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MINIMAX RISK AND UNBIASEDNESS FOR MULTIPLE DECISION PROBLEMS OF TYPE I[1]

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0. Summary. The paper studies Wald's minimax risk (M.R.) criterion and Lehmann's unbiasedness condition for a very general class of Type I problems (Section 2) which contains nearly all nonsequential multiple decision problems (m.d.p.'s) from parametric statistics (at least when no reasonable a priori knowledge is available—apart from knowledge restricting the parameter space in, for example, one-sided problems or in trend situations—and provided that the problems can be formulated by means of a loss function which is *constant* for the various kinds of error.)

Type I problems turn out to behave in a *degenerate* way. Generally the M.R. procedure is not unique (Situation 1, see Theorem 3.1 and the Sections 5, 11, 12); when a unique M.R. procedure exists, then this is trivial and useless (Situation 2, see Corollary 3.1 and Sections 6, $\cdots$, 10).

We try to remedy this by applying Lehmann's unbiasedness condition (Section 4). This has to be done cautiously for the unbiasedness condition might be too restrictive: sometimes the class W of all unbiased procedures is so small (Corollary 4.1) that $W = \emptyset$ (Theorem 8.1 (i)) or such that W contains only poor and useless procedures (Theorem 8.1 (iii) and Section 12).

Fortunately we can show for some problems in Situation 1 that $W \subset M$ where M denotes the class of all M.R. procedures (cf. Theorem 4.1; the unbiasedness restriction seems to be very attractive when the sufficient conditions of Lemma 4.2 are satisfied, see Theorem 5.1 where $W = M$, Lemma 7.1 (ii) and Theorem 11.1; in Section 12 we also have $W \subset M$ but nevertheless the unbiasedness restriction is not attractive).

For problems in Situation 2 the unique M.R. procedure δ^* is trivial and useless. Nevertheless we regard it as an advantage of the unbiasedness restriction when $\delta^* \in W$ (Sections 6, $\cdots$, 10), whereas $\delta^* \notin W$ is regarded as an indication that W might be too small (Theorem 8.1 (iv), Remark 8.1 and Lemma 10.1).

We always try to obtain the procedure with "the most attractive appropriate optimum property." As shown by Lehmann ordinary two-decision testing problems (Section 5) and products of such problems (Section 11) do not present extensive difficulties because the unbiasedness restriction is attractive (Theorems 5.1 and 11.1) and reduces our m.d.p.'s to problems in the Neyman-Pearson formulation. Many three-decision two-sided problems are solved (Section 6) though we have

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to be content with criterions which are not very compelling. Similar results are obtained for some slippage problems (Sections 7, ..., 9) where the "optimum" procedure turns out to be not of the "natural" form considered in literature. For many m.d.p.'s of Type I (except for those in Sections 5, 6 and 11) the actual construction of the "optimum" procedure is forbidding and we have to content ourselves with results relating the classes W and M (cf. Sections 7, ..., 10, 12; the problems can be solved only in very simple special cases).

1. Introduction. Interest in m.d.p.'s was excited by statisticians who proposed m.d. procedures for *ranking the means* in an analysis of variance with one-way classification and where sufficient evidence is required for stating that one mean is (significantly) smaller than another (Fisher (1935), Newman (1939), Keuls (1952), Tukey (1953), Scheffé (1953), Duncan (1955); reviews are given in Miller [22] and David [8]). In Sections 11 and 12 we shall try to solve simplifications of these problems.

In 1948 Mosteller [23] considered *slippage* problems where in the analysis of variance k-sample situation, $k + 1$ decisions are possible; decision d_0 stating that there exists no sufficient evidence for inhomogeneity, and decisions d_i stating that the ith mean has slipped to the right (is the best, or an outlier) ($i = 1, \cdots, k$). Paulson [26] and others ([9], [16] and [33]) proved that a "natural" procedure has certain optimum properties. In Section 9 we shall see that this result is lost when we consider the "most beautiful" decision theoretical formulation of the problem.

Related to slippage problems are the problems (i) for selecting the best or largest mean (decision d_0 is no longer permitted, see [2], [12], [30]); (ii) for selecting the t best populations ([1], [4], [5]); (iii) for *selecting a subset of arbitrary size containing the best* ([2], [13], [21], [24], [29] and [31]; in Example 10.2 we shall give a decision theoretical formulation of this problem); (iv) for selecting (a subset containing) the best when several treatments have been compared with a control or standard situation which is not to be rejected unless sufficient evidence is available for stating that one of the new treatments is better (Paulson [25] and others). In 1950 Bahadur [2] proved for certain formulations of some of these problems, that the "best" procedure is the "natural" one which selects the subset $\{\mu_i\}$ consisting of the ith expectation, if and only if the ith sample mean is the largest.

In 1957 Lehmann [19] gave a general decision theoretical formulation for large classes of m.d.p.'s by defining loss functions which are *constant*, for the various different kinds of error and over the subsets of the parameter space Ω in which this is naturally partitioned. Lehmann applied his unbiasedness condition, constructed uniformly minimum risk unbiased procedures and showed for certain problems that the class W of unbiased procedures is a subclass of the class M, of all M.R. procedures.

Our paper starts from a similar point of view. The dissimilarity to Lehmann's results may be best demonstrated by considering the interesting *three-decision* problem that describes the basic situation occurring when the result of a two-sided test has to be interpreted. Lehmann [19] (see also [18] and [32]) defined the loss for all $\theta \varepsilon \Omega$ and constructed *uniformly minimum risk unbiased* procedures. In our opinion ([27] Chapter 3) a more attractive formulation is obtained when we do not define the loss for all $\theta \varepsilon \Omega$. By adopting this better formulation we have to content ourselves with less compelling but not less reasonable criterions (see Remark 6.1).

2. The general formulation of a class of multiple decision problems. It will be shown in Sections 5, ..., 12 that the following problems are of interest in many situations where the experimenter tries to draw scientific conclusions from his observations. (In many industrial applications one will have doubts about the adequacy of the loss function.)

On the basis of an observation x in the sample space $\mathfrak{X}$ of a random observable X, we have to choose a decision d out of the finite space $\mathfrak{D} = \{d_0, \cdots, d_n\}$ of all interesting decisions (see [6] Chapter 7). It is known that X has a pdf (Radon-Nikodym derivative) p_θ out of the class $\mathcal{P} = \{p_\theta ; \theta \varepsilon \Omega\}$ of admitted pdf's with respect to some σ-finite measure μ over (a σ-field $\mathcal{A}$ of measurable subsets of) $\mathfrak{X}$.

The parameter space Ω is partitioned into $m + 1$ mutually exclusive subsets $\Omega = \Omega_0 \cup \cdots \cup \Omega_m$ where Ω_0 is an *indefiniteness zone* ($\Omega_0 = \emptyset$ is permitted). Let $\Omega' = \Omega - \Omega_0$ and the loss function $L:\Omega' \times \mathfrak{D} \rightarrow [0, \infty)$ be defined by $L(\theta, d) = w_{ij}$ for $d = d_j$ and all $\theta \varepsilon \Omega_i (i = 1, \cdots, m; j = 0, \cdots, n)$. So for d fixed, the loss function is constant over each Ω_i while no loss is defined for $\theta \varepsilon \Omega_0$.

The statistician has to construct a (possibly) randomized decision procedure δ or equivalently (see [6] page 172) an $(n + 1)$-tuple $(\varphi_0, \cdots, \varphi_n)$ of test functions satisfying $\sum_{j=0}^{n} \varphi_j(x) = 1$ for all $x \varepsilon \mathfrak{X}$. A function $\varphi:\mathfrak{X} \rightarrow [0, 1]$ is said to be a test function if and only if φ is (μ)-measurable. The procedure $\delta = \delta(\varphi_0, \cdots, \varphi_n)$ prescribes that if $x \varepsilon \mathfrak{X}$ has been observed, a random experiment has to be performed providing decision d_j with probability $\varphi_j(x) (j = 0, \cdots, n)$. The risk of this procedure δ in $\theta \varepsilon \Omega_i$ is determined by

$$R(\theta, \delta) = E_\theta[L\{\theta, \delta(X)\}] = \textstyle\sum_{j=0}^{n} w_{ij} E_\theta\{\varphi_j(X)\} \quad (2.1)$$

$(i = 1, \cdots, m)$ and a procedure δ^* has minimax risk if

$$\sup_{\theta\varepsilon\Omega'} R(\theta, \delta^*) = \inf_\delta \sup_{\theta\varepsilon\Omega'} R(\theta, \delta). \quad (2.2)$$

Many problems from actual practice satisfy the following definition which expresses that $\Omega_1, \cdots, \Omega_m$ have common boundary points.

DEFINITION. A multiple decision problem is said to be of Type I if the parameter space Ω is a subset of the Euclidean space R^s such that (i) $E_\theta\{\varphi(X)\}$ is a continuous function of $\theta (\theta \varepsilon \Omega)$ for each test function φ and (ii) $[\Omega_1] \cap \cdots \cap [\Omega_m] = \Omega_0' \neq \emptyset$.

Here [ ] denotes the closure of a set and the notation Ω_0' is used because for a number of important applications we have $\Omega_0' = \Omega_0$.

SOME BASIC NOTIONS FOR TYPE I PROBLEMS. Many properties for Type I problems will turn out to be (almost) completely characterized by the $m \times (n + 1)$

matrix (w_{ij}). It is of interest to characterize this matrix by a convex set S in the Euclidean space R^m of points $u = (u_1, \cdots, u_m)$, with inner product $(u, v) = \sum_{i=1}^m u_i v_i$, norm $\|u\| = (u, u)^{\frac{1}{2}}$ and metric $d(u, v) = \|u - v\|$. For that purpose let S be the convex envelope in R^m of the $n + 1$ points $w_{\cdot j} = (w_{1j}, \cdots, w_{mj})$ $(j = 0, \cdots, n)$. Hence

$$S = \{v : v = \textstyle\sum_{j=0}^n p_j w_{\cdot j},\ p_j \geqq 0, \sum_{j=0}^n p_j = 1\}. \tag{2.3}$$

We say that w^* is a *minimax point of* S if (i) $w^* \varepsilon S$ and (ii) if $w^* = (w_1^*, \cdots, w_m^*)$ then for each $v = (v_1, \cdots, v_m) \varepsilon S$ we have $\max v_i \geqq \max w_i^*$. If on the other hand $w^* \varepsilon S$ is such that for each $v \varepsilon S$ we have $\min v_i \leqq \min w_i^*$, then w^* is said to be a *maximin point of* S.

Thus the geometric picture corresponds with that used in the theory for the game with matrix (w_{ij}), where w_{ij} is the pay-off of Player II to Player I when the maximizing Player I uses the pure strategy (row) i while the minimizing Player II uses strategy j. The point $w^* = \sum_{j=0}^n p_j^* w_{\cdot j}$ is a minimax point of S if and only if $(p_0^*, \cdots, p_n^*)$ is a minimax (mixed) strategy for Player II (see Karlin [15] Section 1.4).

3. Minimizing the maximum risk. For Type I problems, much can be said about the M.R. procedures if we know the class of all minimax strategies for Player II in the related game with pay-off matrix (w_{ij}).

THEOREM 3.1. *For Type* I *problems the following holds.*

(i) *If* $(p_0^*, \cdots, p_n^*)$ *is a minimax strategy for Player* II *then the constant procedure* $\delta_1 = \delta(\varphi_0^{(1)}, \cdots, \varphi_n^{(1)})$ *with* $\varphi_j^{(1)}(x) = p_j^*(j = 0, \cdots, n)$ *for all* $x \varepsilon \mathfrak{X}$, *has minimax risk.*

(ii) *If* $\delta^* = \delta(\varphi_0^*, \cdots, \varphi_n^*)$ *is an* M.R. *procedure then for each* $\theta_0 \varepsilon \Omega_0'$ *we have that* $(p_0^*(\theta_0), \cdots, p_n^*(\theta_0))$ *is a minimax strategy for Player* II *when* $p_j^*(\theta_0) = E_{\theta_0}\{\varphi_j^*(X)\}$ $(j = 0, \cdots, n)$.

PROOF. (i) $w^* = \sum_{j=0}^n p_j^* w_{\cdot j} = (w_1^*, \cdots, w_m^*)$ is a minimax point of the convex set S. Let $\bar{w}^* = \max w_i^*$. The problem is of Type I. Hence there exists a sequence $(\theta_1^{(i)}, \theta_2^{(i)}, \cdots)$ of points in Ω_i such that $\theta_k^{(i)} \to \theta_0$ for some $\theta_0 \varepsilon \Omega_0'$ $k \to \infty$ $(i = 1, \cdots, m)$. The risk $R(\theta_k^{(i)}, \delta)$ for the arbirtary procedure $\delta(\varphi_0, \cdots, \varphi_n)$ is given in (2.1) and converges to $\sum_{j=0}^n w_{ij} p_j$ as $k \to \infty$ where $p_j = E_{\theta_0}\{\varphi_j(X)\}$. Obviously $\sum_{j=0}^n p_j w_{\cdot j} \varepsilon S$. But w^* is a minimax point of S. Hence

$$\sup_{\theta \varepsilon \Omega'} R(\theta, \delta) \geqq \max_i \textstyle\sum_{j=0}^n p_j w_{ij} \geqq \max_i w_i^* = \bar{w}^* \tag{3.1}$$

and consequently the right-hand side of (2.2) is not smaller than $\bar{w}^*$. But the constant procedure δ_1 satisfies $\sup_\theta R(\theta, \delta_1) = \bar{w}^*$. Hence δ_1 has minimax risk and the right-hand side of (2.2) is equal to $\bar{w}^*$.

