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MOMENTS OF A STOPPING RULE RELATED TO THE CENTRAL LIMIT THEOREM[1]

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0. Introduction. The main objective of this work has been to try and solve the following problem, which was introduced by Blackwell and Freedman and made the subject of further work by Chow and Teicher, Breiman, and Gundy and Siegmund.

Let $S_n = X_1 + \cdots + X_n$ be the sum of independent random variables $\{X_n\}$ with $EX_n = 0$, $EX_n^2 = 1$, $n = 1, 2, \cdots$, with the stopping time $t_m(c) = \inf \{n : n \geqq m, |S_n| > cn^{\frac{1}{2}}\}$, $0 < c < \infty$, $m = 1, 2, \cdots$

(i) Is $P[t_m(c) < \infty] = 1$? (i.e. is $t_m(c)$ a stopping *rule*?)

(ii) For fixed $k(=1, 2, \cdots)$, is $Et_m^k(c) < \infty$?

We have sought conditions on $\{X_n\}_1^\infty$, and values of m and c, such that (i) respectively (ii) hold. Before reaching a solution, which is given in Section 3, it was necessary to obtain some results in the related areas of martingales, stopping rules, and the Central Limit Theorem, so that Sections 1 and 2 are self-contained and possibly of independent interest.

1. Convergence of moments in the central limit theorem.

1.1. *Summary.* Let $\{X_n\}$ be independent rv's with $EX_n = 0$, $S_n = X_1 + \cdots + X_n$, and $s_n^2 = ES_n^2$ for $n = 1, 2, \cdots$, and $u_{i,j} = EX_j^i$, $v_{i,j} = E|X_j|^i$ for i, $j = 1, 2, \cdots$.
In (1.2) the Lindeberg condition of order ν, L_ν, is defined and in (1.3) it is shown that, when the central limit theorem holds, L_{2k} is necessary and sufficient for the convergence of $E(S_n/s_n)^{2k}$ to the $2k$th moment of a $N(0, 1)$ distribution.

Von Bahr, [11], has delved more fully into the question of convergence of moments in the central limit theorem, but his more involved results do not contain the present ones.

1.2. *Definitions.* A sequence of independent rv's $\{X_n\}$ with $EX_n = 0$ $S_n = X_1 + \cdots + X_n$, $s_n^2 = ES_n^2$, $n = 1, 2, \cdots$, is said to obey a Lindeberg condition of order $\nu \geqq 2$ (i.e. L_ν holds) if $s_n < \infty$, $n = 1, 2, \cdots$, and

$$(1) \qquad \sum_{j=1}^n \int_{[|X_j| \geqq \epsilon s_n]} |X_j|^\nu = o(s_n^\nu) \qquad \text{as } n \to \infty \quad \text{for all} \quad \epsilon > 0.$$

L_2 is the classical Lindeberg condition which is necessary and sufficient for the asymptotic normality of S_n/s_n, and $\max_{1 \leqq k \leqq n} EX_k^2 = o(s_n^2)$, as $n \to \infty$. Consider also

$$(2) \qquad \sum_{j=1}^n \int_{[|X_j| \geqq \epsilon s_j]} |X_j|^\nu = o(s_n^\nu) \qquad \text{as } n \to \infty, \text{ all } \epsilon > 0;$$

and

$$(3) \qquad \sum_{j=1}^n E|X_j|^\nu = o(s_n^\nu) \qquad \text{as } n \to \infty.$$

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When $\nu = 2$, (1) and (2) are equivalent to each other (this is shown in [5] for the case $EX_j^2 = 1, j = 1, 2, \cdots$, and the same method is applicable here) but evidently not to (3). On the other hand, when $\nu > 2$ all three conditions are equivalent as will now be shown.

Clearly (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1). It remains to show that (1) $\Rightarrow$ (3) and this follows directly from the fact that, for $\nu > 2$

$$\sum_{j=1}^n E|X_j|^\nu\{I_{[|X_j|<\epsilon s_n]} + I_{[|X_j|\geq\epsilon s_n]}\}$$
$$\leq \epsilon^{\nu-2}s_n^{\nu-2}\sum_{j=1}^n E|X_j|^2 + \sum_{j=1}^n E|X_j|^\nu I_{[|X_j|\geq\epsilon s_n]}$$
$$= \epsilon^{\nu-2}s_n^\nu + o(s_n^\nu), \qquad \text{from (1), for all } \epsilon > 0.$$

Moreover, if $\alpha > \nu$, then $L_\alpha \Rightarrow L_\nu$, since

$$s_n^{-\alpha}\sum_{j=1}^n \int_{[|X_j|\geq\epsilon s_n]} |X_j|^\alpha \geq \epsilon^{\alpha-\nu}s_n^{-\nu}\sum_{j=1}^n \int_{[|X_j|\geq\epsilon s_n]} |X_j|^\nu.$$

1.3. *Results.*

THEOREM 1.1 *Let $\{X_n\}$ be independent with $EX_n = 0$, $S_n = X_1 + \cdots + X_n$, $s_n^2 = ES_n^2$, for $n = 1, 2, \cdots$. If L_{2k} holds for some $k = 2, 3, \cdots$, then*

$$E(S_n/s_n)^{2k} \rightarrow_{n\to\infty} m_{2k} = \int_{-\infty}^{\infty} (2\pi)^{-\frac{1}{2}} x^{2k} \exp(-x^2/2)\, dx. \tag{4}$$

Conversely, if the central limit theorem ($\mathcal{L}(S_n/s_n) \rightarrow_{n\to\infty} N(0, 1)$) and

$$\max_{1\leq k\leq n} EX_k^2/s_n^2 \rightarrow_{n\to\infty} 0 \tag{5}$$

both hold, then (4) *implies that L_{2k} holds.*

PROOF. $E(S_n^2/s_n^2) = 1$, $n = 1, 2, \cdots$, and L_2 holds (being implied by either L_{2k} or, in the converse, by the central limit theorem).

Assume inductively that $E(S_n/s_n)^{2j} \rightarrow_{n\to\infty} m_{2j}$ for $j = 1, 2, \cdots k - 1$, and that L_{2k-2} holds.

$$\begin{aligned} ES_n^{2k} &= \sum_{r=1}^n \sum_{j=2}^{2k} \tbinom{2k}{j} ES_{r-1}^{2k-j}EX_r^j, \qquad \text{where } S_0 = 0,\\ &= \sum_{r=1}^n EX_r^{2k} + \sum_{r=1}^n \tbinom{2k}{2} ES_{r-1}^{2k-2}EX_r^2\\ &\quad + \sum_{r=1}^n \sum_{j=3}^{2k-2} \tbinom{2k}{j} ES_{r-1}^{2k-j}EX_r^j. \end{aligned} \tag{6}$$

For $j = 3, \cdots, 2k - 2$,

$$\begin{aligned} \sum_{r=1}^n |ES_{r-1}^{2k-j}|\cdot|EX_r^j| &\leq \sum_{r=1}^n E|S_n|^{2k-j}E|X_r|^j\\ &\leq (E|S_n|^{2k-2})^{(2k-j)/(2k-2)} \sum_{r=1}^n E|X_r|^j\\ &= O(s_n^{2k-j})\cdot o(s_n^j) = o(s_n^{2k}). \end{aligned} \tag{7}$$

