *If a mean-continuous Gaussian process* $\bar{X}$ *has eigenvalues* $\lambda_n \sim An^{-p}$, $p > 1$, *then*

$$H_\epsilon(\bar{X}) \gtrsim A^{(p-1)^{-1}}(\pi/(p \sin (\pi/p)))^{p/(p-1)}(p-1)/2 \cdot [2^{2/(p-1)} + p^{-p/(p-1)}]\epsilon^{-2/(p-1)}. \tag{68}$$

PROOF. First we use Lemma 10 to estimate the last term of (16). On a set of measure $1 - \delta$ we have, for ϵ sufficiently small

$$\sum \lambda_k/(\epsilon + \lambda_k q)^2 < (1-\delta)^{-1}.$$

This sum is asymptotically equal to an integral as $q/\epsilon \to \infty$:

$$\sum \lambda_k/(\epsilon + \lambda_k q)^2 \sim \int_0^\infty At^{-p}\, dt/(\epsilon + Aqt^{-p})^2 = A^{p^{-1}}q^{p^{-1}-1}\epsilon^{-p^{-1}-1}\pi/(p^2 \sin (\pi/p)).$$

Hence

$$q \gtrsim A^{(p-1)^{-1}}[\pi(1-\delta)/(p^2 \sin (\pi/p))]^{p/(p-1)}\epsilon^{-(p+1)/(p-1)}.$$

Also, we have

$$\begin{aligned}\sum \lambda_k^2 q^2/(\epsilon + \lambda_k q)^2 &\sim \int_0^\infty A^2q^2t^{-2p}(\epsilon + Aqt^{-p})^{-2}\, dt \\ &= (Aq/\epsilon)^{1/p}(p-1)\pi/(p^2 \sin (\pi/p)) \\ &\gtrsim [A(1-\delta)]^{(p-1)^{-1}}(p-1)[\pi/(p^2 \sin (\pi/p))]^{p/(p-1)}\epsilon^{-2/(p-1)},\end{aligned}$$

off the exceptional set. Then by (61),

$$\sum \lambda_k x_k^2 q^2/(\epsilon + \lambda_k q)^2 > (1-\delta)\sum \lambda_k^2 q^2/(\epsilon + \lambda_k q)^2 \gtrsim (1-\delta)^{p/(p-1)}B_2\epsilon^{-2/(p-1)},$$

where

$$B_2 = A^{(p-1)^{-1}}(p-1)[\pi/(p^2 \sin (\pi/p))]^{p/(p-1)}.$$

This asymptotic inequality holds uniformly on a set of measure at least $1 - \delta$. Hence

$$E \sum \lambda_k x_k^2 q^2/(\epsilon + \lambda_k q)^2 \gtrsim (1-\delta)^{1+p/(p-1)}B_2\epsilon^{-2/(p-1)},$$

and letting $\delta \to 0$,

$$\tfrac{1}{2}E \sum \lambda_k x_k^2 q^2/(\epsilon + \lambda_k q)^2 \gtrsim \tfrac{1}{2}B_2\epsilon^{-2/(p-1)}.$$

Using this estimate for the last term of (16), together with the asymptotic form (56) of $M_\epsilon(\bar{X})$, we obtain (68), and prove Theorem 5.

COROLLARY. *For the Wiener process,*

$$17/(32\epsilon^2) \lesssim H_\epsilon(\underline{\bar{X}}) \lesssim \epsilon^{-2}.$$

PROOF. The lower bound results from putting $p = 2$, $A = \pi^{-2}$ in (68). The upper bound is (55) for this special case. This proves the corollary.

We close the paper with the remark that there is no Gaussian process $\underline{\bar{X}}$ for which we know that $L_{\epsilon/2}(\underline{\bar{X}})$ is not asymptotic to $H_\epsilon(\underline{\bar{X}})$ as $\epsilon \to 0$. Resolution of this question would be extremely interesting.

REFERENCES

[1] LOÉVE, MICHEL, *Probability Theory*, Van Nostrand, New York, 1955.

[2] POSNER, EWARD C., RODEMICH, EUGENE, and RUMSEY, HOWARD, JR. (1967) Epsilon entropy of stochastic processes, *Ann. Math. Statist.* **38** 1000–1020.

[3] POSNER, EDWARD C. and RODEMICH, EUGENE R. Epsilon entropy and data compression. In preparation.

[4] POSNER, EDWARD C., RODEMICH, EUGENE R., and RUMSEY, HOWARD, JR. Product entropy of gaussian distributions. To appear in Ann. Math. Statist.

[5] KOLMOGOROV, ANDREI N. (1956). The Shannon theory of information transmission in the case of continuous signals. *IRE Trans. Information Theory.* **IT-2** 102–108.

[6] NEVEU, JACQUES. (1965). *Mathematical Foundations of the Calculus of Probability*, Holden-Day, San Francisco.