(ii) We have to show that $w^*(\theta_0) = \sum_{j=0}^n p_j^*(\theta_0) w_{\cdot j}$ is a minimax point of S. But with respect to δ^* equality holds everwhere in (3.1). Hence the maximum coordinate of $w^*(\theta_0)$ is equal to $\bar{w}^*$ and $w^*(\theta_0)$ is a minimax point of S.

COROLLARY 3.1. *If* $(p_0^*, \cdots, p_n^*) = (1, 0, \cdots, 0)$ *is the unique minimax* *strategy for Player* II *in the game with matrix* (w_{ij}) *and if* $E_{\theta_0}\{\varphi(X)\} = 0$ *for a test function* φ *implies* $\varphi = 0$ a.e. (μ) *at least for some* $\theta_0 \varepsilon \Omega_0'$, *then each* M.R. *procedure* $\delta^* = \delta(\varphi_0^*, \cdots, \varphi_n^*)$ *for the Type* I *problem satisfies* $\varphi_0^* = 1, \varphi_j^* = 0 \, (j = 1, \cdots, n)$ a.e. (μ).

This corollary is a simple consequence of Theorem 3.1(ii) and constitutes a generalization of [19] page 572 and [27] page 50. It will show for the problems in Sections 6, $\cdots$, 9 that the trivial procedure which assigns decision d_0 with probability 1 to all $x \varepsilon \mathfrak{X}$ is the unique M.R. procedure (procedures are identified when having the same risk function).

The general problem for obtaining all minimax strategies for Player II (and maximin strategies for Player I) in the matrix game, has been dealt with in Karlin [15] Chapter 2. The set of all minimax strategies is the convex envelope of a finite number of extreme-point optimal strategies which can be obtained by considering respectively all square submatrices of the matrix (w_{ij}). Fortunately we can often employ more direct methods by using the special properties of the matrix under consideration. In many cases it is possible to guess what the optimum strategies are. In that case we only need a proof that they are indeed optimal and that we obtained all optimal strategies. The following lemmas are helpful.

Lemma 3.1. $w^* = \sum_{j=0}^n p_j^* w_{\cdot j}$ *is a minimax point of* S $((p_0^*, \cdots, p_n^*)$ *is a minimax strategy for Player* II) *if and only if there exists a strategy* $(g_1, \cdots, g_m)$ *for Player* I *such that* (i) $g_i = 0$ *for all* i *with* $w_i^* < \max w_i^* = \bar{w}^*$ *and* (ii) $(g, w_{\cdot j} - w^*) \geqq 0 \, (j = 0, \cdots, n)$.

Proof. Suppose $(g_1, \cdots, g_m)$, $(p_0^*, \cdots, p_n^*)$ is a pair of optimal strategies for the players I and II respectively. Then $(g, w^*) = \bar{w}^*$ is the value of the game. This shows (i) (see also Karlin [15] Lemma 2.1.2) and (ii), for II will lose not less than $\bar{w}^*$ when using the pure strategy j in case I uses his maximin strategy g.

On the other hand, if the conditions (i) and (ii) are satisfied and $(h_1, \cdots, h_m)$ is an arbitrary mixed strategy for Player I while $(p_0, \cdots, p_n)$ is a strategy for II, then by applying (i) and (ii) respectively, we obtain

$$(h, w^*) \leqq (g, w^*) \leqq (g, \textstyle\sum_{j=0}^n p_j w_{\cdot j})$$

which shows that $(g_1, \cdots, g_m)$ is a maximin strategy for Player I while $(p_0^*, \cdots, p_n^*)$ is a minimax strategy for Player II (see Karlin [15] Corollary 1.3.1).

Lemma 3.2. *If there exists a strategy* $(g_1, \cdots, g_m)$ *for Player* I *such that* (i) $g_i = 0$ *for all* i *with* $w_{i0} < \max w_{i0} = \bar{w}$ *and* (ii) $(g, w_{\cdot j} - w_{\cdot 0}) > 0 \, (j = 1, \cdots, n)$ *then* $(1, 0, \cdots, 0)$ *is the unique minimax strategy for Player* II.

Proof. Lemma 3.1 shows that $w_{\cdot 0}$ is a minimax point of S and that $(g, w_{\cdot 0})$ is the value of the game. For each strategy $(p_0, \cdots, p_n) \neq (1, 0, \cdots, 0)$ of Player II we have $(g, \sum_{j=0}^n p_j w_{\cdot j}) > (g, w_{\cdot 0})$ on account of (ii). This establishes the uniqueness.

Remark concerning another formulation. In [17] Krafft suggested to replace the formulation of Section 2 by the following one which is also based on a matrix $w_{ij} (i = 1, \cdots, m; j = 0, \cdots, n)$ of weights and where one looks for the "best-W" procedure $\delta(\varphi_0^*, \cdots, \varphi_n^*)$ such that

$$\sum \sum w_{ij} \sup_{\theta \varepsilon \Omega_i} E_\theta\{\varphi_j^*(X)\} = \inf_\delta \sum \sum w_{ij} \sup_{\theta \varepsilon \Omega_i} E_\theta\{\varphi_j(X)\}.$$

In our opinion this formulation is less attractive than the formulation of Section 2 because the loss and risk functions of Section 2 have clear interpretations. Moreover "best-W" procedures are unusable for Type I problems. In order to show this, let θ_0 be some point in Ω_0' and let $\bar{w}_{\cdot j} = \sum_{i=1}^m w_{ij}$. Then

$$\sum \sum w_{ij} \sup_{\Omega_i} E_\theta\{\varphi_j(X)\} \geqq \sum \sum w_{ij} E_{\theta_0}\{\varphi_j(X)\}$$
$$= \sum \bar{w}_{\cdot j} E_{\theta_0}\{\varphi_j(X)\} \geqq \min_j \bar{w}_{\cdot j}.$$

But if j^* is an index such that $\bar{w}_{\cdot j^*} = \min_j \bar{w}_{\cdot j}$, then the trivial procedure with $\varphi_{j^*} = 1$ a.e. (μ) has the property that

$$\sum \sum w_{ij} \sup_{\Omega_i} E_\theta\{\varphi_j(X)\} = \bar{w}_{\cdot j^*}$$

with the result that this trivial and unusable procedure is "best-W."

4. Unbiasedness and minimax risk. The conditions of Corollary 3.1 are satisfied for the problems which will be considered in Sections 6, $\cdots$, 10. Thus the M.R. criterion is inappropriate for these problems; for the unique M.R. procedure is the trivial procedure δ_0 which has a constant risk function, with the result that there will exist procedures with a larger maximum risk but with a much smaller risk for the greater part of Ω'. For other interesting m.d.p.'s of Type I (Sections 5, 11 and 12) there exists a large class of M.R. procedures and one will want to restrict this class.

In order to deal with the situations described above, Lehmann's unbiasedness condition may be of interest.

DEFINITION. A procedure δ is said to be unbiased ([20] page 12) if and only if

$$E_\theta[L\{\theta', \delta(X)\}] \geqq E_\theta[L\{\theta, \delta(X)\}] = R(\theta, \delta) \tag{4.1}$$

holds for all $\theta, \theta' \varepsilon \Omega'$.

LEMMA 4.1. *The procedure* $\delta = \delta(\varphi_0, \cdots, \varphi_n)$ *is unbiased if and only if for* $i = 1, \cdots, m$ *the following inequalities hold for all* $\theta \varepsilon \Omega_i$:

$$\sum_{j=0}^n w_{hj} E_\theta\{\varphi_j(X)\} \geqq \sum_{j=0}^n w_{ij} E_\theta\{\varphi_j(X)\} \quad (h = 1, \cdots, m). \tag{4.2}$$

PROOF. For $\theta \varepsilon \Omega_i$ and $\theta' \varepsilon \Omega_h$ we have

$$E_\theta[L\{\theta', \delta(X)\}] = \sum_{j=0}^n w_{hj} E_\theta\{\varphi_j(X)\}. \tag{4.3}$$

COROLLARY 4.1. *For Type* I *problems the unbiasedness of* δ *implies that equality holds in* (4.2) *for all* $\theta \varepsilon \Omega_0'$.

As $\Omega_0' \neq \emptyset$ we see that no unbiased procedure can exist unless the convex set S contains at least one point $w^* = (\bar{w}^*, \cdots, \bar{w}^*)$ with all coordinates equally large. If S contains such a point $w^* = \sum_{j=0}^n p_j^* w_{\cdot j}$ then the constant procedure δ_1 of Theorem 3.1 provides an example of an unbiased procedure. Moreover Corollary 4.1 and Theorem 3.1 (ii) show that no M.R. procedure can be unbiased unless there exists a minimax point w^* of S with all coordinates equally large. In our opinion these remarks show that Lehmann's unbiasedness condition must not be applied too rashly. We shall round off these discussions in Remark 4.2.

THEOREM 4.1. *If for a Type* I *problem there exists a point* $w^* = (\bar{w}^*, \cdots, \bar{w}^*)$ *that is both a minimax and a maximin point of S, then each unbiased procedure has minimax risk.*[2]

PROOF. Suppose $\delta = \delta(\varphi_0, \cdots, \varphi_n)$ is unbiased and $\theta \varepsilon \Omega_i$. Consider the point $w_\delta(\theta) = \sum_{j=0}^n p_j w_{\cdot j} \varepsilon S$ where $p_j = E_\theta\{\varphi_j(X)\}$. Lemma 4.1 shows that no coordinate of $w_\delta(\theta)$ is smaller than the ith which is equal to $R(\theta, \delta)$. But w^* is a maximin point. Hence $R(\theta, \delta) \leqq \bar{w}^*$ for $\theta \varepsilon \Omega_i$ and consequently $\sup_{\theta\varepsilon\Omega'} R(\theta, \delta) \leqq \bar{w}^*$. But the problem is of Type I and w^* is a minimax point. The proof of Theorem 3.1 shows that the right-hand side of (2.2) is equal to $\bar{w}^*$. Hence δ has minimax risk.

The following lemma gives sufficient conditions for the applicability of Theorem 4.1.

LEMMA 4.2. *If* $w^* = (\bar{w}^*, \cdots, \bar{w}^*) \varepsilon S$ *is such that there exists a strategy* $(g_1, \cdots, g_m)$ *for Player* I *with* $g_i > 0 (i = 1, \cdots, m)$ *and* $(g, w_{\cdot j} - w^*) = 0 (j = 0, \cdots, n)$ *then* w^* *is the unique minimax and the unique maximin point of S.*

PROOF. We shall show that w^* is the unique maximin point of S or equivalently that $w = \sum_{j=0}^n p_j w_{\cdot j} \varepsilon S$, $w_i \geqq \bar{w}^* (i = 1, \cdots, m)$ implies that $w = w^*$. But $g_i > 0 (i = 1, \cdots, m)$ implies $\sum_{i=1}^m g_i(w_i - \bar{w}^*) \geqq 0$ with strict inequality unless $w = w^*$. Hence $(g, w) > (g, w^*)$ unless $w = w^*$. But $(g, w) = \sum_{j=0}^n p_j(g, w_{\cdot j}) = (g, w^*)$. Hence $w = w^*$.

REMARK 4.1. The conditions of Lemma 4.2 do not imply that w^* has a unique convex representation (see Lemma 7.1 (ii) and Remark 11.1). The following counter-example shows that the sufficient conditions of Lemma 4.2 are not necessary. Take $m = 2$, $n = 2$, $w_{\cdot 0} = (2, 2)$, $w_{\cdot 1} = (3, 0)$, $w_{\cdot 2} = (3, 1)$ then $w_{\cdot 0}$ is the unique minimax and also the unique maximin point.

REMARK 4.2. We propose to apply the unbiasedness condition (i) when the M.R. procedure is not unique while S contains a point $w^* = (\bar{w}^*, \cdots, \bar{w}^*)$ that is the *unique* minimax point and (one of) the maximin points, for in that case Theorem 4.1 shows $W \subset M$ where W denotes the class of all unbiased and M that of all M.R. procedures (in Sections 5 and 11 the conditions of Lemma 4.2 are satisfied and accordingly we apply the unbiasedness restriction; in Section 12 there exists a point w^* that is the unique maximin point but the worst of the minimax points: though $W \subset M$ on account of Theorem 4.1, the unbiasedness condition is not appropriate because it is too restrictive, W contains only bad and useless M.R. procedures), (ii) when the M.R. procedure δ^* is unique and trivial (Theorem 3.1 (i)) while the class W of unbiased procedures is "not too small"; in this case we regard $\delta^* \not\varepsilon W$ as an indication that W might be too small though this argument is not sufficient because δ^* is an uninteresting procedure from the practical point of view (see Sections 5, $\cdots$, 10).

[2] In the meantime, Mrs. W. Stefansky, Department of Statistics, Berkeley, obtained a theorem describing necessary and sufficient conditions for each unbiased procedure having minimax risk.