(5) holds by hypothesis in the necessity part, and is implied by L_{2k} in the sufficiency. (5) implies that $\Delta_n = o(s_n)$ as $n \rightarrow \infty$, where $\Delta_j = s_j - s_{j-1}$, and thus

$$\begin{aligned} &\sum_{r=1}^n \tbinom{2k}{2} E(S_{r-1}/s_{r-1})^{2k-2}\cdot EX_r^2\cdot s_{r-1}^{2k-2}\\ &= O(1) + \{\tbinom{2k}{2}m_{2k-2} + o(1)\} \sum_{r=1}^n (s_r^2 - s_{r-1}^2)s_{r-1}^{2-2k}\\ &\sim O(1) + \{\tbinom{2k}{2}m_{2k-2} + o(1)\} \sum_{r=1}^n 2\Delta_r s_r^{2k-1}\\ &\sim O(1) + \{\tbinom{2k}{2}m_{2k-2} + o(1)\} \sum_{r=1}^n (s_r^{2k} - s_{r-1}^{2k})/k\\ &= O(1) + \{(2k - 1)m_{2k-2} + o(1)\}s_n^{2k} \sim m_{2k}s_n^{2k}. \end{aligned} \tag{8}$$

(7) and (8) enable (6) to be written as

$$ES_n^{2k} = \sum_{r=1}^n EX_r^{2k} + m_{2k}s_n^{2k} + o(s_n^{2k}), \qquad \text{as} \quad n \to \infty.$$

Therefore L_{2k} implies $E(S_n/s_n)^{2k} \to_{n\to\infty} m_{2k}$, and conversely, when the central limit theorem holds, $E(S_n/s_n)^{2k} \to_{n\to\infty} m_{2k}$ implies that $\sum_{r=1}^n EX_r^{2k} = o(s_n^{2k})$, as $n \to \infty$; i.e. L_{2k} holds.

2. Moments of stopped martingales.

2.1. *Notation, summary.* Let $(\Omega, \mathfrak{F}, P)$ be a probability space, $\{S_n\}_{n=1}^\infty$ a sequence of rv's on Ω, and $\{\mathfrak{F}_n\}_{n=1}^\infty$ an increasing sequence of sub-σ-fields of $\mathfrak{F}$, such that S_n is $\mathfrak{F}_n$-measurable, with $S_0 = 0$, $\mathfrak{F}_0 = \{\phi, \Omega\}$. Assume that $\{S_n, \mathfrak{F}_n, n \geqq 1\}$ is a martingale with $X_n = S_n - S_{n-1}$ and $E|X_n|^m < \infty$ for some fixed positive integer $m \geqq 2$ and $n = 1, 2, \cdots$. Write $u_{i,j} = E(X_j^i \mid \mathfrak{F}_{j-1})$, and $v_{i,j} = E(|X_j|^i \mid \mathfrak{F}_{j-1})$. Thus $u_{1,j} = 0$ a.s. for $j = 2, 3, \cdots$ and we assume $u_{1,1} = 0$ without loss of generality.

If t is a stopping rule (i.e. t is an integer valued rv such that $[t \leqq n] \varepsilon \mathfrak{F}_n$, $n = 1, 2, \cdots$, and $P[t = \infty] = 0$) then for each fixed $m \geqq 2$ there is a natural identity connecting moments of S_t and t; this is called the "moment identity of order m" and is given by

$$0 = EZ(t, m), \tag{1}$$

where

$$Z(n, m) = S_n^m + \sum_{r=2}^m m!((m-r)!)^{-1} S_n^{m-r} \sum_{Q_r} (-1)^l (\omega_1! \cdots \omega_l!)^{-1} \cdot \sum_{1 \leqq i_1 \leqq \cdots \leqq i_l \leqq n} u_{\omega_1, i_1} \cdots u_{\omega_l, i_l}, \tag{2}$$

with $Q_r = \{(\omega_1, \cdots \omega_l)\colon \text{each } \omega_j \text{ is an integer} \geqq 2, \omega_1 + \cdots + \omega_l = r\}$, i.e. Q_r is the set of ordered partitions of r into integers > 1.

Theorems 2.1 and 2.2 give conditions under which the moment identity (1) holds; it was first given for $m = 2$ by Chow, Robbins and Teicher (Theorem 1 of [3]) under conditions equivalent to that of Theorem 2.1, and for $m = 2k$, $k = 1, 2, \cdots$ by Teicher ([10]; where the notation is different to ours) under a condition which is stronger than the one obtained here.

2.2. *Lemmas, main theorems.*

LEMMA 1. *If* $E|S_t|^m < \infty$ *and* $E(\sum_{i=1}^t v_{j,i})^{m/j} < \infty$ *for* $2 \leqq j \leqq m$, *then*

$$E \sum_{k=1}^t |S_{k-1}^{m-j} u_{j,k}| < \infty \qquad \text{for} \quad 2 \leqq j \leqq m.$$

PROOF. $S_n^2 \leqq E(S_t^2 \mid \mathfrak{F}_n)$ a.e. on $[t > n]$ by Doob's submartingale theorem ([4a], p. 302), since $\int_{t>n} S_n^2 \to 0$ as $n \to \infty$ (from Theorem 1 of [3]). Thus $|S_n|^m \leqq E(|S_t|^m \mid \mathfrak{F}_n)$ a.e. on $[t > n]$, by Jensen's inequality for conditional expectations ([4a], p. 34), and

$$\begin{aligned} E \sum_{k=1}^t |S_{k-1}^{m-j} u_{j,k}| &\leqq E \sum_{k=1}^t |S_{k-1}|^{m-j} v_{j,k} \\ &\leqq E \sum_{k=1}^t E(|S_t|^{m-j} \mid \mathfrak{F}_{k-1}) v_{j,k} \\ &= E \sum_{k=1}^t |S_t|^{m-j} v_{j,k}, \qquad \text{by Lemma 6 of [3],} \\ &\leqq (E|S_t|^m)^{1-j/m} [E(\textstyle\sum_{k=1}^t v_{j,k})^{m/j}]^{j/m} < \infty, \end{aligned}$$

by Holder's inequality.

COROLLARY. *If* $E|S_t|^m < \infty$ *and* $E(\sum_{i=1}^t v_{j,i})^{m/j} < \infty$ *for* $2 \leqq j \leqq m$, *then* $EY_t = 0$, *where*

$$Y_n = S_n^m - \sum_{k=1}^n E(S_k^m - S_{k-1}^m \mid \mathfrak{F}_{k-1}) = S_n^m - \sum_{k=1}^n \sum_{j=2}^m \binom{m}{j} S_{k-1}^{m-j} u_{j,k}. \tag{3}$$

PROOF. As in the proof of Lemma 1, $|S_n|^m \leqq E(|S_t|^m \mathfrak{F}_n)$ a.e. on $[t > n]$, giving $\int_{[t>n]} |S_n|^m \leqq \int_{[t>n]} |S_t|^m \to_{n\to\infty} 0$, since $E|S_t|^m < \infty$. Thus (from (3)) $\int_{[t>n]} |Y_n| \to_{n\to\infty} 0$, and $E|Y_t| < \infty$, since $E\sum_{k=1}^t |S_{k-1}^{m-j} u_{j,k}| < \infty$ for $2 \leqq j \leqq m$, from Lemma 1.

