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A DECOMPOSITION THEOREM FOR VECTOR VARIABLES WITH A LINEAR STRUCTURE

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0. Summary. A vector variable $\mathbf{X}$ is said to have a linear structure if it can be written as $\mathbf{X} = \mathbf{AY}$ where $\mathbf{A}$ is a matrix and $\mathbf{Y}$ is a vector of independent random variables called structural variables. In earlier papers the conditions under which a vector random variable admits different structural representations have been studied. It is shown, among other results, that complete non-uniqueness, in some sense, of the linear structure characterizes a multivariate normal variable. In the present paper we prove a general decomposition theorem which states that any vector variable $\mathbf{X}$ with a linear structure can be expressed as the sum $(\mathbf{X}_1 + \mathbf{X}_2)$ of two independent vector variables $\mathbf{X}_1$, $\mathbf{X}_2$ of which $\mathbf{X}_1$ is non-normal and has a unique linear structure, and $\mathbf{X}_2$ is multivariate normal variable with a non-unique linear structure.

1. Introduction. In two previous papers (Rao, 1966, 1967), the author proved a number of results characterizing the distribution of structural variables in linear structural relations. An important result is the characterization of the multivariate normal variable through non-uniqueness of its linear structure. The object of the present paper is to prove a general theorem which characterizes a vector variable with a linear structure.

DEFINITION 1. A vector variable $\mathbf{X}$ is said to have a linear structure if it can be expressed as

$$\mathbf{X} = \boldsymbol{\mu} + \mathbf{AY} \tag{1.1}$$

where $\boldsymbol{\mu}$ is a constant vector, $\mathbf{Y}$ is a vector of non-degenerate independent one dimensional variables (called structural variables) and $\mathbf{A}$ is a matrix such that, without loss of generality, there are no two columns of which one is a multiple of the other.

DEFINITION 2. Two structural representations

$$\mathbf{X} = \boldsymbol{\mu}_1 + \mathbf{AY}, \qquad \mathbf{X} = \boldsymbol{\mu}_2 + \mathbf{BZ} \tag{1.2}$$

are said to be equivalent if every column of $\mathbf{A}$ is a multiple of some column of $\mathbf{B}$ and vice versa. Otherwise, they are non-equivalent.

As a necessary condition for equivalence, matrices $\mathbf{A}$ and $\mathbf{B}$ must be of the same order.

DEFINITION 3. A variable $\mathbf{X}$ is said to have an essentially unique structure, or simply a unique structure, if all its linear structural representations are equivalent.

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We prove a lemma which enables us to drop the constant vector in the structural representation (1.1).

LEMMA 0. *Let* $\mathbf{X} = \boldsymbol{\mu}_1 + \mathbf{AY}$ *and* $\mathbf{X} = \boldsymbol{\mu}_2 + \mathbf{BZ}$ *be two structural representations of* $\mathbf{X}$. *Then the linear manifolds generated by the columns of* $\mathbf{A}$ *and* $\mathbf{B}$ *are the same and* $\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2$ *belongs to this common linear manifold.*

Let $\boldsymbol{\alpha}$ be a column vector such that $\boldsymbol{\alpha}'\mathbf{A} = \mathbf{0}$. Then

$$\boldsymbol{\alpha}'\mathbf{X} = \boldsymbol{\alpha}'\boldsymbol{\mu}_1 = \boldsymbol{\alpha}'\boldsymbol{\mu}_2 + \boldsymbol{\alpha}'\mathbf{BZ} \tag{1.3}$$

which shows that $\boldsymbol{\alpha}'\mathbf{BZ}$ is a degenerate random variable, which is not possible unless $\boldsymbol{\alpha}'\mathbf{B} = 0$, observing that the elements of $\mathbf{Z}$ are non-degenerate variables. Thus $\boldsymbol{\alpha}'\mathbf{A} = \mathbf{0} \Leftrightarrow \boldsymbol{\alpha}'\mathbf{B} = \mathbf{0}$, i.e., the linear manifolds generated by the columns of $\mathbf{A}$ and $\mathbf{B}$ are the same.