REMARK 4.3. We apply the unbiasedness restriction in order to obtain an optimum procedure with respect to the class of all unbiased procedures. We define some optimum properties with respect to an arbitrary class W of decision procedures (see [27]). Let $R_W^*(\theta) = \inf_{\delta \varepsilon W} R(\theta, \delta)$ denote the *envelope risk* function in $\theta \varepsilon \Omega'$. Then $S_W(\theta, \delta) = R(\theta, \delta) - R_W^*(\theta)$ is called the *regret* function in $\theta \varepsilon \Omega'$ of δ with respect to the class W. The procedure δ^* has U.M.R. (W) (*uniformly minimum risk with respect to* W) if (i) $\delta^* \varepsilon W$ and (ii) $S_W(\theta, \delta) = 0$ for all $\theta \varepsilon \Omega'$ We say that δ^* has *minimax regret* (W) if (i) $\delta^* \varepsilon W$ and (ii)

$$\sup_{\Omega'} S_W(\theta, \delta^*) = \inf_{\delta \varepsilon W} \sup_{\Omega'} S_W(\theta, \delta). \tag{4.4}$$

A procedure δ is said to have S.M.R. (W) (*somewhere minimum risk* with respect to W) if (i) $\delta \varepsilon W$ and (ii) $S_W(\theta, \delta) = 0$ for some $\theta \varepsilon \Omega'$. Let U denote the class of all S.M.R. (W) procedures. We say that δ_0 has *minimax regret* S.M.R. (W) if (i) $\delta_0 \varepsilon U$ and (ii)

$$\sup_{\Omega'} S_W(\theta, \delta_0) = \inf_{\delta \varepsilon U} \sup_{\Omega'} S_W(\theta, \delta). \tag{4.5}$$

Only a few m.d.p.'s admit U.M.R. (W) procedures (Sections 5 and 11). If there does not exist a U.M.R. (W) procedure then we try to apply the much less compelling criterions minimax regret (W) and minimax regret S.M.R. (W) which are modifications of the criterions most stringent (D) and most stringent S.M.P. (D) in the Neyman-Pearson theory ([27] Chapter 3; see also Sections 6 and 9). That these criterions are far from compellng is unfortunate but natural: compelling criterions will only be applicable to very simple problems (Sections 5 and 11) and to more difficult problems when their formulation is oversimplified.

5. The general two-decision hypothesis testing problem as an example with $m = 2, n = 1$. Let the observation $x \varepsilon \mathfrak{X}$ be obtained in order to decide upon one of the two statements.

d_0: *both H and K are neither rejected nor accepted* ("$\theta \varepsilon \Omega$")

d_1: *H is rejected and K is accepted* ("$\theta \varepsilon \Omega_2$")

where H is the hypothesis that $\theta \varepsilon \Omega_1$ and K is the alternative that $\theta \varepsilon \Omega_2$ ($\Omega_1 \cap \Omega_2 = \varnothing$; $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$; $\Omega_0 = \varnothing$ in many cases). The two-decision procedure $\delta(\varphi_0, \varphi_1)$ is completely determined by the test function φ_1.

In the Neyman-Pearson theory the statistician controls the *error of the first kind* (d_1 is made whereas $\theta \varepsilon \Omega_1$) by restricting the attention to the class of all size-α tests where α is the predetermined level of significance. In many cases this class is further restricted by applying Neyman's unbiasedness condition. Within such a restricted class the statistician looks for an optimum test for which the power function $\beta_{\varphi_1}(\theta) = E_\theta\{\varphi_1(X)\} = 1 - E_\theta\{\varphi_0(X)\}$ over Ω_2 is in some sense as large as possible. Here $E_\theta\{\varphi_0(X)\}$ is the probability of an *error of the second kind* (d_0 is made whereas $\theta \varepsilon \Omega_2$).

The following example shows that the Neyman-Pearson formulation is not completely satisfactory. Suppose an observation x is obtained from the normal $N(\mu, 1)$ distribution in order to test the hypothesis $H: \mu = 0$ against the alternative $K: \mu = \mu_1$ where μ_1 is some positive constant. In order to protect ourselves against the serious error of the first kind we use a small value of α, say $\alpha = .05$. Then the Neyman-Pearson theory provides the procedure $\delta(\varphi_0, \varphi_1)$ where φ_1 is the characteristic function of the interval (u_α, ∞) where $u_\alpha = 1.645$. This result is not reasonable when $\mu_1 > 2u_\alpha$, for when an error of the first kind is regarded as much more serious than an error of the second kind then it is reasonable to require that $E_{\mu=0}\{\varphi_1(X)\} < E_{\mu_1}\{\varphi_0(X)\}$. Hence α has in some sense to depend on μ_1 but the Neyman-Pearson theory does not solve this problem.

A more attractive decision-theoretical formulation is obtained by introducing Lehmann's loss function ([20] page 12)

$$(5.1) \qquad \begin{aligned} L(\theta, d_0) &= w_{10} = 0; & L(\theta, d_1) &= w_{11} = b & (\theta \varepsilon \Omega_1) \\ L(\theta, d_0) &= w_{20} = a; & L(\theta, d_1) &= w_{21} = 0 & (\theta \varepsilon \Omega_2) \end{aligned}$$

which expresses that the loss resulting from an error of the first kind is b/a times the loss resulting from an error of the second kind. In this formulation b/a is a predetermined constant which plays a similar part as α in the Neyman-Pearson formulation. By adopting this formulation with loss function (5.1), the above-mentioned difficulty is remedied. Problems with a simple hypothesis and a simple alternative are not of Type I. We can easily construct the M.R. procedure as the Bayes procedure with respect to the least favorable a priori distribution and this procedure turns out to be very attractive.

Most testing problems from actual practice are of Type I. For such problems the following theorem of Lehmann shows that there will exist a class of M.R. procedures.

THEOREM 5.1. (*Lehmann*). *If the two-decision testing problem with loss function* (5.1) *is of Type* I ($[\Omega_1] \cap [\Omega_2] \neq \emptyset$) *then the following three classes of procedures coincide*: (i) *the class of all* M.R. *procedures,* (ii) *the class of all unbiased procedures,* (iii) *the class of all unbiased size-α tests where* $\alpha = a/(a + b)$.

PROOF. (see [20] pages 12, 24, 25). The equivalence of (ii) and (iii) is a consequence of Lemma 4.1. In order to establish the equivalence of (i) and (ii) we remark that S is the line segment joining $w_{\cdot 0} = (0, a)$ and $w_{\cdot 1} = (b, 0)$. Lemma 4.2 may be used to show that $w^* = (\bar{w}^*, \bar{w}^*)$ with $\bar{w}^* = ab/(a + b)$ is the unique minimax and maximin point of S. Hence each unbiased procedure has minimax risk (Theorem 4.1). On the other hand, if one coordinate of $w \varepsilon S$ is smaller than $\bar{w}^*$, then this coordinate is the smallest one. This argument shows that each M.R. procedure is unbiased (the argument does not hold in more dimensions: Theorem 4.1 cannot be sharpened).

REMARK. Theorem 5.1 was the starting point of Lehmann's paper [19] and of our results in [27] Chapter 3. The theorem shows that for many two-decision testing problems of Type I both the Neyman-Pearson and the decision-theoretical approach will provide the same "optimum" two-decision procedure provided that the predetermined constants α and b/a satisfy $\alpha = a/(a + b)$. Of course the optimum properties of the Neyman-Pearson theory will have to be reformulated in the decision-theoretic approach (see [27] Chapter 3 and [28]).

6. The general three-decision two-sided hypothesis testing problem as an example with $m = n = 2$. Let the observation $x \varepsilon \mathfrak{X}$ be obtained in order to decide upon one of the three statements

d_0: *no sufficient evidence for* $\theta \varepsilon \Omega_1$, *nor for* $\theta \varepsilon \Omega_2$

d_1: *the statement* "$\theta \varepsilon \Omega_1$" *is accepted*

d_2: *the statement* "$\theta \varepsilon \Omega_2$" *is accepted*

where for the partition $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$ in a certain sense $\Omega_0 = \Omega_0'$ is situated between Ω_1 and Ω_2. An important simple example is obtained when x is the outcome of the normal $N(\mu, 1)$-distribution and where $\Omega_0 = \{0\}$, $\Omega_1 = (-\infty, 0)$ and $\Omega_2 = (0, \infty)$. In this case d_2 may be reformulated by saying that there is sufficient evidence for μ being positive (d_0 and d_1 can be reformulated similarly).

The Neyman-Pearson approach to these problems is rather artificial. The statistician constructs an optimum unbiased size-α test φ^* for testing $\theta \varepsilon \Omega_0$ against the two-sided alternative $\theta \varepsilon \Omega'$. In the most important applications φ^* is the indicator function of the union of two mutually exclusive simply connected subsets of $\mathfrak{X}$. In that case the statistician proposes to use the three-decision procedure $\delta(\varphi_0, \varphi_1, \varphi_2)$ where φ_1 is the indicator function of one of these subsets and φ_2 of the other. In Section 7 we shall meet problems where the optimum test φ^* is the indicator function of one simply connected subset of $\mathfrak{X}$ so that the above-described natural interpretation of $\varphi^* = 1$ does not exist. Moreover for many other problems with a natural interpretation of the optimum unbiased size-α test, the tail probabilities $E_{\theta_0}\{\varphi_1(X)\}$ and $E_{\theta_0}\{\varphi_2(X)\}$ are not (both) equal (to $\frac{1}{2}\alpha$). This is not very reasonable from the three-decision point of view. Moreover it is unpractical, for one would need many new tables. However in practice one does not use the optimum unbiased size-α test but its analogue with equal tail probabilities. In Remark 6.3 we shall see that this is reasonable from our theoretical point of view.

In our opinion, the most attractive formulation of the three-decision problem is obtained by defining the following loss function

$$\begin{aligned}(6.1)\quad & L(\theta, d_0) = w_{10} = a; \quad L(\theta, d_1) = w_{11} = 0; \quad L(\theta, d_2) = w_{12} = b \quad (\theta \varepsilon \Omega_1)\\ & L(\theta, d_0) = w_{20} = a; \quad L(\theta, d_1) = w_{21} = b; \quad L(\theta, d_2) = w_{22} = 0 \quad (\theta \varepsilon \Omega_2)\end{aligned}$$

which just like (5.1) expresses that the loss resulting from an error of the first kind (the explicit statement d_1 (d_2) is made whereas $\theta \varepsilon \Omega_2$ (Ω_1)) is b/a times the loss resulting from an error of the second kind (the uninformative statement d_0 is made whereas actually $\theta \varepsilon \Omega_1$ or $\theta \varepsilon \Omega_2$). The theory of Section 5 suggests that the experimenter might use (6.1) with $b/a = -1 + 1/\alpha$ if he intended to use the Neyman-Pearson size-α restriction. We remark that the above-mentioned formulation of the error of the first kind is not consistent with that used in the Neyman-Pearson approach because there an error of the first kind is committed when $\theta \varepsilon \Omega_0 = \Omega_0'$ whereas one of the decisions d_1 and d_2 is made.

In this section we restrict the attention to three-decision problems of Type I where (i) in (6.1) we have $b > 2a$ and (ii) $E_{\theta_0}\{\varphi(X)\} = 0$ for some $\theta_0 \varepsilon \Omega_0 = \Omega_0'$ implies $\varphi = 0$ a.e. (μ) when φ is a test function. The symmetry of (6.1) suggests that $(g_1, g_2) = (\frac{1}{2}, \frac{1}{2})$ is a maximin strategy for Player I in the game with 2×3 matrix (w_{ij}) (the maximin strategy is not unique). On account of $b > 2a$ this strategy $(\frac{1}{2}, \frac{1}{2})$ satisfies the conditions of Lemma 3.2. Hence $(1, 0, 0)$ is the unique minimax strategy for Player II and Corollary 3.1 shows that the trivial procedure δ_0 which assigns d_0 with probability 1 to all $x \varepsilon \mathfrak{X}$ is the unique M.R. procedure (see [27] page 50). Hence the M.R. criterion is inappropriate.

We shall see that the unbiasedness restriction provides relief. It follows immediately from Lemma 4.1 that $\delta(\varphi_0, \varphi_1, \varphi_2)$ is unbiased if and only if

$$E_{\theta i}\{\varphi_i(X)\} \geqq E_{\theta i}\{\varphi_{3-i}(X)\} \quad \text{for all} \quad \theta_i \varepsilon \Omega_i \tag{6.2}$$

holds for $i = 1, 2$. We shall apply one of the criterions minimax regret (W) and minimax regret S.M.R. (W).

REMARK 6.1. The above-described approach to three-decision problems was introduced in [27] where *minimax regret* S.M.R. (W) procedures were constructed for very large classes of problems where Ω_1 and Ω_2 are each other's image under the transformation $f(\theta) = -\theta$ in R^s while Ω_1 (and consequently Ω_2) is a subset of R^s (a polyhedral angle or cone) which is defined by a number of homogeneous linear inequalities. (For example, consider the two-sided trend problem where Ω_1 consists of all $\theta = (\theta_1, \cdots, \theta_s)$ satisfying $\theta_1 \geqq \theta_2 \geqq \cdots \geqq \theta_s$ (with at least one inequality strong), while $\theta \varepsilon \Omega_2$ if and only if $\theta_1 \leqq \theta_2 \leqq \cdots \leqq \theta_s$, with at least one inequality strong).) If the number of inequalities is larger than one, then these problems are so difficult that there do not exist completely satisfactory compelling criterions (see [27] and [28]). We shall now show that in the case of only one inequality, *the same procedure has minimax regret* S.M.R. (W) *and minimax regret* (W) at least when certain conditions are satisfied.