Since $(Y_n, \mathfrak{F}_n, n \geqq 1)$ is a martingale (obvious, from (3)), this implies $EY_t = 0$ by a theorem of Doob (Lemma 1 of [3]).

LEMMA 2. *If* m *is an even integer and* $E(\sum_{i=1}^t v_{j,i})^{m/j} < \infty$ *for* $2 \leqq j \leqq m$, *then* $E|S_t|^m < \infty$.

PROOF. Since $\{Y_n\}$ is a martingale, $EY_{t'} = 0$, where $t' = \min(t, n)$, i.e.

$$\begin{aligned} ES_{t'}^m &= E\sum_{k=1}^{t'} \sum_{j=2}^m \binom{m}{j} S_{k-1}^{m-j} u_{j,k} \\ &\leqq \sum_{j=2}^m \binom{m}{j} E\sum_{k=1}^{t'} |S_{k-1}|^{m-j} v_{j,k} \\ &\leqq \sum_{j=2}^m \binom{m}{j} (E|S_{t'}|^m)^{1-j/m} [E(\sum_{k=1}^{t'} v_{j,k})^{m/j}]^{j/m}, \end{aligned}$$

as in the proof of Lemma 1.
But $ES_{t'}^m = E|S_{t'}|^m$, so that $E|S_{t'}|^m \leqq$ some $K < \infty$ for all $n \geqq 1$, giving

$$E|S_t|^m \leqq \limsup_{n\to\infty} E|S_{t'}|^m < \infty.$$

THEOREM 2.1 *If* m *is an even integer and* $E(\sum_{i=1}^t v_{j,i})^{m/j} < \infty$ *for* $2 \leqq j \leqq m$, *then* $EY_t = 0$. *If* m *is odd, and in addition* $E|S_t|^m < \infty$, *then* $EY_t = 0$.

PROOF. We can suppose $E|S_t|^m < \infty$, since it is postulated in the odd case, and holds, by Lemma 2, in the even case. Then the theorem follows by the corollary to Lemma 1.

THEOREM 2.2. *Under the hypotheses of Theorem* 2.1, *the moment identity* (1) *holds.*

PROOF. By Theorem 1, $EY_t = 0$, which from (3) may be written

$$ES_t^m = \sum_{j=2}^m \binom{m}{j} E\sum_{k=1}^t S_{k-1}^{m-j}(U_{j,k} - U_{j,k-1}),$$

where

$$U_{j,k} = \sum_{r=1}^k u_{j,r}.$$

Therefore

$$\begin{aligned} ES_t^m &= \sum_{j=2}^m \binom{m}{j} \{E\sum_{k=1}^t (S_{k-1}^{m-j} - S_k^{m-j}) U_{j,k} + ES_t^{m-j} U_{j,t}\} \\ &= \sum_{j=2}^m \sum_{r=2}^{m-j} m!(j!r!(m-j-r)!)^{-1} E\sum_{1\leq l\leq k\leq t} u_{j,l} u_{r,k} S_{k-1}^{m-j-r} \\ &\quad + \sum_{j=2}^m \binom{m}{j} ES_t^{m-j} \sum_{k=1}^t u_{j,k}, \end{aligned}$$

by taking conditional expectations within the summation in the previous line (Lemma 6 of [3]).

The procedure can be repeated by writing $\sum_{l=1}^{k} u_{j,l}u_{r,k} = U_{j,r,k} - U_{j,r,k-1}$ in the first term; after $[m/2]$ repetitions we arrive at (1). At each stage, absolute integrability is needed to justify taking conditional expectations inside summations, and this is provided since the terms of (1) are absolutely integrable, as follows (cf. (2)):

$$\begin{aligned} E|Z(t, m)| &\leqq E|S_t|^m + \sum_{r=2}^{m} m!((m-r)!)^{-1}E|S_t|^{m-r} \sum_{Q_r} (\omega_1 ! \cdots \omega_l !)^{-1} \\ &\qquad \cdot \prod_{j=1}^{l} (\sum_{i=1}^{t} v_{\omega_j,i}) \\ &\leqq E|S_t|^m + \sum_{r=2}^{m} m!((m-r)!)^{-1}(E|S_t|^m)^{1-r/m} \\ &\qquad \cdot \sum_{Q_r} O\{\prod_{j=1}^{l} [E(\sum_{i=1}^{t} v_{\omega_j,i})^{m/\omega_j}]^{\omega_j/r}\}, \qquad \text{by Holder's inequality,} \\ &< \infty. \end{aligned}$$

The initial assumption of $S_0 = 0$, $\mathfrak{F}_0 = \{\phi, \Omega\}$ may not always hold, and S_0 may be a non-degenerate rv which is $\mathfrak{F}_0$-measurable, $\mathfrak{F}_0$ being a non-trivial σ-field. In that case we can follow our working through with practically no change, arriving at:

COROLLARY. *If* $E|S_0|^m < \infty$, $E(\sum_{i=1}^{t} v_{j,i})^{m/j} < \infty$ *for* $2 \leqq j \leqq m$, *and in addition* $E|S_t|^m < \infty$ *if* m *is odd, then*

$$E(Y_t \mid \mathfrak{F}_0) =_{\text{a.e.}} S_0^m =_{\text{a.e.}} E(Z(t, m) \mid \mathfrak{F}_0), \quad \textit{and} \quad EY_t = ES_0^m = EZ(t, m).$$

2.3. *Remarks.* The moment identity of order m has been presented in two different forms in Theorems 2.1 and 2.2. The first form is the simpler, while the second corresponds to Teicher, [10], where it is given for $m = 2k$, $k = 1, 2, \cdots$, under the assumption that $Et^{k-1} \sum_{i=1}^{t} v_{2k,i} < \infty$. This condition implies the one used here, for, using the inequality

$$(\sum_1^n |a_i|)^r \leqq n^{r-1} \sum_1^n |a_i|^r, \qquad r \geqq 1,$$

and the conditional Holder inequality, we obtain

$$E(\sum_{i=1}^{t} v_{\omega,i})^{2k/\omega} \leqq Et^{2k/\omega - 1} \sum_{i=1}^{t} v_{\omega,i}^{2k/\omega} \leqq Et^{k-1} \sum_{i=1}^{t} v_{2k,i}.$$

3. On moments of a stopping rule.

3.1. *Introduction and summary.* On a probability space $(\Omega, \mathfrak{F}, P)$, let $\{X_n\}$ be independent rv's with $EX_n = 0$, $EX_n^2 = 1$, $S_n = X_1 + \cdots + X_n$, $n = 1, 2, \cdots$, and $u_{i,j} = EX_j^i$, $v_{i,j} = E|X_j|^i$. Define $t = t_m(c) = \inf\{n : n \geqq m, |S_n| > cn^{\frac{1}{2}}\}$, $0 < c < \infty$, $m \geqq 1$. Then it is known that

(i) when $c \geqq 1$, $Et = \infty$ (Chow, Robbins, Teicher, [3]),

(ii) when $c < 1$ and the $\{X_n\}$ are uniformly bounded, then $Et < \infty$ (Chow, Robbins, [3]),

(iii) when $c < 1$ and the $\{X_n\}$ obey the Lindeberg condition (ensuring $\mathcal{L}(S_n/n^{\frac{1}{2}}) \to_{n\to\infty} N(0, 1)$) then $Et < \infty$ (Gundy, Siegmund, [5]), and

(iv) when the $\{X_n\}$ are uniformly bounded, $c^2 < 3 - 6^{\frac{1}{2}}$ implies $Et^2 < \infty$ for all m and $c^2 > 3 - 6^{\frac{1}{2}}$ implies that $Et^2 = \infty$ for m sufficiently large (Chow, Teicher, [4]).