Further $\boldsymbol{\alpha}'\mathbf{A} = \mathbf{0} \Rightarrow \boldsymbol{\alpha}'(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2) = 0$, i.e., $\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2$ belongs to the same manifold generated by the columns of $\mathbf{A}$ or of $\mathbf{B}$.

It follows from Lemma 0 that, by subtracting a suitable constant vector from $\mathbf{X}$, we can express a structural representation simply as $\mathbf{AY}$. We shall use such a representation in all subsequent work.

We shall state a theorem which follows from the results of the previous papers (Rao, 1966, 1967) and which will be used in the present paper.

THEOREM 1. *Consider a structural representation* $\mathbf{X} = \mathbf{AY}$ *of a vector random variable* $\mathbf{X}$. *Let* $\mathbf{Y}_1$, $\mathbf{Y}_2$ *be two subsets of* $\mathbf{Y}$ *such that the elements of* $\mathbf{Y}_1$ *are non-normal and those of* $\mathbf{Y}_2$ *are normal variables. Further let* $\mathbf{A}_1$, $\mathbf{A}_2$ *be the corresponding partition of* $\mathbf{A}$ *so that*

$$\mathbf{X} = \mathbf{A}_1\mathbf{Y}_1 + \mathbf{A}_2\mathbf{Y}_2 . \tag{1.4}$$

Then any other structure of $\mathbf{X}$ *is of the form*

$$\mathbf{X} = \mathbf{A}_1\mathbf{U}_1 + \mathbf{B}_2\mathbf{U}_2 \tag{1.5}$$

where, after suitable scaling, the elements of $\mathbf{U}_1$ *are non-normal with the same structural matrix* $\mathbf{A}_1$ *as for* $\mathbf{Y}_1$, *and those of* $\mathbf{U}_2$ *are normal variables with a structural matrix* $\mathbf{B}_2$ *which may be different from* $\mathbf{A}_2$ *in the number of columns and which may not be deducible from* $\mathbf{A}_2$ *by suitable scaling of columns.*

Note that in all structural representations of $\mathbf{X}$, a part of the structure is unique and the other part can vary both with respect to the structural coefficients and the number of structural variables. The number of non-normal variables is the same in all structural representations; hence we have the following theorem.

THEOREM 2. *Let* $\mathbf{X} = \mathbf{AY}$ *be a structural representation of* $\mathbf{X}$ *and let the elements of* $\mathbf{Y}$ *be all non-normal variables. Then there does not exist a non-equivalent structure involving the same number or a smaller number of structural variables than that of* $\mathbf{Y}$.

It also follows from Theorem 1, that if $\mathbf{X} = \mathbf{AY}$ and $\mathbf{X} = \mathbf{BZ}$ are two structural representations such that no column of $\mathbf{A}$ is a multiple of any column of $\mathbf{B}$, then $\mathbf{X}$ is multivariate normal.

The main theorem proved in this paper is as follows.

THEOREM 3. *Let* $\mathbf{X}$ *be a p-vector random variable with a linear structure* $\mathbf{X} = \mathbf{AY}$. *Then* $\mathbf{X}$ *admits the decomposition*

$$\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2 \tag{1.6}$$

where $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *are independent,* $\mathbf{X}_1$ *has an essentially unique linear structure and* $\mathbf{X}_2$ *is p-variate normal (with a non-unique linear structure). It is possible that* $\mathbf{X}_1$ *or* $\mathbf{X}_2$ *is a null vector.*

We need to establish some preliminary lemmas.

LEMMA 1. *Let* $\mathbf{G}_n$ *for each* n *be a vector of* k *independent random variables. Consider the sequence of p-vector random variables* $\mathbf{X}_n = \mathbf{BG}_n$ *where* $\mathbf{B}$ *is* $p \times k$ *matrix. If* $\mathbf{X}_n \to_L \mathbf{X}$, *then* $\mathbf{X}$ *has also the structure,* $\mathbf{X} = \mathbf{BG}$ *where* $\mathbf{G}$ *is a vector of* k *independent random variables.*