In order to be able to consider the uniqueness of certain results, we identify tests when they have the same power function $E_\theta\{\varphi(X)\}$ over Ω and we consider procedures as identical when they have the same risk function over Ω'. A subclass C' of a class C of procedures is called a complete subclass of C, if for each $\delta \varepsilon C$ there exists a procedure $\delta' \varepsilon C'$ such that $R(\theta, \delta') \leqq R(\theta, \delta)$ for all $\theta \varepsilon \Omega'$.

THEOREM 6.1. *Suppose* $\Omega_0 = \Omega_0' = \{\theta_0\}$ *is simple and that for each* $\alpha \varepsilon [0, \frac{1}{2}]$ *there exists a pair of test functions* $\varphi_1^{(\alpha)}, \varphi_2^{(\alpha)}$ *such that* (i) *for all* $\theta_i \varepsilon \Omega_i$, $\varphi_i^{(\alpha)}$ *is the unique* M.P. *size-*α *test for testing* $H{:}\theta = \theta_0$ *against the simple alternative* $K{:}\theta = \theta_i (i = 1, 2)$; (ii) $\varphi_i^{(\alpha)}$ *is the unique least powerful test for* $H{:}\theta = \theta_0$ *against the simple alternative* $K{:}\theta = \theta_{3-i}$ *among the similar size-*α *tests (satisfying* $E_{\theta_0}\{\varphi(X)\} = \alpha$), *for all* $\theta_{3-i} \varepsilon \Omega_{3-i} (i = 1, 2)$; *and* (iii) $\varphi_0^{(\alpha)}(x) = 1 - \varphi_1^{(\alpha)}(x) - \varphi_2^{(\alpha)}(x) \geqq 0$ *for all* $x \varepsilon \mathfrak{X}$.

Then $\delta^{(\alpha)} = \delta(\varphi_0^{(\alpha)}, \varphi_1^{(\alpha)}, \varphi_2^{(\alpha)})$ *is a procedure and the following holds when* W *is the class of all unbiased procedures,* U *is the class of all* S.M.R. (W) *procedures and* $U' = \{\delta^{(\alpha)}; 0 \leqq \alpha \leqq \frac{1}{2}\}$: (a) $W \supset U' \supset U$; (b) U' *is a complete subclass of* W; (c) *if* δ^* *is a minimax regret* (U') *procedure, then* δ^* *is a minimax regret* (W)

procedure and (d) *if moreover* $\delta^* \varepsilon U$, *then* δ^* *is a minimax regret* S.M.R. (W) *procedure.*

PROOF. (a) With respect to the arbitrary procedure $\delta^{(\alpha)} \varepsilon U'$, the conditions (i) and (ii) show that $E_{\theta_i}\{\varphi_i^{(\alpha)}(X)\} \geqq \alpha \geqq E_{\theta_i}\{\varphi_{3-i}^{(\alpha)}(X)\}$ holds for all $\theta_i \varepsilon \Omega_i$ ($i = 1, 2$). Lemma 4.1 shows that $\delta^{(\alpha)}$ is unbiased (see (6.2)). Hence $W \supset U'$.

In order to show $U' \supset U$ suppose $\delta = \delta(\varphi_0, \varphi_1, \varphi_2) \varepsilon U$ or in other words $\delta \varepsilon W$ and there exists $\theta_i \varepsilon \Omega_i$ ($i = 1$ or 2) with $R(\theta_i, \delta) = \inf_W R(\theta_i, \delta)$ where

$$(6.3) \qquad R(\theta_i, \delta) = a - aE_{\theta_i}\{\varphi_i(X)\} + (b - a)E_{\theta_i}\{\varphi_{3-i}(X)\}.$$

Now $\delta \varepsilon W$ shows that $E_{\theta_0}\{\varphi_1(X)\} = E_{\theta_0}\{\varphi_2(X)\} = \alpha$ for some $\alpha \varepsilon [0, \frac{1}{2}]$ (Corollary 4.1). Consider the corresponding procedure $\delta^{(\alpha)}$. We already established $W \supset U'$ with the result that $\delta^{(\alpha)} \varepsilon W$. But δ has minimum risk (W) in θ_i. Hence $R(\theta_i, \delta) \leqq R(\theta_i, \delta^{(\alpha)})$.

On the other hand (i) shows that $E_{\theta_i}\{\varphi_i(X)\} \leqq E_{\theta_i}\{\varphi_i^{(\alpha)}(X)\}$ with strict inequality, unless $E_\theta\{\varphi_i(X)\} = E_\theta\{\varphi_i^{(\alpha)}(X)\}$ holds for all $\theta \varepsilon \Omega$. Moreover (ii) shows that $E_{\theta_i}\{\varphi_{3-i}(X)\} \geqq E_\theta\{\varphi_{3-i}^{(\alpha)}(X)\}$ for all $\theta \varepsilon \Omega$. By using (6.3) we obtain that $R(\theta_i, \delta) \geqq R(\theta_i, \delta^{(\alpha)})$ holds with strict inequality, unless δ and $\delta^{(\alpha)}$ have the same risk function in which case they are identified. But we already established $R(\theta_i, \delta) \leqq R(\theta_i, \delta^{(\alpha)})$. Hence $\delta = \delta^{(\alpha)} \varepsilon U'$ and we have proved that $U' \supset U$.

(b) Suppose $\delta \varepsilon W$ then $E_{\theta_0}\{\varphi_1(X)\} = E_{\theta_0}\{\varphi_2(X)\} = \alpha$ for some $\alpha \varepsilon [0, \frac{1}{2}]$ and $\delta^{(\alpha)} \varepsilon U'$ has the property that $R(\theta, \delta^{(\alpha)}) \leqq R(\theta, \delta)$ holds for all $\theta \varepsilon \Omega'$.

(c) On account of (b) we have $R_W^*(\theta) = R_{U'}^*(\theta)$ for all $\theta \varepsilon \Omega'$. Hence $S_W(\theta, \delta) = S_{U'}(\theta, \delta)$ holds for the regret function. Next suppose δ^* does not have minimax regret (W). Then there exists a procedure $\delta \varepsilon W$ and correspondingly on account of (b) a procedure $\delta' \varepsilon U'$ such that

$$\sup S_{U'}(\theta, \delta') \leqq \sup S_W(\theta, \delta) < \sup S_{U'}(\theta, \delta^*)$$

with the result that δ^* does not have minimax regret (U').

(d) Apply similar arguments using $U' \supset U$ and $\delta^* \varepsilon U$.

The class U' is a *one-parameter* family. The construction of the minimax regret (U') procedure δ^* is feasible. First we have to determine the envelope risk $R_{U'}^*(\theta)$ choosing the best $\alpha(\theta)$ for each θ. Then for each $\delta \varepsilon U'$ the regret function $S_{U'}(\theta, \delta)$ is determined and we have to look for the parameter α^* corresponding with the minimax regret (U') procedure δ^*. In many cases it can be shown easily that δ^* is the *unique* minimax regret (U') procedure and that there exists a parameter, say $\theta_i \varepsilon \Omega_i$, such that $S_{U'}(\theta_i, \delta^*) = \sup_{\Omega'} S_{U'}(\theta, \delta^*)$. In that case it follows easily that δ^* is the *unique* minimax regret (W) procedure (uniqueness for (d) follows similarly).

EXAMPLE 6.1. We shall elucidate Theorem 6.1 by working out the example mentioned at the beginning of this section. We remark that much more general situations can be dealt with, particularly those where the family $\{p_\theta ; \theta \varepsilon \Omega\}$ is a one-parameter exponential family or a family with monotone likelihood-ratio. However we restrict the attention to the above-mentioned example because for this example certain computational tasks have already been accomplished in [27]. Hence let X have the normal $N(\mu, 1)$ distribution; $\Omega_0 = \{0\}, \Omega_1 = (-\infty, 0)$ and $\Omega_2 = (0, \infty)$. Here and in following examples we shall use the following notation.

NOTATION. Φ denotes the cumulative distribution function of the $N(0, 1)$ distribution. Hence $\Phi(x) = P(X \leqq x)$ when X has the $N(0, 1)$ distribution. Moreover $u_\alpha = \Phi^{-1}(1 - \alpha)$ will denote the number ($u_{.05} = 1.645$) for which $\Phi(u_\alpha) = 1 - \alpha$.

With respect to our example let δ_c denote the three-decision procedure $\delta(\varphi_0, \varphi_1, \varphi_2)$ where φ_1, φ_0 and φ_2 successively are the indicator functions of $(-\infty, -c)$, $[-c, +c]$ and (c, ∞). The Neyman-Pearson fundamental lemma shows that δ_c is the procedure $\delta^{(\alpha)}$ of Theorem 6.1 when $c = u_\alpha$. Hence $U' = \{\delta_c; 0 \leqq c \leqq \infty\}$ is a complete subclass of the class W of all unbiased procedures (Theorem 6.1 (b)). The S.M.R. (W) procedure with minimum risk in $\theta_i = \mu \in \Omega_i$ is consequently of the form δ_c where c is determined in such a way that for $\rho = |\mu|$ the risk (see (6.3))

$$R(\theta_i, \delta_c) = (b - a)\Phi(-c - \rho) + \Phi(c - \rho) \tag{6.4}$$

is minimized as a function of c. Differentiation shows that the minimum is obtained when $c = \tilde{c}(\rho)$ where

$$\tilde{c}(\rho) = \{\ln(b - a) - \ln a\}/(2\rho). \tag{6.5}$$

We have $0 < \rho < \infty$ and consequently $\infty > \tilde{c}(\rho) > 0$. Hence $U = \{\delta_c; 0 < c < \infty\}$. The minimax regret ($U'$) procedure δ^* is obtained by minimizing

$$\sup_{\rho>0}[(b - a)\{\Phi(-c - \rho) - \Phi(-\tilde{c}(\rho) - \rho)\} + a\{\Phi(c - \rho) - \Phi(\tilde{c}(\rho) - \rho)\}] \tag{6.6}$$

as a function of c. Let $c^*(b/a)$ be the value of c minimizing (6.6). Then $\delta^* = \delta_{c^*(b/a)}$ is the unique minimax regret (U'), the unique minimax regret (W) and the unique minimax regret S.M.R. (W) procedure (Theorem 6.1). The values of $c^*(b/a)$ are given by the line $\cos \Psi_0 = 1$ in [27] Figure 7. For $\alpha = .10(.05)$ the Neyman-Pearson approach of the beginning of this section, provides the procedure δ_c with $c = u_{\frac{1}{2}\alpha} = 1.645$ (1.960). The related decision-theoretical approach based on the loss function (6.1) with $b/a = 9(19)$ provides $\delta_{c^*(b/a)}$ where $c^*(b/a) = 1.73(2.03)$.

In Theorem 6.1 we restricted the attention to the case that $\Omega_0 = \Omega_0' = \{\theta_0\}$ is *simple*. It is possible to generalize parts of Theorem 6.1 in order to deal with the case that $\Omega_0 = \Omega_0'$ is *composite* (see [27] Section 3.4) but we encountered difficulties in generalizing Theorem 6.1 (c). The following lemma may be helpful in order to overcome these difficulties.

Suppose that with respect to the same random observable X we consider two related three-decision problems (see [28] Lemma 2) Π and π both with loss function (6.1). Π is defined by the partition $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$ and π by $\omega = \omega_0 \cup$

$\omega_1 \cup \omega_2$ where $\omega_i \subset \Omega_i$ $(i = 0, 1, 2)$. Let W denote a class of procedures for Π and w for π (in the applications W and w will be the classes of unbiased procedures for Π and π respectively so that Lemma 4.1 shows that, on account of $\omega_i \subset \Omega_i$ $(i = 0, 1, 2)$, holds that $W \subset w$). Obviously a procedure $\delta(\varphi_0, \varphi_1, \varphi_2)$ may both be interpreted as a procedure for Π and as one for π.

LEMMA 6.1. *Suppose that* (i) $W \subset w$; (ii) δ^* *has minimax regret* (w) *for problem* π; (iii) $\delta^* \varepsilon W$; (iv) $R_W^*(\theta) = R_w^*(\theta)$ *for all* $\theta \varepsilon \omega' = \omega_1 \cup \omega_2$ *and* (v) $\sup_{\Omega'} S_W(\theta, \delta^*) = \sup_{\omega'} S_W(\theta, \delta^*)$. *Then* δ^* *has minimax regret* (W) *for problem* Π.

PROOF. We have

$$\inf_{\delta \varepsilon W} \sup_{\theta \varepsilon \Omega'} S_W(\theta, \delta) \overset{\text{(iii)}}{\leqq} \sup_{\theta \varepsilon \Omega'} S_W(\theta, \delta^*) \overset{\text{(v),(iv)}}{=} \sup_{\theta \varepsilon \omega'} S_w(\theta, \delta^*)$$

$$\overset{\text{(ii)}}{=} \inf_{\delta \varepsilon w} \sup_{\theta \varepsilon \omega'} S_w(\theta, \delta) \overset{\text{(iv)}}{\leqq} \inf_{\delta \varepsilon w} \sup_{\theta \varepsilon \Omega'} S_W(\theta, \delta) \overset{\text{(i)}}{\leqq} \inf_{\delta \varepsilon W} \sup_{\theta \varepsilon \Omega'} S_W(\theta, \delta).$$

Hence equality holds everywhere and δ^* has minimax regret (W).

We already remarked that condition (i) will hold when W and w are the classes of unbiased procedures. Condition (ii) can sometimes be fulfilled by taking ω so small that $\omega_0 = \omega \cap \Omega_0$ is simple so that Theorem 6.1 can be applied; (iv) can sometimes be proved by showing that each S.M.R. (w) procedure for problem π is a S.M.R. (W) procedure for problem Π. We elucidate Lemma 6.1 by working out the following very simple example.