The stopping rule $t_m(c)$ was first introduced by Blackwell and Freedman, [1], for coin-tossing rv's. In (3.2) it is shown that $t_m(c)$ is a bona-fide stopping rule (i.e. $P[t_m(c) = \infty] = 0$) as long as the Lindeberg condition holds on the $\{X_n\}$. In (3.3) the definition of the Lindeberg condition of order $2k$ given in Section 1 is repeated ($k = 1, 2, \cdots$) and it is shown (Theorem 3.2) that if c_k is the smallest positive root of the Hermite polynomial of order $2k$ and the $\{X_n\}$ obey a Lindeberg condition of order $2k$, then $Et_m^k(c) < \infty$ for all $m \geqq 1$ if $c < c_k$, while $Et_m^k(c) = \infty$ for all sufficiently large m, if $c > c_k$. Then, use of a truncation enables Theorem 3.3 to give the same result as Theorem 3.2, assuming a Lindeberg condition of order 2, and hence the central limit theorem, instead of one of order $2k$. The result is similar to one of Shepp, [8], for the continuous analogue of the present (discrete) problem, and Shepp's conjecture ([8]) is verified, in conjunction with a related result of Breiman, [2], for iid rv's with $E|X_1|^3 < \infty$.

3.2. *The stopping rule* $t = t_m(c)$. It is not *a priori* clear if or when $t_m(c)$ is a bona-fide stopping rule. Evidently some conditions on the $\{X_n\}$ are needed, for if $P[X_n = 0] = 1 - a_n^{-2}$, $P[X_n = -a_n] = (2a_n^2)^{-1} = P[X_n = a_n]$, with $\sum a_n^{-2} < \infty$ and $a_n > cn^{\frac{1}{2}}$, $n = 1, 2, \cdots$, then $P[t_1(c) > n] = P[X_1 = 0, X_2 = 0, \cdots X_n = 0] = \prod_{j=1}^{n} (1 - a_j^{-2}) \downarrow b > 0$, as $n \to \infty$; and $t_1(c)$ is *not* a bona-fide stopping rule.

THEOREM 3.1. *Let* $\{X_n\}$ *obey the Lindeberg condition. Then* $P[t_m(c) < \infty] = 1$, i.e. $t_m(c)$ *is a bona-fide stopping rule, for all* $c > 0$ *and* $m \geqq 1$.

(This is already known for $c < 1$, since it is shown in [5] that then $Et_m(c) < \infty$).

PROOF.

$$P \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} [|S_k| > ck^{\frac{1}{2}}] = \lim_{m\to\infty} P \bigcup_{k=m}^{\infty} [|S_k| > ck^{\frac{1}{2}}] \geqq \lim_{m\to\infty} P[|S_m| > cm^{\frac{1}{2}}].$$

Therefore $P[|S_n| > cn^{\frac{1}{2}}$ i.o.$] = 1$ by the zero-one law and the central limit theorem.

3.3. *Definition, results and corollaries.* A sequence of independent rv's $\{X_n\}$ with $EX_n = 0$, $EX_n^2 = 1$, $n = 1, 2, \cdots$, is said to obey L_{2k}, a Lindeberg condition of order $2k$, if

$$\sum_{j=1}^{n} \int_{[|X_j| \geqq \epsilon n^{\frac{1}{2}}]} |X_j|^{2k} = o(n^k) \quad \text{as} \quad n \to \infty \quad \text{for all} \quad \epsilon > 0.$$

When $k = 1$, this is the classical Lindeberg condition.

For $k > 1$, the condition was shown in Section 1 to be equivalent to

$$(1) \qquad \sum_{j=1}^{n} v_{2k,j} = \sum_{j=1}^{n} E|X_j|^{2k} = o(n^k), \quad \text{as} \quad n \to \infty.$$

Before giving Theorem 3.2, we need three preliminary lemmas.

LEMMA 1. *If* t *is any stopping rule with* $Et^k < \infty$, *and the* $\{X_n\}$ *obey* L_{2k}, *then*

$$(2) \qquad 0 = ES_t^{2k} + \sum_{r=2}^{2k} (2k)!((2k - r)!)^{-1} ES_t^{2k-r} t^{r/2} A(t, r),$$

where

(3) $A(n, r) = n^{-r/2} \sum_{Q_r} (-1)^l (\omega_1 ! \cdots \omega_l !)^{-1} \sum_{1 \leqq i_1 \leqq \cdots \leqq i_l \leqq n} u_{\omega_1, i_1} \cdots u_{\omega_l, i_l}$,

and where Q_r is the set of ordered partitions of the integer r into integers >1, with $(\omega_1, \cdots \omega_l)$ a typical element, i.e. $Q_r = \{(\omega_1, \cdots \omega_l)$: *each ω_j is an integer* $\geqq 2$, $\omega_1 + \cdots + \omega_l = r\}$.

PROOF. By L_{2k}, $\sum_{i=1}^{n} v_{\omega,i} = O(n^{\omega/2})$ as $n \to \infty$ for $2 \leqq \omega \leqq 2k$. Therefore $E(\sum_{i=1}^{t} v_{\omega,i})^{2k/\omega} = O(Et^k) < \infty$ for $2 \leqq \omega \leqq 2k$. Theorem 2.2 (Section 2) is applicable, and (2) follows.

LEMMA 2. *Let $a \geqq 0, b > 0$ be integers with $a + b/2 = k$, and let $\{s_j\}_{j=1}^{\infty}$ be a sequence of stopping rules such that $Es_j^k < \infty, j = 1, 2, \cdots$ and $Es_j^k \uparrow \infty$ as $j \to \infty$. If $\{X_n\}$ obey L_{2k}, then*

$$Es_j^a |X_{s_j}|^b = o(Es_j^k) \qquad as \quad j \to \infty.$$

PROOF.

$$E|X_{s_j}|^{2k} \leqq E \sum_{r=1}^{s_j} |X_r|^{2k} = E \sum_{r=1}^{s_j} E|X_r|^{2k} \qquad \text{(Lemma 6 of [3])}$$
$$= o(Es_j^k) \qquad \text{as} \quad j \to \infty, \qquad \text{since } L_{2k} \text{ holds.}$$

Then by Holder's inequality,

$$Es_j^a |X_{s_j}|^b \leqq (Es_j^k)^{a/k} (E|X_{s_j}|^{2k})^{b/2k} = (Es_j^k)^{a/k} (o(Es_j^k))^{b/2k}$$
$$= o(Es_j^k), \qquad \text{as} \quad j \to \infty.$$

COROLLARY. *If $t = t_m(c)$, $t_n = \min(t, n)$, $Et_n^k \uparrow Et^k = \infty$ as $n \to \infty$, and $\{X_n\}$ obeys L_{2k}, then*

$$Et_n^j |X_{t_n}|^b |S_{t_n-1}|^{2k-2j-b} I_{[t>m]} = o(Et_n^k) \qquad as \quad n \to \infty,$$

for integers j, b with $0 < b \leqq 2k - 2j$.