We may assume, without loss of generality, that $\mathbf{B}$ has no column of all zeroes. Then the condition $\mathbf{X}_n \to_L \mathbf{X}$ implies, by a slight extension of a theorem due to Parthasarathy, Ranga Rao and Varadhan (1962) that $\mathbf{G}_n$ is shift compact, i.e., there exists a subsequence $\mathbf{G}_m$ with a sequence of centering vectors $\mathbf{C}_m$, such that $(\mathbf{G}_m - \mathbf{C}_m) \to_L \mathbf{G}$. Now consider

$$\mathbf{X}_m = \mathbf{B}(\mathbf{G}_m - \mathbf{C}_m) + \mathbf{BC}_m. \tag{1.7}$$

Since $\mathbf{X}_m$ and $(\mathbf{G}_m - \mathbf{C}_m)$ have limiting distributions, it follows that $\mathbf{BC}_m \to \mathbf{C}$ (a constant vector). Then

$$\mathbf{X} = \mathbf{BG} + \mathbf{C}. \tag{1.8}$$

Let $\mathbf{b}$ be a vector orthogonal to the columns of $\mathbf{B}$. Then

$$0 = \mathbf{b}'\mathbf{BC}_m \to \mathbf{b}'\mathbf{C}, \quad \text{i.e.,} \quad \mathbf{b}'\mathbf{C} = 0, \tag{1.9}$$

i.e., the constant $\mathbf{C}$ can be absorbed in the random variable $\mathbf{G}$ in (1.8), so that the structure of $\mathbf{X}$ can be simply written as $\mathbf{X} = \mathbf{BG}$.

LEMMA 2. *Let* $\mathbf{X}$ *be any p-vector variable. Then* $\mathbf{X}$ *admits the decomposition*

$$\mathbf{X} = \mathbf{X}_1 + \mathbf{X}_2 \tag{1.10}$$

where $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *are independent, and* $\mathbf{X}_2$ *is p-variate normal with a maximal dispersion matrix, i.e., there is no other decomposition*

$$\mathbf{X} = \mathbf{Y}_1 + \mathbf{Y}_2 \tag{1.11}$$

where $\mathbf{Y}_1$ *and* $\mathbf{Y}_2$ *are independent, and* $\mathbf{Y}_2$ *is p-variate normal with its dispersion matrix greater than that of* $\mathbf{X}_2$.

Let $C(\mathbf{t})$ be the characteristic function (ch.f.) of $\mathbf{X}$ and let S be the set of all non-negative definite matrices such that for any member $A \varepsilon S$

$$C(\mathbf{t}) \exp [\tfrac{1}{2}\mathbf{t}'\mathbf{A}\mathbf{t}] \tag{1.12}$$

is a ch.f. It is easy to see that the set of matrices in S is bounded above.

Consider the set $\{a_{11}^{\alpha}\}$ of the first diagonal elements of the members of S. It is easy to see that there is an upper bound a_{11}^{*} belonging to the set. Now consider the set $\{a_{11}^*, a_{22}^\beta\}$, where a_{22}^β represents the second diagonal element of a matrix with a_{11}^* as the first diagonal element. The set $\{a_{22}^\beta\}$ has similarly an upper bound a_{22}^* belonging to the set. Finally we arrive at a matrix with diagonal elements $a_{11}^*, \cdots, a_{pp}^*$ which is obviously a maximal element in S. The associated decomposition (1.10) satisfies the requirements of Lemma 2.

2. Proof of the main theorem. Consider the structural representation $\mathbf{X} = \mathbf{AY}$. Let us partition the vector variable $\mathbf{Y}$ into $\mathbf{Y}_1, \mathbf{Y}_2$ where the elements of $\mathbf{Y}_1$ are non-normal and those of $\mathbf{Y}_2$ are normal variables. We have the corresponding partition of $\mathbf{A}$ giving the structural relationship

$$\mathbf{X} = \mathbf{A}_1\mathbf{Y}_1 + \mathbf{A}_2\mathbf{Y}_2 = \mathbf{U}_1 + \mathbf{U}_2 \tag{2.1}$$

where $\mathbf{U}_1$ and $\mathbf{U}_2$ are independent and $\mathbf{U}_2$ is p-variate normal. The equation (2.1) provides a decomposition of $\mathbf{X}$ but $\mathbf{U}_1$ may not have a unique structure. However, from Theorems 1 and 2, it follows that if $\mathbf{U}_1$ does not have a unique structure, it has an alternative structure of the form

$$\mathbf{U}_1 = \mathbf{A}_1\mathbf{Y}_{1\alpha} + \mathbf{B}_\alpha\mathbf{Z}_\alpha = \mathbf{X}_{1\alpha} + \mathbf{X}_{2\alpha} \tag{2.2}$$

where $\mathbf{Z}_\alpha$ is a vector of $N(0, 1)$ variables.