EXAMPLE 6.2. Suppose $X = (X_1, X_2)$ where X_1 and X_2 have independent normal $N(\mu_1, 1)$ and $N(\mu_2, 1)$ distributions respectively. Ω_1 will be $\{\mu_1, \mu_2);$ $\mu_1 < 0\}$, $\Omega_0 = \{(\mu_1, \mu_2); \mu_1 = 0\}$ and $\Omega_3 = \{(\mu_1, \mu_2); \mu_1 > 0\}$. In order to solve this problem, we first consider the auxiliary problem where $\omega_1 = \{(\mu_1, 0);$ $\mu_1 < 0\}$, $\omega_0 = \{(0, 0)\}$ and $\omega_2 = \{(\mu_1, 0); \mu_1 > 0\}$. This problem π is a slight modification of Example 6.1. We can apply Theorem 6.1 because $\omega_0 = \omega_0' =$ $\{(0, 0)\}$ is simple. By doing so we easily obtain that $\delta^* = \delta_{c^*(b/a)}$ is the unique minimax regret unbiased procedure for problem π when $c^*(b/a)$ is again the value of c minimizing (6.6) and with $\delta_c = \delta(\varphi_0, \varphi_1, \varphi_2)$ where φ_1, φ_0 and φ_2 are respectively the indicator functions of the sets of sample points which are defined by $x_1 < -c$, $-c \leqq x_1 \leqq c$ and $c < x_1$ respectively. The conditions of Lemma 6.1 can be verified easily. Hence $\delta^* = \delta_{c^*(b/a)}$ is the unique minimax regret unbiased procedure for problem Π.

REMARK 6.2. Application of Theorem 6.1 together with Lemma 6.1 will only be successful if there exist uniformly M.P. size-α tests for testing the possibly composite hypothesis $\theta \varepsilon \Omega_0'$ against the composite alternative $\theta \varepsilon \Omega_i$ $(i = 1, 2)$ (see Example 6.2). Fortunately we can deal with many other problems (namely those where uniformly M.P. similar size-α tests exist) if we modify the criterions of Theorem 6.1 a little bit. First we remark that the unbiasedness of a procedure $\delta(\varphi_0, \varphi_1, \varphi_2)$ implies $E_{\theta_0}\{\varphi_1(X)\} = E_{\theta_0}\{\varphi_2(X)\}$ for all $\theta_0 \varepsilon \Omega_0 = \Omega_0'$ but not that the *similarity* condition holds which states that $E_{\theta_0}\{\varphi_j(X)\}$ has the *same* value for all $\theta_0 \varepsilon \Omega_0 (j = 0, 1, 2)$. A counter example is obtained easily by considering

Example 6.2. Let φ_0, φ_1 and φ_2 be the indicator functions of the sets defined by $x_2 \geqq 0$; $x_1, x_2 < 0$ and $x_1 \geqq 0$, $x_2 < 0$ respectively. Then $\delta(\varphi_0, \varphi_1, \varphi_2)$ is unbiased on account of Lemma 4.1 but $E_{\theta_0}\{\varphi_j(X)\}$ depends on $\theta_0 = (0, \mu_2) \varepsilon \Omega_0$. Let $\bar{W}$ denote the class of all *unbiased similar* procedures which are defined by (6.2) ($i = 1, 2$) and the existence of a number $\alpha \varepsilon [0, \frac{1}{2}]$ such that $E_{\theta_0}\{\varphi_1(X)\} = E_{\theta_0}\{\varphi_2(X)\} = \alpha$ for all $\theta_0 \varepsilon \Omega_0 = \Omega_0'$. Then one can easily formulate and prove a modification of Theorem 6.1 which shows that one may be able to construct minimax regret ($\bar{W}$) procedures if uniformly M.P. *similar* size-α tests exist. Thus one can deal with many problems with nuisance parameters (see [27] Section 3.5 where minimax regret S.M.R. ($\bar{W}$) procedures were outlined for very large classes of problems).

REMARK 6.3. Obviously we can obtain results in an easier way, if we restrict our class W (or the class $\bar{W}$) further, by applying a *size-restriction*. At first sight this approach is not very attractive from the theoretical point of view because it seems to provide an over-simplification, for "which size is compatible with the coefficient b/a in the loss function (6.1)." On the other hand, if one restricts the attention to the class C_α of all unbiased procedures $\delta(\varphi_0, \varphi_1, \varphi_2)$ satisfying the condition that they are "similar size-α": $E_{\theta_0}\{\varphi_0(X)\} = 1 - \alpha$ for all $\theta_0 \varepsilon \Omega_0$ then procedure $\delta^{(\frac{1}{2}\alpha)}$ will have U.M.R. (C_α) *for each loss function* (6.1) provided that the conditions of Theorem 6.1 are satisfied (Remark 6.2 can be supplemented similarly). In some sense this result establishes in a clear way that we have indeed over-simplified the formulation of our problem for it is not natural when the same procedure is always optimal, no matter which coefficient b/a is chosen in the loss function. On the other hand, the simplicity of the result makes it very attractive from the practical point of view. We remark that $\delta^{(\frac{1}{2}\alpha)}$ corresponds with the natural interpretation of the *uniformly* M.P. *equal tails size-α* test $\varphi = \varphi_1 + \varphi_2$ for testing the hypothesis $H:\theta \varepsilon \Omega_0$ against the two-sided alternative $K_2:\theta \varepsilon (\Omega_1 \cup \Omega_2)$ (the criterion "uniformly M.P. equal tails size-α" corresponds with the U.M.R. (C_α) criterion in the special case $a = b = 1$ in the loss function (6.1)).

7. Selecting the best in slippage situations and selecting the unknown time point at which a shift occurs, as examples with $m = n \geqq 2$. The following problem may be regarded as a generalization of the problem of Section 6 and may provide a reasonable reformulation of certain problems of Doornbos and Prins [9], Kander and Zacks [14] and Paulson [26].

With respect to a partition $\Omega = \Omega_0 \cup \cdots \cup \Omega_m$ with the corresponding hypotheses $H_i:\theta \varepsilon \Omega_i$ where $i = 1, \cdots, m$, we consider the problem that is defined by the decisions $d_0, \cdots, d_m$ where d_0 corresponds with the statement that all H_i are neither rejected nor accepted ("$\theta \varepsilon \Omega$") while d_i is the assertion that H_i is accepted and H_j is rejected for all $j \neq i$ ("$\theta \varepsilon \Omega_i$") ($i = 1, \cdots, m$). The loss function is defined as follows

$$(7.1) \quad w_{i0} = a(i = 1, \cdots, m); \qquad w_{ij} = b(1 - \delta_{ij}) \quad (i, j = 1, \cdots, m)$$

where $\delta_{ij} = 0$ (or 1) in case $i \neq j$ (or $i = j$). Just like (5.1) and (6.1) this loss function expresses that the loss resulting from an error of the first kind (the explicit statement d_i("$\theta \varepsilon \Omega_i$") is made whereas $\theta \varepsilon \Omega_j$ $(i, j = 1, \cdots, m; i \neq j)$) is b/a times the loss resulting from an error of the second kind (the uninformative statement d_0 is made whereas actually $\theta \varepsilon \Omega_i$ for some index i $(i = 1, \cdots, m)$). Here b/a is a predetermined constant and the theory of Section 5 suggests that the experimenter might use (7.1) with $b/a = -1 + 1/\alpha$ if he is in the habit of applying the Neyman-Pearson size-α restriction.

LEMMA 7.1. (i) $b > m(m - 1)^{-1}a$ *implies that* $(1, 0 \cdots 0)$ *is the unique minimax strategy for Player* II *in the game with matrix* (w_{ij}).

(ii) *If* $b = m(m - 1)^{-1}a$ *then* $w_{\cdot 0}$ *is the unique minimax and the unique maximin point of* S. *The convex representations* $w_{\cdot 0} = \sum_{j=0}^{m} p_j w_{\cdot j}$ *of* $w_{\cdot 0}$ *are determined by* $p_0 = \rho$, $p_j = (1 - \rho)/m$ $(j = 1, \cdots, m)$ *where* ρ *is arbitrary* $(0 \leqq \rho \leqq 1)$.

(iii) *If* $b < m(m - 1)^{-1}a$ *then* $(0, 1/m, \cdots, 1/m)$ *is the unique minimax strategy for Player* II.

PROOF. The symmetry of (7.1) shows that $(g_1, \cdots, g_m) = (1/m, \cdots, 1/m)$ is a maximin strategy for Player I. We shall have to use this strategy when applying Lemma 3.1, Lemma 3.2 or Lemma 4.2. By doing so, (i) is a simple consequence of Lemma 3.2, (ii) of Lemma 4.2 while (iii) follows from Lemma 3.1 (the uniqueness has to be established separately, for example by considering the convex hull of $w_{\cdot 1}, \cdots, w_{\cdot m}$ to which Lemma 4.2 may be applied).

In actual situations we shall have $b > m(m - 1)^{-1}a$ and Lemma 7.1 shows that the conditions of Corollary 3.1 are satisfied.

THEOREM 7.1. *If for a Type* I *problem with loss function* (7.1) *holds that* (i) $b > m(m - 1)^{-1}a$ *and* (ii) $E_{\theta_0}\{\varphi(X)\} = 0$ *for some* $\theta_0 \varepsilon \Omega_0'$ *implies* $\varphi = 0$ a.e. (μ) *for each test function* φ; *then the trivial procedure* δ_0 *with* $\varphi_0 = 1$ a.e. (μ) *is the unique* M.R. *procedure.*

This generalization of [19] page 572 and [27] page 50 shows that the M.R. criterion is inappropriate. According to Remark 4.2 we try to apply Lehmann's unbiasedness restriction. Lemma 4.1 and Corollary 4.1 provide the following result.

LEMMA 7.2. *The procedure* $\delta(\varphi_0, \cdots, \varphi_m)$ *is unbiased for a problem with loss function* (7.1), *if and only if*

$$E_{\theta i}\{\varphi_i(X)\} = \max_{j=1,\cdots,m} E_{\theta i}\{\varphi_j(X)\} \tag{7.2}$$

holds for all $\theta_i \varepsilon \Omega_i$ $(i = 1, \cdots, m)$ (*see* (6.2)).

For a Type I *problem the unbiasedness of* $\delta(\varphi_0, \cdots, \varphi_m)$ *implies*

$$E_{\theta_0}\{\varphi_1(X)\} = \cdots = E_{\theta_0}\{\varphi_m(X)\} \quad \textit{for all} \quad \theta_0 \varepsilon \Omega_0'. \tag{7.3}$$

In Section 9 we shall construct the *minimax regret unbiased* procedure for a very simple slippage problem similar to those considered by Paulson. In the following example we illustrate Theorem 7.1 by considering a problem similar to those of Kander and Zacks in [14].

EXAMPLE. Let $X_{i1}, \cdots, X_{in_i}$ be a sample from the normal $N(\mu_i, \sigma^2)$ distribution $(i = 1, \cdots, m)$. The experimenter aims at $\mu_1 = \cdots = \mu_m = \mu$ where μ is *known*, but it is possible that there occurs an increasing shift at an unknown time point (index i). He assumes that *no more than one shift* can occur. Hence $\Omega = \Omega_0 \cup \cdots \cup \Omega_m$ where $\Omega_i = \{\theta; \theta = (\mu_1, \cdots, \mu_m, \sigma^2); \mu_1 = \cdots = \mu_{m-i} = \mu < \mu_{m-i+1} = \cdots = \mu_m, \sigma^2 > 0\}$ $(i = 0, \cdots, m)$. An observation of $X = \{X_{ij}; j = 1, \cdots, n_i; i = 1, \cdots, m\}$ is made in order to decide whether a shift has occurred and if this is the case, at which time point. Thus the experimenter wishes to make one of the decisions $d_0, \cdots, d_m$ described at the beginning of this section. The sets Ω_i $(i = 0, \cdots, m)$ in R^{m+1} are such that the problem is of Type I. If the experimenter decides to use the loss function (7.1) with $b > m(m-1)^{-1}a$ then the trivial procedure δ_0 is the unique M.R. procedure. This result is also true when the loss function (7.1) is replaced by

$$(7.4) \quad w_{i0} = a(i = 1, \cdots, m); \quad w_{ij} = b|i - j| \qquad (i, j = 1, \cdots, m)$$

which expresses that the experimenter "makes $|i - j|$ errors of the first kind" when he decides upon a shift at index j whereas actually the shift occurred at index i.

In Section 6 we remarked that the Neyman-Pearson approach to the problems of this section, results in a most stringent size-α test (see [28] Section 4) or in some other size-α test (see [14]) for testing the hypothesis $H: \theta \in \Omega_0$ against the alternative $K: \theta \in \bigcup_{i=1}^m \Omega_i$. The rejection regions of these tests turn generally out to be simply connected subsets of the sample space. Consequently no natural interpretation is possible when H is rejected (the problems of Section 6 constitute an exception).

8. A modification of Section 7 which is obtained by defining the loss all over Ω. In Section 6 and Section 7 we did not define the loss over the indefiniteness zone Ω_0. This has the following advantages (i) it is not necessary to give motivations for a particular choice of $L(\theta, d)$ for $\theta \in \Omega_0$, (ii) the corresponding optimum procedure remains the same whether (a) Ω_0 is omitted from the parameter space Ω (because it is "practically impossible that θ belongs to the set Ω_0 of Lebesgue measure 0") or (b) Ω_0 is included in one of the regions $\Omega_1, \cdots, \Omega_m$.