PROOF. Use the inequality $|S_{t_n-1}| < ct_n^{\frac{1}{2}}$ on $[t > m]$, then apply Lemma 2 with $\{s_n\}$ replaced by $\{t_n\}$.

LEMMA 3. *Let $t = t_m(c)$, $t_n = \min(t, n)$, $2 \leqq r \leqq 2k$, and let $\{b_n\}$ be a bounded sequence which $\to 0$ as $n \to \infty$. If $\{X_n\}$ obeys L_{2k}, then*

(i) *$Et^k = \infty$ implies $ES_{t_n}^{2k-r} t_n^{r/2} b_{t_n} = o(Et_n^k)$ as $n \to \infty$;*

(ii) *$Et^k < \infty$ for all $m = 1, 2, \cdots$ implies that $ES_t^{2k-r} t^{r/2} b_t = o(Et^k)$ as $m \to \infty$.*

PROOF.

$$|ES_{t_n}^{2k-r} t_n^{r/2} b_{t_n}| \leqq b_m m^{r/2} \int_{[t=m]} |S_m|^{2k-r} + E|S_{t_n-1} + X_{t_n}|^{2k-r} t_n^{r/2} b_{t_n} I_{[t>m]}$$
$$\leqq O(1) + o(Et_n^k) + E|S_{t_n-1}|^{2k-r} t_n^{r/2} b_{t_n} I_{[t>m]},$$

by expanding and using the corollary to Lemma 2,

$$= O(1) + o(Et_n^k), \qquad \text{since } |S_{t_n-1}| < ct_n^{\frac{1}{2}} \quad \text{on} \quad [t > m],$$
$$= o(Et_n^k).$$

For (ii), choose $m > n_0$ where $n > n_0 \Rightarrow b_n < \epsilon$. Then

$$\begin{aligned} |ES_t^{2k-r}t^{r/2}b_t| &\leqq \epsilon m^{r/2} \int_{[t=m]} |S_m|^{2k-r} + E|S_{t-1} + X_t|^{2k-r}t^{r/2}b_t I_{[t>m]} \\ &\leqq \epsilon m^{r/2}E|S_m|^{2k-r} + \epsilon E|ct^{\frac{1}{2}} + X_t|^{2k-r}t^{r/2}I_{[t>m]} \\ &\leqq \epsilon O(m^k) + \epsilon c^{2k-r}Et^k I_{[t>m]} + o(Et^k), \end{aligned}$$

since $E|S_m|^{2k} = O(m^k)$ by Theorem 1.1 (Section 1), and by expanding the second term and applying Lemma 2 with s_m replaced by t,

$$= o(Et^k) \text{ as } m \to \infty, \text{ since } t \geqq m.$$

THEOREM 3.2. *Let c_k be the smallest positive zero of the Hermite polynomial of order $2k$ ($k = 1, 2, \cdots$). If $\{X_n\}$ obey L_{2k}, then $Et_m^k(c) < \infty$ for all $m \geqq 1$ if $c < c_k$, while $Et_m^k(c) = \infty$ for all sufficiently large m if $c > c_k$.*

PROOF. Let $t = t_m(c)$, $t_n = \min(t, n)$, $c < c_k$ and assume that $Et_n^k \uparrow Et^k = \infty$ as $n \to \infty$. For $2 \leqq r \leqq 2k$, define $A_1(n, r) = A(n, r)$ if r is odd, or

$$= n^{-r/2} \sum_{Q_r'} (-1)^l (\omega_1! \cdots \omega_l!)^{-1} \sum_{1 \leqq i_1 \leqq \cdots \leqq i_l \leqq n} u_{\omega_1, i_1} \cdots u_{\omega_l, i_l}$$

if r is even, where

$$Q_r' = Q_r - \underbrace{\{(2, 2, \cdots 2)\}}_{r/2 \text{ entries}}.$$

Therefore

$$|A_1(n, r)| \leqq \sum_Q (\omega_1! \cdots \omega_l!)^{-1} \prod_{j=1}^l (n^{-\omega_j/2} \sum_{i=1}^n v_{\omega_j, i}), \tag{4}$$

where $Q \equiv Q_r$, r odd, or $\equiv Q_r'$, r even, $= \sum_Q o[\prod_{j=1}^l O(1)]$, as $n \to \infty$ since L_{2k} is obeyed, and at least one $\omega_j > 2$, $= o(1)$ as $n \to \infty$.

By Theorem 2.2 (Section 3), (2) holds with t replaced by t_n, since t_n is a bounded stopping rule. By grouping together those terms of (2) with $r = 2j$, $l = j$, and $\omega_1 = \omega_2 = \cdots = \omega_j = 2$, we can write

$$\begin{aligned} 0 &= ES_{t_n}^{2k} + \sum_{j=1}^k (-1)^j (2k)! ((2k - 2j)! (2!)^j)^{-1} ES_{t_n}^{2k-2j} \\ &\quad \cdot \sum_{1 \leqq i_1 \leqq \cdots \leqq i_j \leqq t_n} 1 + \sum_{r=2}^{2k} (2k)! ((2k - r)!)^{-1} ES_{t_n}^{2k-r} t_n^{r/2} A_1(t_n, r) \\ &= \sum_{j=0}^k (-1)^j (2k)! ((2k - 2j)! j! (2!)^j)^{-1} ES_{t_n}^{2k-2j} \\ &\quad \cdot (t_n + j - 1)! ((t_n - 1)!)^{-1} \\ &\quad + \sum_{r=2}^{2k} (2k)! ((2k - r)!)^{-1} ES_{t_n}^{2k-r} t_n^{r/2} A_1(t_n, r) \\ &= \sum_{j=0}^k (-1)^j (2k)! ((2k - 2j)! j! (2!)^j)^{-1} ES_{t_n}^{2k-2j} t_n^j + o(Et_n^k), \end{aligned} \tag{5}$$

by applying (i), Lemma 3 to both terms, with $b_n = n^{-1}, n^{-2}, \cdots$ in the first, and in the second, $b_n = A_1(n, r) = o(1)$, from (4). But

$$\begin{aligned} Et_n^j S_{t_n}^{2k-2j} &= Et_n^j (S_{t_n-1} + X_{t_n})^{2k-2j} I_{[t>m]} + m^j \int_{[t=m]} S_m^{2k-2j} \\ &= Et_n^j S_{t_n-1}^{2k-2j} I_{[t>m]} + o(Et_n^k) + O(1), \qquad \text{as } n \to \infty, \end{aligned}$$

by expanding and using lemma (2), with t_n replacing s_n, and its corollary. Therefore (5) becomes

$$0 = o(Et_n^{\,k}) + (-)^{k+1} \sum_{j=0}^{k} (-1)^j (2k)!\,((2k-2j)!\,j!\,(2!)^j)^{-1} Et_n^{\,j} S_{t_n-1}^{2k-2j} I_{[t>m]}$$
$$= o(Et_n^{\,k}) + (-1)^{k+1} E(t_n^{\,k} H_{2k}(S_{t_n-1}/t_n^{\frac{1}{2}}) I_{[t>m]}),$$

where H_{2k} is the Hermite polynomial of order $2k$ (see Appendix, (A2))

$$\leqq o(Et_n^{\,k}) + (-1)^{k+1} H_{2k}(c) \int_{[t>m]} t_n^{\,k},$$

since $|S_{t_n-1}| < ct_n^{\frac{1}{2}}$ on $[t > m]$, and since $(-1)^{k+1}H_{2k}(x)$ is an even function, increasing in $(0, c_k)$, by the Proposition of the Appendix. By the same proposition, $(-1)^{k+1}H_{2k}(c) < 0$ for $0 < c < c_k$, thus contradicting the assumption that $Et_n^{\,k} \uparrow Et^k = \infty$, as $n \to \infty$. Therefore $Et^k < \infty$ when $c < c_k$.