Consider the set S of non-negative definite matrices $\{\mathbf{D}_\alpha\} = \{\mathbf{B}_\alpha\mathbf{B}_\alpha'\}$ for which a decomposition such as (2.2) exists. Then applying Lemmas 1 and 2, we find that there is a maximal element $\mathbf{G}$ in the set S leading to the decomposition

$$\mathbf{A}_1\mathbf{Y}_1 = \mathbf{U}_1 = \mathbf{A}_1\mathbf{V}_1 + \mathbf{HV}_2 \tag{2.3}$$

where $\mathbf{HH}' = \mathbf{G}$. Let $\mathbf{X}_1 = \mathbf{A}_1\mathbf{V}_1$. Then $\mathbf{X}_1$ has a unique structure. If not let

$$\mathbf{X}_1 = \mathbf{A}_1\mathbf{W}_1 + \mathbf{FW}_2 \tag{2.4}$$

where $\mathbf{W}_2$ is a vector of $N(0, 1)$ variables. In such a case

$$\mathbf{U}_1 = \mathbf{A}_1\mathbf{W}_1 + \mathbf{FW}_2 + \mathbf{HV}_2 \tag{2.5}$$

where the dispersion matrix of the normal components $(\mathbf{W}_2, \mathbf{V}_2)$ is $\mathbf{FF}' + \mathbf{HH}' \geqq \mathbf{HH}' = \mathbf{G}$ leading to a contradiction.

From (2.1)

$$\begin{aligned} \mathbf{X} &= \mathbf{A}_1\mathbf{Y}_1 + \mathbf{A}_2\mathbf{Y}_2 \\ &= (\mathbf{A}_1\mathbf{V}_1 + \mathbf{HV}_2) + \mathbf{A}_2\mathbf{Y}_2 = \mathbf{A}_1\mathbf{V}_1 + (\mathbf{HV}_2 + \mathbf{A}_2\mathbf{Y}_2) = \mathbf{X}_1 + \mathbf{X}_2 \end{aligned} \tag{2.6}$$

where $\mathbf{X}_1$ and $\mathbf{X}_2$ are independent, $\mathbf{X}_1$ has a unique structure and $\mathbf{X}_2$ is multivariate normal.

Thus we have proved that given a vector variable with a linear structure, it can be expressed as the sum of two independent variables one of which has a unique linear structure and the other is multivariate normal (with a non-unique linear structure). The non-uniqueness of the linear structure of $\mathbf{X}$ is due to the (multivariate) normal component in it.

In general, the decomposition (2.6) may not be unique. An alternative decomposition $\mathbf{Z}_1 + \mathbf{Z}_2$ may exist such that $\mathbf{X}_1$ and $\mathbf{Z}_1$ both have *unique* linear structure but may have different distributions. A sufficient condition for unique decomposition is that rank $\mathbf{A}_1 =$ the numbers of columns of $\mathbf{A}_1$ where $\mathbf{A}_1$ is as defined in (2.1) (see Rao, 1967).

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REFERENCES

[1] PARTHASARATHY, K., RANGA RAO, R. and VARADHAN, S. R. S. (1962). On the category of indecomposable distributions on topological groups. *Trans. Amer. Math. Soc.* **102** 202–217.

[2] RADHAKRISHNA RAO, C. (1966). Characterization of the distribution of random variables in linear structural relations. *Sankhyā, Ser. A* **28** 252–260.

[3] RADHAKRISHNA RAO, C. (1967). On vector variables with a linear structure and a characterization of the multivariate normal distribution. To be published in *Bull. Inst. Internat. Statist.* **42**.