Nevertheless suppose that the experimenter wants to define the loss for all $\theta \in \Omega$ by means of (7.1) and

$$(8.1) \quad L(\theta, d_0) = w_{00} = a'; \quad L(\theta, d_j) = w_{0j} = b'(j = 1, \cdots, m) \quad (\theta \in \Omega_0)$$

where it is reasonable to assume $0 \leqq a' \leqq a$ and $0 \leqq b' \leqq b$. The resulting problem is of the form discussed in the general Sections 2, 3 and 4 when the index 0 is replaced by $(m + 1)$.

LEMMA 8.1. *If* $a' \leqq a$ *and* $b' > m\,(m-1)^{-1}a$ *then* $(1, 0, \cdots, 0)$ *is the unique minimax strategy for Player* II *in the game with matrix* w_{ij} $(i = 0, \cdots, m; j = 0, \cdots, n)$.

This lemma is an obvious modification of Lemma 7.1 (i) and can be proved by applying Lemma 3.2 and using the strategy $(0, 1/m, \cdots, 1/m)$ which is obviously maximin for Player I. We are interested in Type I problems with $\Omega_0 = \Omega_0' = [\Omega_0] \cap \cdots \cap [\Omega_m] \neq \emptyset$. It follows from Corollary 3.1 that under the conditions of Lemma 8.1 the trivial procedure with $\varphi_0 = 1$ a.e. (μ) is the unique M.R. procedure. This is a generalization of the $m = 2, a' = 0, b' = b - a$ case in Lehmann [19] page 572. We shall investigate Lehmann's unbiasedness condition which now states that (4.1) holds for all $\theta, \theta' \varepsilon \Omega$.

THEOREM 8.1. *For Type* I *problems with* $\Omega_0 = \Omega_0' = [\Omega_0] \cap \cdots \cap [\Omega_m] \neq \emptyset$ *and loss function* {(7.1), (8.1)} *the following results hold.*

(i) $a' < a, b' < b - b/m$ *implies that no unbiased procedure exists;*

(ii) $a' = a, b' = b - b/m$ *implies that* $\delta(\varphi_0, \cdots, \varphi_m)$ *is unbiased if and only if* (7.2) *holds for all* $\theta_i \varepsilon \Omega_i$ $(i = 1, \cdots, m)$;

(iii) $a' = a, b' \neq b - b/m$ *implies that the trivial procedure with* $\varphi_0 = 1$ a.e. (μ) *is the unique unbiased procedure;*

(iv) $a' < a, b' \geqq b - b/m$ *implies that* $\delta(\varphi_0, \cdots, \varphi_m)$ *is unbiased if and only if the conditions A and* B_i $(i = 1, \cdots, m)$ *hold where A is satisfied if and only if* $E_{\theta_0}\{\varphi_0(X)\} = 1 - \alpha$, $E_{\theta_0}\{\varphi_j(X)\} = \alpha/m$ $(j = 1, \cdots, m)$ *holds for all* $\theta_0 \varepsilon \Omega_0$ *while*

$$\alpha = (a - a')/(a - a' + b' - b + b/m) \tag{8.2}$$

and where B_i *is satisfied if and only if* (7.2) *and*

$$bE_{\theta i}\{\varphi_i(X)\} \geqq (b - b') + (a - a' - b + b')E_{\theta i}\{\varphi_0(X)\} \tag{8.3}$$

holds for all $\theta_i \varepsilon \Omega_i$.

This theorem is a corollary of Lemma 4.1 where $h, i = 0, \cdots, m$. In our opinion a completely satisfactory restriction is obtained only in case (ii) where the class W of all unbiased procedures is exactly the same as in Section 7 (see Section 9). The requirements (7.2) are very reasonable and the unique trivial M.R. procedure δ_0 with $\varphi_0 = 1$ a.e. (μ) satisfies $\delta_0 \varepsilon W$. In case (i) the unbiasedness restriction is completely unfit for use because $W = \emptyset$. In case (iii) it does not provide relief, for δ_0 is the unique unbiased procedure. Case (iv) is very interesting. Here the class W is so small that W does not contain the unique M.R. procedure δ_0. Though δ_0 is a useless procedure, this might indicate that the unbiasedness restriction is too restrictive in case (iv); we are not sure of not throwing out the baby with the bath-water. On the other hand the size condition A may make it rather easy to obtain a "best" unbiased procedure. This will be elucidated by considering the three-decision problem of Section 6 where the loss function is defined by (6.1) and

$$\begin{aligned} L(\theta, d_0) &= w_{00} = a'; \\ L(\theta, d_1) &= w_{01} = L(\theta, d_2) = w_{02} = b' \qquad (\theta \varepsilon \Omega_0) \end{aligned} \tag{8.4}$$

where we assume

$$0 \leqq a' < a < \tfrac{1}{2}b \leqq b' \leqq b \quad \text{and} \quad b - b' \geqq a - a'. \tag{8.5}$$

Next let α be defined by (8.2) where $m = 2$ is substituted.

THEOREM 8.2. *If* $\Omega_0 = \Omega_0' = [\Omega_1] \cap [\Omega_2] \neq \emptyset$ *and* φ_1 *and* φ_2 *are two test functions such that* (i) $\varphi_0 = 1 - \varphi_1(x) - \varphi_2(x) \geqq 0$ *holds for all* $x \varepsilon \mathfrak{X}$, (ii) φ_i *is the unique*

U.M.P. *similar size* $-\frac{1}{2}\alpha$ *test for testing* $H:\theta \varepsilon \Omega_0$ *against* $K_i:\theta \varepsilon \Omega_i$, $(i = 1, 2)$ *while* (iii) φ_{3-i} *is the unique uniformly least powerful similar size-*$\frac{1}{2}\alpha$ *test for testing* H *against* K_i $(i = 1, 2)$. *Then* $\delta(\varphi_0, \varphi_1, \varphi_2)$ *is the unique* U.M.R. (W) *procedure where* W *is the class of all unbiased procedures.*

PROOF. Let W' denote the class of all procedures with $E_{\theta_0}\{\varphi_0(X)\} = 1 - \alpha$ and $E_{\theta_0}\{\varphi_1(X)\} = E_{\theta_0}\{\varphi_2(X)\} = \frac{1}{2}\alpha$ for all $\theta_0 \varepsilon \Omega_0$. Then $W \subset W'$ on account of Theorem 8.1 (iv). Moreover (6.3) shows that $\delta(\varphi_0, \varphi_1, \varphi_2)$ has U.M.R. (W') (this is Theorem 11.1 in Lehmann [18]). Next we show that $\delta \varepsilon W$ by applying Theorem 8.1(iv). It follows from (ii) and (iii) that

$$(8.6) \qquad E_{\theta i}\{\varphi_i(X)\} \geqq \tfrac{1}{2}\alpha \geqq E_{\theta i}\{\varphi_{3-i}(X)\} \quad \text{for all} \quad \theta_i \varepsilon \Omega_i$$

holds for $i = 1, 2$. Hence (7.2) or equivalently (6.2) is satisfied. Condition (8.3) is equivalent to the condition $(a - a' + b')E_{\theta i}\{\varphi_i(X)\} - \{(b - b') - (a - a')\} \cdot E_{\theta i}\{\varphi_{3-i}(X)\} \geqq a - a'$ where on account of (8.5) the coefficients of $E_{\theta i}\{\varphi_j(X)\}$ are nonnegative. The proof of this condition follows by using (8.6) and (8.2) with $m = 2$ for α. Hence $\delta \varepsilon W$ and consequently δ has U.M.R. (W).

REMARK 8.1. In the fundamental paper [19] Lehmann considered two-sided testing problems as three-decision problems with loss function {(6.1), (8.4)} where $a' = 0$ and $b' = b - a > \frac{1}{2}b$ (apart from a change of notation, this loss function is given by [19] equation (9.4)). He constructed U.M.R. unbiased procedures along the lines of a generalization of Theorem 8.2. Though U.M.R. (W) is a more compelling criterion than those used in Section 6, we like the formulation and the results of Section 6 more than those based on Theorem 8.2 with $a' = 0$ and $b' = b - a$ because (i) the unbiasedness restriction is open to some suspicion as $a' = 0$, $b' = b - a$ leads to case (iv) in Theorem 8.1, (ii) the particular choice of a' and b' plays a very important part in the determination of α whereas it seems to be reasonable that the optimum procedure does not depend (strongly) on the special construction of the loss function over the region Ω_0 which will have measure 0 (Lehmann motivated the choice $a' = 0$, $b' = b - a$ leading to $\alpha = 2a/b$; next suppose that the experimenter wants to use $a' = 0$, $b' = \frac{1}{2}b$ because he considers an error of the first kind to be twice as wrong as the error which is committed when one decides upon d_1 or d_2 whereas $\theta \varepsilon \Omega_0$, then the experimenter arrives at the unusable value $\alpha = 1$ which may differ much from $2a/b$ because in many inference situations one will use $b/a = 9$, 19 or 99 on account of Section 5).

9. A method for constructing minimax regret unbiased procedures for some problems of Section 7. Theorem 6.1 and Remark 6.2 show that for many three-decision problems related with two-sided testing problems, minimax regret (W or $\bar{W}$) procedures can be constructed rather easily. Unfortunately this does not hold for the more general problems of Section 7 because for such problems the compatibility condition (iii) in Theorem 6.1 will generally be violated. This may be elucidated by considering the following *slippage* problem.

EXAMPLE 9.1. Let X_1 and X_2 have independent normal $N(\mu_i, 1)$ distribution with $\mu_i \geqq 0$ $(i = 1, 2)$ and μ_1 or μ_2 or both equal to zero. Consider the partition

$\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$ where $\Omega_0 = \{(0, 0)\}$ and Ω_i is the positive μ_i-axis ($i = 1, 2$). With respect to Theorem 6.1 the test $\varphi_i^{(\alpha)}$ satisfying (i) rejects if and only if $x_i \geqq u_\alpha$. This test does not satisfy (ii) and also (iii) is violated. Nevertheless we shall construct the minimax regret unbiased procedure, assuming that the loss function is given by (6.1) or (7.1).

Consider the general problem of Section 7 with loss function (7.1). We modify this problem by defining the loss over Ω_0 according to (8.1) where $a' = a$ and $b' = b - b/m$. Assuming that the problem is of Type I with $\Omega_0 = \Omega_0'$, Theorem 8.1 (ii) and Lemma 7.2 show that the class W of all unbiased procedures is the same for both the original and the modified problem. The unbiasedness does also imply continuity of the risk function. The envelope risk function and the regret function will also be continuous for each $\delta \varepsilon W$, where W is the class of all unbiased procedures. Thus the original problem and the modified problem will lead to the same minimax regret (W) procedure. The modification has the advantage that we need not consider "wide sense Bayes procedures" in the following discussion.

It is well known that Bayes procedures whose maximum risk is assumed and constant on a set of a priori probability 1, are minimax risk. Similarly it can be shown easily that a Bayes procedure δ^* has minimax regret (W) if (i) $\delta^* \varepsilon W$ and (ii) δ^* assumes its maximum regret with respect to W in each point of a set of a priori probability 1 (similar results for the criterion "most stringent size-α" were considered in [28]).

Application to Example 9.1. The symmetry of the problem suggests that the "least favorable" a priori distribution assigns a priori probabilities $1 - 2p$, p p and p to $\theta_0 = (0, 0) \varepsilon \Omega_0$, $\theta_1 = (\kappa, 0) \varepsilon \Omega_1$ and $\theta_2 = (0, \kappa) \varepsilon \Omega_2$ respectively where p and κ have to be chosen in the right way ($\kappa > 0; 0 \leqq p \leqq \frac{1}{2}$). A straightforward application of the classical theory ([20] page 23 or [27] page 9) to the above-described a priori distribution, produces the Bayes procedure $\delta(p, \kappa)$ with

$$(9.1) \quad \begin{aligned} \varphi_i(x_1, x_2) &= 1 \qquad \text{for all } (x_1, x_2) \text{ with } x_i > f(x_{3-i}); \quad (i = 1, 2), \\ \varphi_0(x_1, x_2) &= 1 \qquad \text{for all other points } (x_1, x_2); \end{aligned}$$

where the function f is determined by

$$(9.2) \qquad f(x) = \kappa^{-1} \ln \{(b/a - 1)e^{\kappa x} + (1/p - 2)(\tfrac{1}{2}b/a - 1)e^{\frac{1}{2}\kappa^2}\}$$

provided that the usual assumption $b/a > 2$ holds (in the case $b/a \leqq 2$ the Bayes procedure is a trivial one which assigns d_i to all points (x_1, x_2) with $x_i > x_{3-i}$ ($i = 1, 2$)).

Symmetry arguments show that $\delta(p, \kappa)$ always satisfies the unbiasedness condition (6.2). Hence $\delta(p, \kappa) \varepsilon W$. The envelope risk $R_W^*(\theta)$ is equal to a when $\theta = (0, 0)$, for the trivial procedure δ_0 with $\varphi_0 = 1$ a.e. (μ) is unbiased while each other procedure has the risk $a + \frac{1}{2}b[1 - E\{\varphi_0(X)\}] > a$. If $\theta = (\nu, 0)$ or $\theta = (0, \nu)$ then

$$(9.3) \qquad R_W^*(\theta) = (b - a)\Phi\{-\tilde{c}(\nu) - \nu\} + a\Phi\{\tilde{c}(\nu) - \nu\}$$

where $\tilde{c}(\nu)$ is determined by (6.5). For $\theta = (\nu, 0)$ this results from the fact that each procedure δ_c of Example 6.2 is *unbiased* and a discussion similar to that in Example 6.1 and in the proof of Theorem 6.1 (see (6.3)).