Next consider the case $c > c_k$. Assume that $Et^k < \infty$ for infinitely many, and therefore all, m. By lemma (1), (2) holds and may be written (cf. (5))

$$0 = \sum_{j=0}^{k} (-1)j(2k)!\,((2k-2j)!\,j!\,(2!)^j)^{-1} ES_t^{2k-2j}(t+j-1)!\,((t-1)!)^{-}$$
$$+ \sum_{r=2}^{2k} (2k)!\,((2k-r)!)^{-1} ES_t^{2k-r} t^{r/2} A_1(t, r)$$
$$= \sum_{j=0}^{k} (-1)^j (2k)!\,((2k-2j)!\,j!\,(2!)^j)^{-1} ES_t^{2k-2j} t^j + o(Et^k),$$

as $m \to \infty$, by applying (ii), Lemma 3 to both terms, with $b_n = n^{-1}, n^{-2}, \cdots$, in the first, and $b_n = A_1(n, r)$ in the second; thus

$$(6) \qquad 0 = (-1)^{k+1} Et^k H_{2k}(S_t/t^{\frac{1}{2}}) + o(Et^k), \qquad m \to \infty.$$

Now $ct^{\frac{1}{2}} < |S_t| \leqq ct^{\frac{1}{2}} + |X_t|$ on $[t > m]$, so if $|S_t| = ct^{\frac{1}{2}} + e_t$ (e_t is the "excess"), $e_t \leqq |X_t|$ on $[t > m]$, and Lemma (2) holds with s_m replaced by t and X_t replaced by $e_t I_{[t>m]}$. Therefore

$$(-1)^{k+1} E(t^k H_{2k}(S_t/t^{\frac{1}{2}}) I_{[t>m]})$$
$$= (-1)^{k+1} Et^k I_{[t>m]} \{H_{2k}(c) + (e_t/t^{\frac{1}{2}}) H_{2k}'(c) + \cdots + ((2k)!)^{-1} (e_t/t^{\frac{1}{2}})^{2k} H_{2k}^{(2k)}(c)\},$$

using the Taylors series expansion of $H_{2k}((ct^{\frac{1}{2}} + e_t)/t^{\frac{1}{2}})$,

$$(7) \qquad = (-1)^{k+1} H_{2k}(c) \int_{[t>m]} t^k + o(Et^k),$$

using Lemma 2, cited above. Furthermore

$$(8) \quad (-1)^{k+1} E(t^k H_{2k}(S_t/t^{\frac{1}{2}}) I_{[t=m]}) = m^k \int_{[|S_m|>cm^{\frac{1}{2}}]} (-1)^{k+1} H_{2k}(S_m/m^{\frac{1}{2}}).$$

But from Theorem 1.1 (Section 1), $E(S_m/m^{\frac{1}{2}})^{2r} \to EY^{2r}$, $m \to \infty$, for integers r, $1 \leqq r \leqq k$, where Y has a $N(0, 1)$ distribution. Therefore

$$EH_{2k}(S_m/m^{\frac{1}{2}}) \to_{m\to\infty} EH_{2k}(Y).$$

Since $S_m/m^{\frac{1}{2}}$ converges to Y in distribution (central limit theorem) the Helly-

Bray lemma (see [6], p. 180) ensures that

$\int_{[|S_m| \leqq cm^{\frac{1}{2}}]} H_{2k}(S_m/m^{\frac{1}{2}}) \rightarrow_{m\to\infty} \int_{[|Y| \leqq c]} H_{2k}(Y)$,

whence

$\int_{[|S_m| > cm^{\frac{1}{2}}]} H_{2k}(S_m/m^{\frac{1}{2}}) \rightarrow_{m\to\infty} \int_{[|Y| > c]} H_{2k}(Y) = -(2/\pi)^{\frac{1}{2}} H_{2k-1}(c)e^{-c^2/2}$

((A7)) of Appendix).

Therefore (8) becomes

$$(9) \qquad (-1)^{k+1}E(t^k H_{2k}(S_t)t^{\frac{1}{2}})I_{[t=m]}) = Bm^k + o(m^k), \qquad m \to \infty,$$

where $B = (-)^k(2/\pi)^{\frac{1}{2}}H_{2k-1}(c)e^{-c^2/2}$. In view of (7) and (9), (6) becomes

$$(10) \qquad 0 = A \int_{[t>m]} t^k + Bm^k + o(Et^k) + o(m^k),$$

where $\qquad A = (-1)^{k+1}H_{2k}(c).$

When $c_k < c \leqq c_{k-1}$, $B > 0$ (since $(-)^k H_{2k-1}(c) > 0$, (A4) of Appendix). But $A > 0$ for $c_k < c_k \leqq c_{k-1}$ by the Proposition of the Appendix, giving a contradiction in (10) by letting $m \to \infty$. Therefore $Et^k = \infty$ for m sufficiently large when $c_k < c \leqq c_{k-1}$, and since inductively, $Et^r = \infty$ for m large enough, $c_r < c \leqq c_{r-1}$, $r = 1, 2, \cdots k - 1$, we conclude that $Et^k = \infty$ for m large enough when $c_k < c$.

COROLLARY. *If* $\{X_n\}$ *obey* L_{2k}, *then* $Et_m^{\ k}(c_k) = \alpha_{m,k}m^k$, *where* $\alpha_{m,k} \rightarrow_{m\to\infty} \infty$ *(possibly,* $\alpha_{m,k} = \infty$ *for some finite* m*).*

PROOF. The statement of the corollary is equivalent to

$$m^k = o(Et^k) \qquad \text{as} \quad m \to \infty, \qquad \text{where} \quad t = t_m(c_k);$$

which follows from (10) since $A = 0$.