Hence the risk function $R\{\theta, \delta(p, \kappa)\}$ and the regret function $S_W\{\theta, \delta(p, \kappa)\}$ can be computed numerically (we used Gauss-Hermite m-point quadrature formulas and checked the results, by varying m). We are looking for the Bayes procedure $\delta(p^*, \kappa^*)$ which assumes its maximum regret in each of the three points (0, 0), $(\kappa^*, 0)$ and $(0, \kappa^*)$ if such a procedure exists. In order to find the the point (p^*, κ^*) we first determine $p(\kappa)$ such that

$$S_W[(0, 0), \delta\{p(\kappa), \kappa\}] = S_W[(\kappa, 0), \delta\{p(\kappa), \kappa\}] = S_W^*(\kappa)$$

and next we determine κ^* such that $S_W^*(\kappa^*) = \sup S_W^*(\kappa)$ (a similar method was applied in [28]). The computations showed that the resulting procedure $\delta\{p(\kappa^*), \kappa^*\}$ assumes its maximum regret in each of the points (0, 0), $(\kappa^*, 0)$ and $(0, \kappa^*)$.

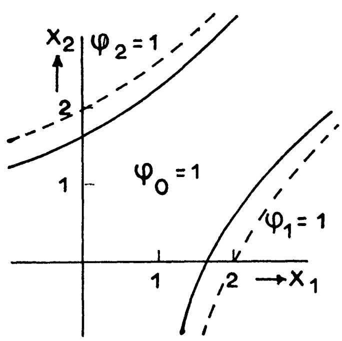

fig 1

Hence this procedure has indeed minimax regret with respect to the class W of all unbiased procedures. For $b/a = 9$ we obtained $\kappa^* = 1.667$; $p(\kappa^*) = .3992$. The corresponding partition of the sample space $\mathfrak{X} = R^2$ is drawn in Figure 1. For $b/a = 19$ we obtained $\kappa^* = 1.856$; $p(\kappa^*) = .4141$ (see Figure 1, dotted lines).

Previous publications ([9], [16], [23] and [33]) and especially Paulson's fundamental paper [26] suggest that it is quite natural to consider procedures δ_c^{nat} where decision d_i is taken with probability 1 for all points (x_1, x_2) with $x_i = \max(x_1, x_2) > c$ $(i = 1, 2)$ while d_0 is taken for all other points (see Figure 2). We determined the *best natural procedure* $\delta_{c^*}^{\text{nat}}$ that has minimax regret (W) among the procedures of the class $U = \{\delta_c; c > 0\}$. For $b/a = 9$ we obtained $c^* = 1.758$ (see Figure 2) and for $b/a = 19$ we calculated $c^* = 2.255$ (Figure 2, dotted lines). In Figures 3 and 4 the regret functions (divided by a) are plotted for $b/a = 9$ and $b/a = 19$ respectively. Of course $\delta_{c^*}^{\text{nat}}$ has a larger maximum regret than the minimax regret (W) procedure $\delta^* = \delta\{p(\kappa^*), \kappa^*\}$ because $\delta_{c^*}^{\text{nat}} \varepsilon U \subset W$. This disadvantage is counterbalanced to some extent, because $\delta_{c^*}^{\text{nat}}$ has a smaller regret in the neighborhood of the origin but nevertheless the *minimax regret*

(W) procedure δ^* *seems to provide a worthwhile improvement upon the best natural procedure* $\delta_{c^*}^{\text{nat}}$ *when the loss function* (6.1) *is used.*

REMARK 9.1. Let C_α denote the class of all $\delta = \delta(\varphi_0, \varphi_1, \varphi_2)$ with (i) $E_{\theta_0}\{\varphi_0(X)\} = 1 - \alpha$ for $\theta_0 = (0, 0)$ and (ii) $\varphi_1(x_1, x_2) = \varphi_2(x_2, x_1)$ for all x_1, x_2 (a symmetry

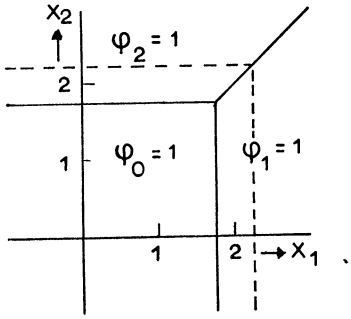

fig 2

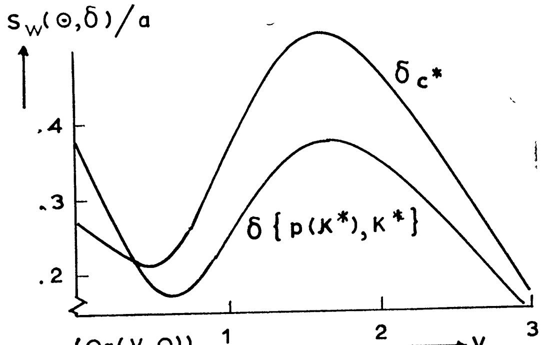

fig 3 (b' = 9)

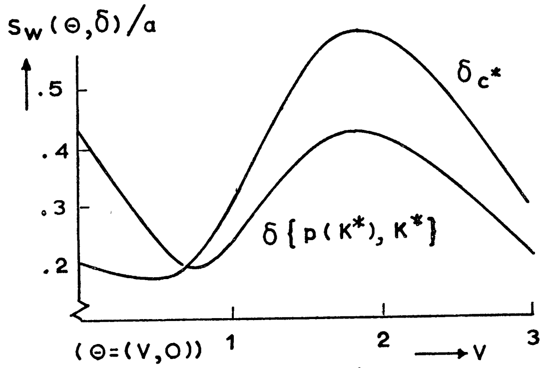

fig 4 (b' = 19)

property). In Remark 6.3 a similar restriction was very attractive because there existed a simple procedure with U.M.R. (C_α) *for each loss function* (6.1). It will turn out that these advantages of the size-restriction get lost for the more general problems of Section 7 and particularly for Example 9.1 with loss function (6.1). First we can apply Paulson's method [26] in order to show that *the natural procedure* $\delta^{\text{nat}}_{c(\alpha)}$ *with* $c(\alpha) = u_\beta$ *where* $\beta = 1 - (1-\alpha)^{\frac{1}{2}}$ *is the* U.M.R. (C_α) *procedure in the special case* $a = b$ (the proof is deleted). Next we remark that the application to Example 9.1 shows that this result will not hold for $b/a > 2$. If b/a is fixed and larger than 2, then there even will not exist a U.M.R. (C_α) procedure. We might try to obtain the minimax regret (C_α) procedure but this will depend on b/a. Hence the size-restriction will not provide relief unless we combine this restriction with the rather unreasonable assumption $a = b$. We have the opinion that in that case the problem is oversimplified.

REMARK 9.2. The method described above in the application to Example 9.1 will apply to many problems from Section 7. Unfortunately it may be very difficult to determine the envelope risk function $R_W{}^*(\theta)$. This may be illustrated by modifying Example 9.1, changing Ω_2 for the half-line of all points $(r \cos \Psi, r \sin \Psi)$ $(r > 0)$ where Ψ is a fixed angle $(0 < \Psi \leqq 180°)$. Then (9.3) determines the envelope risk in $\theta_1 = (\nu, 0)$ and $\theta_2 = (\nu \cos \Psi, \nu \sin \Psi)$ provided that $\Psi \geqq 90°$. If $\psi < 90°$ (as will be the case when we try to deal with certain simplifications of the example of Section 7) then the procedures δ_c mentioned under (9.3) are not unbiased and the envelope risk is unknown.

10. Other problems for selecting the best. Example 9.1 and the example of Section 7 were formulated by means of a parameter space which was restricted by using an unreasonable assumption. In Example 9.1 for instance, we assumed that μ_1 and μ_2 cannot both be positive. Of course one can try to repair this by extending the parameter space and the regions in which this space is partitioned. The general results of the Sections 7 and 8 remain true because the problem remains of Type I, but (i) the procedures constructed (or suggested) in Section 9 lose their optimum property (Example 10.1), (ii) one might want to reformulate the loss function (Example 10.1) and (iii) one might want to enlarge the decision space (Example 10.2).

EXAMPLE 10.1. *Repairing Example* 9.1. Suppose that we try to remedy Example 9.1 by defining Ω_0 as the set of all (μ_1, μ_2) with $\mu_1 = \mu_2 \geqq 0$ and Ω_i as the set of all (μ_1, μ_2) with $\mu_i > \mu_{3-i} \geqq 0$ $(i = 1, 2)$. Everything of the application to Example 9.1 gets lost. Even the envelope risk function changes because the procedures δ_c mentioned under (9.3) are no longer unbiased (see Remark 9.2). Fortunately one can prove easily by modifying the theory of Section 6 and applying Lemma 6.1 that there exists a unique minimax regret unbiased procedure $\delta^*(\varphi_0{}^*, \varphi_1{}^*, \varphi_2{}^*)$ and that $\varphi_i{}^*$ is the indicator function of the set of sample points (x_1, x_2) satisfying $x_i - x_{3-i} \geqq c^*(b/a)2^{\frac{1}{2}}$ where $c^*(b/a)$ is the value of c minimizing (6.6) $(i = 1, 2)$. This procedure can be compared with those of Application to Example 9.1 (see Figures 1 and 2). $\delta\{p(\kappa^*), \kappa^*\}$ is much more attractive than $\delta^{\text{nat}}_{c^*}$ from this point of view.

We have the opinion that $\delta^*(\varphi_0^*, \varphi_1^*, \varphi_2^*)$ is not a very reasonable procedure in many actual situations, notwithstanding its attractive optimum property. The explanation is that the loss function (6.1) is not very attractive: deciding upon d_i whereas $\mu_i = 0 < \mu_{3-i}$ has to be regarded as more serious than deciding upon d_i while $0 < \mu_i < \mu_{3-i}$ $(i = 1, 2)$, because $\mu = 0$ will correspond with a certain standard situation. This might be expressed by constructing a new loss function. Let Ω_i denote the positive μ_i-axis as in Example 9.1 and let Ω_{2+i} denote the region defined by $\mu_i > \mu_{3-i} > 0$ $(i = 1, 2)$. The following extension of loss function (6.1) seems to be reasonable.

$$\begin{aligned} &L(\theta_i, d_0) = a; \quad L(\theta_i, d_i) = 0; \quad L(\theta_i, d_{3-i}) = b \qquad (\theta_i \varepsilon \Omega_i) \\ (10.1)\quad &L(\theta_{2+i}, d_0) = 2a; \quad L(\theta_{2+i}, d_i) = 0; \quad L(\theta_i, d_{3-i}) = c \qquad (\theta_{2+i} \varepsilon \Omega_{2+i}), \\ &(i = 1, 2). \end{aligned}$$

Of course the problem is again of Type I ($\Omega_0' = \{(0, 0)\}$) but now $m = 4, n = 2$. The loss function (10.1) expresses that we make two errors of the second kind when we decide upon d_0 whereas both μ_1 and μ_2 are positive $(\theta \varepsilon \Omega_3 \cup \Omega_4)$. We assume $0 < 4a < c < b$. Straightforward geometric arguments (consider the matrix (w_{ij}) as 4 vectors in R^3 and determine the maximin point) show that $(0, 0, \frac{1}{2}, \frac{1}{2})$ is the unique maximin strategy for Player I. Using this auxiliary strategy in Lemma 3.2, we can prove that $(1, 0, 0)$ is the unique minimax strategy for Player II with the result that once more the trivial procedure δ_0 with $\varphi_0 = 1$ a.e. (μ) is the unique M.R. procedure. By applying Lemma 4.1 we can characterize the class W of all unbiased procedures.

LEMMA 10.1 *$\delta(\varphi_0, \varphi_1, \varphi_2) \varepsilon W$ if and only if for all $\theta = \theta_i \varepsilon \Omega_i$*

$$(10.2)\quad E_\theta\{\varphi_{3-i}(X)\} \leqq E_\theta\{\varphi_i(X)\} = 1 - (b - c + a)E_\theta\{\varphi_{3-i}(X)\}/a \quad (i = 1, 2)$$

holds, while for all $\theta = \theta_{2+i} \varepsilon \Omega_{2+i}$

$$(10.3)\quad E_\theta\{\varphi_i(X)\} \geqq \max\,[E_\theta\{\varphi_{3-i}(X)\},\ 1 - (b - c + a)E_\theta\{\varphi_{3-i}(X)\}/a] \quad (i = 1, 2)$$

Hence necessary for $\delta \varepsilon W$ is that $E_{\theta_0}\{\varphi_0(X)\} = 1 - \alpha$, $E_{\theta_0}\{\varphi_i(X)\} = \frac{1}{2}\alpha$ $(i = 1, 2)$ holds for $\theta_0 = (0, 0)$ and with $\alpha = 2a/(2a + b - c)$.

From the last part follows $\delta_0 \not\varepsilon W$; the unbiasedness condition is open to suspicion because it may be too restrictive. This does also follow from the equality in (10.2), the reasonableness of which is questionable and which shows that the procedures constructed in the application to Example 9.1 are not unbiased and cannot be modified easily such that they become unbiased.