THEOREM 3.3. *Let* $\{X_n\}$ *be independent,*

$$EX_n = 0, \quad EX_n^{\ 2} = 1, \quad S_n = X_1 + \cdots + X_n, \quad n = 1, 2, \cdots.$$

If $\{X_n\}$ *obey the central limit theorem* (i.e. $\mathfrak{L}(S_n/n^{\frac{1}{2}}) \rightarrow_{m\to\infty} N(0, 1)$) *then* $Et_m^{\ k}(c) < \infty$ *for all* $m \geqq 1$ *if* $c < c_k$, *while* $Et_m^{\ k}(c) = \infty$ *for all sufficiently large* m *if* $c > c_k$, $k = 1, 2, \cdots$. $Et_m^{\ k}(c_k) = \alpha_{m,k}m^k$ *where* $\alpha_{m,k} \rightarrow_{m\to\infty} \infty$ *(possibly,* $\alpha_{m,k} = \infty$ *for some finite* m*).*

PROOF. Let $t = t_m(c)$ and $\Omega^* = [t > m]$. On Ω^* set $Z_0 = S_m I_{[t>m]}$ and $Z_n = S_{m+n}I_{[t>m]}$, $n = 1, 2, \cdots$. Let $\mathfrak{F}_n^{\ *}$, $\mathfrak{F}^*$ be the restrictions of $\mathfrak{F}_n$ and $\mathfrak{F}$ to Ω^*, where $\mathfrak{F}_n = \mathfrak{B}(X_1, \cdots, X_n)$ and $\mathfrak{F} = \mathfrak{B}(X_1, X_2, \cdots)$, and let P^* be the restriction of the underlying probability measure P to Ω^*. Then $(Z_n, \mathfrak{F}_n^{\ *}, n \geqq 1)$ is a martingale on $(\Omega^*, \mathfrak{F}^*, P^*)$ and $s = s_m(c) = \inf\{n \geqq 1 : |Z_n| > c(n + m)^{\frac{1}{2}}\}$ is a stopping rule on $(Z_n, \mathfrak{F}_n^{\ *}, n \geqq 1)$.

Let $z_n = Z_n - Z_{n-1}$, $n = 1, 2, \cdots$. Therefore $z_n = X_{n+m}I_{[t>m]}$, $n = 1, 2, \cdots$. The proof of Theorem 3.2 can be applied to $\{\Omega^*, \mathfrak{F}^*, P^*, Z_n, \mathfrak{F}_n^{\ *}, n \geqq 1; s_m(c)\}$ instead of $\{\Omega, \mathfrak{F}, P, S_n, \mathfrak{F}_n, n \geqq 1; t_m(c)\}$ with a modification of the analogue of (2) to allow for the initial term of the martingale (see the Corollary of Section 2).

The details are a repetition of the proof of Theorem 3.2 and are omitted; the ensuing equations are identical to the original ones and we obtain the result that: If $\sum_1^n Ez_j^{2k} = o(n+m)^k$ (i.e. L_{2k} holds) then

$$
\begin{aligned}
& c < c_k \Rightarrow Es_m^k(c) < \infty, \qquad \text{all} \quad m \geqq 1 \\
(11) \qquad & c > c_k \Rightarrow Es_m^k(c) = \infty, \qquad \text{for large enough} \quad m \\
& m^k = o(Es_m^k(c_k)), \qquad \text{as} \quad m \to \infty.
\end{aligned}
$$

This statement, because of

$$(12) \qquad t = t_m(c) = mI_{[t=m]} + (m + s_m(c))I_{[t>m]},$$

is equivalent to the statement of Theorem 3.2. A further allowable modification is that (11) can be obtained when the properties $Ez_n = 0$, $Ez_n^2 = 1$ are replaced by

$$(13) \qquad \textstyle\sum_{j=1}^n Ez_j = o((n+m)^{\frac{1}{2}}) \qquad \text{as} \quad n \to \infty,$$

and

$$(14) \qquad \textstyle\sum_{j=1}^n Ez_j^2 = n + o(n+m) \qquad \text{as} \quad n \to \infty.$$

Next, we set

$$
\begin{aligned}
Y_n = -2c(m+n)^{\frac{1}{2}}I_{[z_n \leqq -2c(m+n)^{\frac{1}{2}}]} &+ z_n I_{[|z_n| < 2c(m+n)^{\frac{1}{2}}]} \\
&+ 2c(m+n)^{\frac{1}{2}}I_{[z_n \geqq 2c(m+n)^{\frac{1}{2}}]}, \quad n = 1, 2, \cdots
\end{aligned}
$$

Since $[s > n-1, |z_n| \geqq 2c(m+n)^{\frac{1}{2}}] \subset [s = n]$, $n = 1, 2, \cdots$, we have

$$s = s_m(c) = \inf\{n \geqq 1 : |Y_1 + \cdots + Y_n| > c(m+n)^{\frac{1}{2}}\}$$

i.e. $s_m(c)$ is unaltered by replacing $\{z_n\}_1^\infty$ by $\{Y_n\}_1^\infty$. Therefore the theorem will be proved if we can apply (11) and (12) to $\{Y_n\}_1^\infty$, and to do this we only need show that

$$(15) \qquad \textstyle\sum_{j=1}^n EY_j = o((n+m)^{\frac{1}{2}}) \qquad \text{as} \quad n \to \infty,$$

$$(16) \qquad \textstyle\sum_{j=1}^n EY_j^2 = n + o((n+m)^{\frac{1}{2}}) \qquad \text{as} \quad n \to \infty$$

(by the remark preceding (13) and (14)); and

$$(17) \qquad \textstyle\sum_{j=1}^n EY_j^{2k} = o(n+m)^k \qquad \text{as} \quad n \to \infty;$$

i.e. that L_{2k} holds. Because of the independence of $z_1, z_2, \cdots$ and $I_{[t>m]}$, we can work with P instead of P^*, without losing generality.

For (15), we have

$$
\begin{aligned}
EY_j &= 2c(j+m)^{\frac{1}{2}}\{P[z_j \geqq 2c(j+m)^{\frac{1}{2}}] - P[z_j \leqq -2c(j+m)^{\frac{1}{2}}]\} \\
&\quad - \textstyle\int_{[|z_n| \geqq 2c(j+m)^{\frac{1}{2}}]} z_j \\
&= 2c(j+m)^{\frac{1}{2}}\{P[X_{j+m} \geqq 2c(j+m)^{\frac{1}{2}}] - P[X_{j+m} \leqq -2c(j+m)^{\frac{1}{2}}]\} \\
&\quad - \textstyle\int_{[|X_{j+m}| \geqq 2c(j+m)^{\frac{1}{2}}]} X_{j+m}
\end{aligned}
$$

and

$$|\sum_{j=1}^{n} EY_j| \leqq c^{-1} \sum_{j=1}^{n} (j+m)^{-\frac{1}{2}} \int_{[|X_{j+m}| \geqq 2c(j+m)^{\frac{1}{2}}]} X_{j+m}^2$$
$$= c^{-1} \sum_{j=m+1}^{n+m} j^{-\frac{1}{2}} \int_{[|X_j| \geqq 2cj^{\frac{1}{2}}]} X_j^2$$
$$= o(n+m)^{\frac{1}{2}}) \quad \text{as} \quad n \to \infty,$$

because of the equivalence of L_2 to

(18) $$\sum_{j=1}^{n} \int_{[|X_n| \geqq \epsilon j^{\frac{1}{2}}]} X_j^2 = o(n) \quad \text{as} \quad n \to \infty, \quad \text{all} \quad \epsilon > 0;$$

which was shown by Gundy and Siegmund, [5].

For (16), we have

$$\sum_{j=1}^{n} Ez_j^2 - \sum_{j=1}^{n} EY_j^2 = \sum_{j=1}^{n} \int_{[|X_{j+m}| \geqq 2c(j+m)^{\frac{1}{2}}]} [X_{j+m}^2 - 4c^2(j+m)],$$
$$= o((n+m)^{\frac{1}{2}}),$$

invoking (18); and for (17),

$$\sum_{j=1}^{n} EY_j^{2k} = \sum_{j=m+1}^{m+n} \{\int_{[|X_j| < \epsilon j^{\frac{1}{2}}]} X_j^{2k} + \int_{[\epsilon j^{\frac{1}{2}} \leqq |X_j| < 2cj^{\frac{1}{2}}]} X_j^{2k}$$
$$+ (4c^2 j)^k P[|X_j| \geqq 2cj^{\frac{1}{2}}]\}$$
$$\leqq \sum_{j=m+1}^{m+n} \{ (\epsilon^2(m+n))^{k-1} \int_{[|X_j| < \epsilon j^{\frac{1}{2}}]} X_j^2 + (4c^2(m+n))^{k-1}$$
$$\cdot \int_{[\epsilon j^{\frac{1}{2}} \leqq |X_j| < 2cj^{\frac{1}{2}}]} X_j^2 + (4c^2(m+n))^{k-1} \int_{[|X_j| \geqq 2cj^{\frac{1}{2}}]} X_j^2\}$$
$$\leqq (\epsilon^2(m+n))^{k-1} \cdot n + (4c^2(m+n))^{k-1} \cdot o(m+n)$$
$$+ (4c^2(m+n))^{k-1} \cdot o(m+n),$$

(invoking (18) twice),

$$= o(n+m)^k,$$ by choosing ϵ small; thus completing the proof.

THEOREM 3.4. *If* $\{X_n\}$ *obey the central limit theorem and*

$$u_m(c) = \inf\{n : n \geqq m, |S_n| > ca_n n^{\frac{1}{2}}\},$$

then $Eu_m^k(c) < \infty$ *for all positive* $c < \infty$, *all* $m \geqq 1$ *and all* $k \geqq 1$ *when* $a_n \to 0$, *while* $Eu_m^k(c) = \infty$ *for* m *sufficiently large, for all* $c > 0$ *and all* $k \geqq 1$ *when* $a_n \to \infty$.

PROOF. If $a_n \to 0$, $ca_n < c_k$ for any positive finite c if n is large enough, so $u_m(c) \leqq t_m(c)$, hence $Eu_m^k(c) < \infty$ (by Theorem 3.3) for m sufficiently large. Therefore $Eu_m^k(c) < \infty$ for all m. Similarly, when $a_n \to \infty$, $ca_n > c_k$ if $c > 0$ and n is large enough, so $Eu_m^k(c) = \infty$ if m is sufficiently large, whether $u_m(c)$ is a bona-fide stopping rule or not.

3.4. *Remarks.* Theorem 3.4 is strikingly similar to the result of Shepp, [8], that, if $\{W(t), 0 \leqq t < \infty\}$ is a Wiener process and

$$T_{a,c} = \inf\{t : |W(t)| > c(t+a)^{\frac{1}{2}}, a > 0, c > 0\},$$

then $ET_{a,c}^k < \infty$ if and only if $c < c_k$. The rationale for the similarity is that, under the hypothesis of Theorem 3.4 of the central limit theorem, $S_n/n^{\frac{1}{2}}$ behaves like a Wiener process as $n \to \infty$. Thus, on first sight there is a flaw in our result for the case $c = c_k$, when we only have $m^k = o(Et_m^{\ k}(c_k))$ as $m \to \infty$, compared to $ET^k_{a,c_k} = \infty$ for the Wiener process. However the two results are essentially the same by the same rationale as above, if we bear in mind that $W(zt)/z^{\frac{1}{2}}$ is still a Wiener process, and apply similar scale transformations to $S_n/n^{\frac{1}{2}}$, and thence to $t_m(c)$. In the case that $\{X_n\}_{n=1}^{\infty}$ are iid $(EX_n = 0, EX_n^{\ 2} = 1)$ with $E|X_1|^3 < \infty$, it is known (Breiman, [2]) that $Et_m^{\ k}(c_k) = \infty$ for sufficiently large m; the methods used are different to ours.

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APPENDIX

Properties of Hermite polynomials. Define Hermite Polynomials $H_j(x)$ of order j by

(A1) $$H_j(x)e^{-x^2/2} = (d/dx)^j e^{-x^2/2}, \qquad j = 0, 1, 2, \cdots.$$

This definition differs by a scale factor from that of Szego, [9], but is more convenient in the present context. Thus $H_0(x) = 1$, $H_1(x) = -x$, $H_2(x) = x^2 - 1$, and in general

(A2) $$H_{2k}(x) = \sum_{j=0}^{k} (-1)^j (2k)! ((2k-2j)! j! (2!)^j)^{-1} x^{2k-2j}, \quad k = 0, 1, 2, \cdots,$$

and

(A3) $$H_{2k+1}(x) = \sum_{j=0}^{k} (-1)^{j+1} (2k+1)! ((2k+1-2j)! j! (2!)^j)^{-1} x^{2k+1-2j}, \quad k = 0, 1, 2, \cdots.$$

The odd order polynomials H_{2k+1} are odd functions, while those of even order are even functions; with $H_{2k+1}(0) = 0$, and

$$(-1)^{k+1} H_{2k}(0) = -(2k)! (k! (2!)^k)^{-1} < 0, \qquad k = 0, 1, \cdots.$$

Thus, if c_k is the smallest positive zero of $H_{2k}(x)$, then $(-1)^{k+1}H_{2k}(x)$ is negative in $0 \leqq x < c_k$ by continuity. We define $c_0 = +\infty$.

PROPOSITION. *$(-1)^{k+1}H_{2k}$ is increasing, and has a single simple zero, in $0 < x < c_{k-1}$. Consequently, this zero is c_k and $c_k \downarrow$ as $k \to \infty$.*

The elementary properties of Hermite polynomials can be used either to give a direct proof, or to show that $\{H_{2r}(x), r = 0, 1, \cdots, k\}$ is a *Sturm system* of polynomials in $0 \leqq x \leqq c_{k-1}$ ([7], p. 7), whence the proposition follows from Sturm's theorem on zeroes of a polynomial ([7], p. 9).

The corollary

(A4) $$(-1)^k H_{2k-1}(x) > 0 \quad \text{for} \quad 0 < x \leqq c_{k-1} \quad \text{follows,}$$

because of

$$H_j'(x) = -jH_{j-1}(x), \qquad j = 1, 2, \cdots . \tag{A5}$$

We know $c_0 = +\infty$, $c_1 = 1$, $c_2 = (3 - 6^{\frac{1}{2}})^{\frac{1}{2}}$, and in general

$$\pi(8k + 2)^{-\frac{1}{2}} \leqq c_k \leqq \{5/(4k + 1)\}^{\frac{1}{2}}, \qquad k = 1, 2, \cdots \quad \text{(Szego, [9])}. \tag{A6}$$

Finally,

$$\int_{[|y| \geqq c]} H_{2k}(y)e^{-y^2/2}\, dy(2\pi)^{-\frac{1}{2}} = -(2/\pi)^{\frac{1}{2}} H_{2k-1}(c)e^{-c^2/2}, \quad \text{by using (A1)}. \tag{A7}$$

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