CONCLUSION 10.1. The problem has become so difficult that we can only obtain certain general results relating and characterizing the classes of minimax risk and unbiased procedures: optimum results are unattainable.

EXAMPLE 10.2. *On selecting a subset containing the best.* When both μ_1 and μ_2 are admitted to be positive, then it may be very difficult to separate them and the decision space $\mathfrak{D} = \{d_0, d_1, d_2\}$ is only appropriate in the rather unscientific situation that no further experimentation is allowed and that the experimenter has been charged to take one of the three decisions where d_0 for example corresponds with proceeding under standard conditions while d_i results in converting from standard treatment to treatment i $(i = 1, 2)$. In many situations one will like to introduce the decision d_3, stating for example that the best of the treatments 1 and 2 is better than the standard treatment but that one does not decide which one of the two nonstandard treatments is actually the best. Now our problem may be regarded as a problem to select a subset containing the best: d_0 corresponds with selecting the void set; d_i with selecting the set $\{\mu_i\}$ consisting of the ith mean $(i = 1, 2)$ while d_3 selects the set $\{\mu_1, \mu_2\}$ of all means. One is interested in selecting the largest mean; the decision states that the corresponding subset contains this largest mean. In Example 11.1 we shall deal with the problem where one is not primarily interested in selecting the subset containing the best but in selecting the subset containing the means that are significantly positive.

We shall now formulate a very general problem of selecting the subset containing the largest. Selecting the void set is not permitted in this problem (this modification is introduced for the sake of Remark 10.1). Let X_i have the normal $N(\mu_i, 1)$ distribution $(i = 1, \cdots, m)$. The investigator tries to find a subset of the m expectations or indices i such that this subset is likely to contain the largest μ_i; on the other hand the subset is as small as possible (this problem can be modified and extended in many ways, see [1], [5], [13], [21], [24], [31]). $\Omega = \Omega_0 \cup \cdots \cup \Omega_m$ will consist of all possible $\theta = (\mu_1, \cdots, \mu_m)$; Ω_i is defined by $\mu_i > \max_{j \neq i} \mu_j$ (μ_i is the largest expectation) $(i = 1, \cdots, m)$. The decision space $\mathfrak{D}$ consists of all $2^m - 1$ decisions $d(h_1, \cdots, h_m)$ $\{h_i = 0, 1$ $(i = 1, \cdots, m)$, $\sum h_i > 0\}$ where $d(h_1, \cdots, h_m)$ corresponds with selecting the subset of those expectations μ_i for which $h_i = 1$. If $\theta \in \Omega_i$ and $h_i = 0$ in $d(h_1, \cdots, h_m)$ then an error of the first kind is committed, for the selected subset does not contain the largest expectation μ_i. This serious error results in a loss equal to b units. If $\theta \in \Omega_i$ and $h_j = 1$ in $d(h_1, \cdots, h_m)$ for a certain index $j \neq i$, then the selected subset unnecessarily contains μ_j. This error of the second kind results in losing a units. Accordingly our loss function becomes

$$(10.4) \qquad L\{\theta, d(h_1, \cdots, h_m)\} = b(1 - h_i) + a(\textstyle\sum_{j=1}^{m} h_j - h_i) \qquad (\theta \in \Omega_i) \quad (i = 1, \cdots, m).$$

The symmetry shows that $(1/m, \cdots, 1/m)$ is a maximin strategy for Player I which may be used when we want to apply Lemma 3.2 or Lemma 3.1. By doing so we obtain the following results (i) and (ii).

(i) If $b > (m - 1)a$ then Player II has a unique minimax strategy and this uses $d(1, \cdots, 1)$ with probability 1. Hence there exists a unique M.R. procedure and this assigns $d(1, \cdots, 1)$ with probability 1 to (almost) all $x \in \mathfrak{X}$. This trivial procedure is not attractive because the risk is everywhere equal to $(m - 1)a$. Lehmann's unbiasedness condition will be attractive and not too restrictive, for the above-mentioned M.R. procedure is unbiased. Unfortunately the problems with $m > 2$ seem to be rather forbidding (Conclusion 10.1 holds); the case $m = 2$ will be discussed in Remark 10.1.

(ii) If $b < (m - 1)a$ then Player II has a unique minimax strategy and this uses the m columns $d(0 \cdots 010 \cdots 0)$ each with probability $1/m$. Hence an M.R. procedure never selects a subset consisting of more than 1 mean. The natural procedure which selects the subset $\{\mu_i\}$ if and only if $x_i = \max (x_1, \cdots, x_m)$ will have certain optimum properties among the M.R. procedures ([2]) and seems to be the proper procedure if $b < (m - 1)a$, though this result is somewhat suspect for it implies that the "optimum" procedure does not depend on the special value of b/a in (10.4), as long as $b/a < m - 1$. In practice the experimenter will choose $b > (m - 1)a$ or try to find a more adequate partition of Ω (see Example 10.1), for the above-mentioned natural procedure will not be attractive for an investigator who considers all $2^m - 1$ decisions of $\mathfrak{D}$ as reasonable.

REMARK 10.1. The theory of Section 6 applies to the case $m = 2$; the decisions d_0, d_1 and d_2 are analogs of $d(1, 1)$, $d(1, 0)$ and $d(0, 1)$; Ω_0 consists of all (μ_1, μ_2) with $\mu_1 = \mu_2$ and is not simple (Lemma 6.1 has to be used in order to apply Theorem 6.1); Ω_i is the set of all (μ_1, μ_2) with $\mu_i > \mu_{3-i}$ $(i = 1, 2)$. The loss functions (6.1) and (10.4) $(m = 2)$ are not in complete agreement; the coefficient b in (6.1) has to be changed for $a + b$. This shows that the notions of errors of the first and second kind of this section are *not equivalent* to those of Section 6.

11. Unbiasedness and minimax risk for unrestricted products of two-decision testing problems. Apart from the two- and three-decision problems of Section 5 and Section 6, the most important problems seem to be those where the means or variances have to be ranked in an analysis of variance with one-way classification (first paragraph of Section 1). In this and the following section we shall try to attack these problems (Example 11.2 and 12.2). First we consider *unrestricted* products of problems of the form studied in Section 5. Lehmann remarked that this concept is not general enough to consider the most interesting m.d.p.'s and he introduced the more general concept of a *restricted* product. In [19] he showed that even for such general problems there often exists a U.M.R. unbiased procedure. Nevertheless we confine the attention to unrestricted products because (i) for such problems Theorem 11.1 can be shown, suggesting that the unbiasedness restriction is very attractive (see also Remark 11.1) whereas for many restricted products the unbiasedness restriction is open to suspicion (see Remark 8.1 concerning Lehmann's formulation of the three-decision problem), (ii) many m.d.p.'s formulated as a restricted product by Lehmann may obtain another more beautiful formulation along the lines of this paper (see Remark 8.1).

In order to define the notion of an unrestricted product, let $\Pi_1, \cdots, \Pi_k$ be k two-decision problems for the random observable X with pdf p_θ $(\theta \varepsilon \Omega)$ over $\mathfrak{X}$. Problem Π_h is of the form described in Section 5. Let $\Omega = \Omega_1(h) \cup \Omega_2(h)$ be the corresponding partition of Ω (we assume $\Omega_0(h) = \varnothing$) and let $d_0(h)$ and $d_1(h)$ denote the decisions. The loss function for Problem Π_h is given by (see (5.1))

$$(11.1)\quad \begin{aligned} L\{\theta, d_0(h)\} &= w_{10}(h) = 0; \quad L\{\theta, d_1(h)\} = w_{11}(h) = b_h \quad (\theta \varepsilon \Omega_1(h)) \\ L\{\theta, d_0(h)\} &= w_{20}(h) = a_h; \quad L\{\theta, d_1(h)\} = w_{21}(h) = 0 \quad (\theta \varepsilon \Omega_2(h)). \end{aligned}$$

Suppose that all 2^k intersections $\Omega(i_1, \cdots, i_k) = \bigcap_{h=1}^k \Omega_{i_h}(h)$ are nonempty $\{i_h = 1, 2 (h = 1, \cdots, k)\}$ and that $d(j_1, \cdots, j_k)$ denotes the compound statement composed of the k decisions $d_{j_h}(h)$ $(h = 1, \cdots, k)$. The unrestricted product problem $\Pi = \Pi_1 \times \cdots \times \Pi_k$ is defined by the partition of Ω into the 2^k nonempty intersections described above, the 2^k compound decisions and the loss function

$$(11.2) \quad L(\theta, d(j_1, \cdots, j_k)) = \sum_{h=1}^k \{a_h\epsilon_1(i_h, j_h) + b_h\epsilon_2(i_h, j_h)\} \quad (\theta \in \Omega(i_1, \cdots, i_k))$$

where by definition $\epsilon_1(2, 0) = 1$ and $\epsilon_2(1, 1) = 1$ while in all other cases $\epsilon_\nu(i, j) = 0$.

A procedure δ for Problem Π is determined by 2^k test-functions $\varphi_{j_1 \cdots j_k}$ $\{j_h = 0, 1$ $(h = 1, \cdots, k)\}$ determining the probability of taking the corresponding decision when the observation $x \in \mathfrak{X}$ has been obtained. The procedure δ defines in a natural way a procedure $\delta_h = \delta(\varphi_0^{(h)}, \varphi_1^{(h)})$ for the component Problem Π_h when

$$(11.3) \qquad \varphi_{j_h}^{(h)} = \sum_{j_1=0}^1 \cdots \sum_{j_{h-1}=0}^1 \sum_{j_{h+1}=0}^1 \cdots \sum_{j_k=0}^1 \varphi_{j_1 \cdots j_k}$$

On the other hand if $\delta_h = \delta(\varphi_0^{(h)}, \varphi_1^{(h)})$ is a procedure for Problem Π_h then the family $\{\delta_1, \cdots, \delta_k\}$ corresponds with a product procedure $\delta = \delta_1 \times \cdots \times \delta_k$ for Problem Π where

$$(11.4) \qquad \varphi_{j_1 \cdots j_k} = \prod_{h=1}^k \varphi_{j_h}^{(h)}.$$

It can be shown easily that the relations (11.3) and (11.4) establish a 1:1 correspondence between the *product procedures* for Problem Π and the families $\{\delta_1, \cdots, \delta_k\}$ of procedures for the respective Problems Π_h. In the case of nonrandomized ($\varphi = 0$ or 1) procedures, Lehmann ([19] Theorem 1) proved that each procedure for Problem Π is a product procedure. Unfortunately this result does not hold in the case of randomized procedures ([21] page 995), but the following lemma shows that it is very reasonable to restrict the attention to the class of all product procedures for Problem Π.

LEMMA 11.1. *Suppose δ' is an arbitrary procedure for Problem Π and that $\{\delta_1, \cdots, \delta_k\}$ is the corresponding family defined by* (11.3). *Then δ' and $\delta = \delta_1 \times \cdots \times \delta_k$ have the same risk function. Moreover δ' is unbiased if and only if δ is unbiased.*

PROOF. The result is obtained easily by showing that for $\theta' \in \Omega(i_1, \cdots, i_k)$,

$$\begin{aligned} &E_\theta[L\{\theta', \delta'(X)\}] \\ (11.5) \quad &= \sum_{j_1=0}^1 \cdots \sum_{j_k=0}^1 \sum_{h=1}^k \{a_h\epsilon_1(i_h, j_h) + b_h\epsilon_2(i_h, j_h)\} E_\theta\{\varphi_{j_1 \cdots j_k}(X)\} \\ &= \sum_{h=1}^k E_\theta[L\{\theta', \delta_h(X)\}] = E_\theta[L\{\theta', \delta(X)\}] \end{aligned}$$

for then $\theta = \theta'$ shows $R(\theta, \delta') = R(\theta, \delta)$ while the unbiasedness result follows immediately from the definition (4.1).

THEOREM 11.1. *If* $\Pi = \Pi_1 \times \cdots \times \Pi_k$ *is of Type* I, W_1 *is the class of all* M.R. *product procedures,* W_2 *that of all unbiased products and* W_3 *that of all products* $\delta_1 \times \cdots \times \delta_k$ *where* $\delta_h = \delta(\varphi_0^{(h)}, \varphi_1^{(h)})$ *and such that* $\varphi_1^{(h)}$ *is an unbiased size-*α_h *test for Problem* Π_h *with* $\alpha_h = a_h/(a_h + b_h)(h = 1, \cdots, k)$. *Then* $W_3 \subset W_2 \subset W_1$.

PROOF. We first remark that each component Problem Π_h will be of Type I with the result that Theorem 5.1 shows that W_3 is the class of all $\delta = \delta_1 \times \cdots \times \delta_k$ where δ_h is unbiased for Problem Π_h. By applying (11.5) we obtain $W_3 \subset W_2$. Next let W_1' denote the class of all M.R. procedures and W_2' that of all unbiased procedures for Problem Π. By applying Lemma 4.2 and Theorem 4.1 we shall show that $W_2' \subset W_1'$ with as a consequence $W_2 \subset W_1$. For that purpose consider $w^* = (\bar{w}^*, \cdots, \bar{w}^*)$ with $\bar{w}^* = \sum_{h=1}^{k} a_h b_h/(a_h + b_h)$. Then (Lemma 4.2) we have to show $w^* \varepsilon S$. Let $w(j_1, \cdots, j_k)$ denote the vector in $R^m (m = 2^k)$ with the coordinates (11.2) $\{i_h = 1, 2 (h = 1, \cdots, k)\}$. We have to show that w^* is a convex combination of these 2^k vectors. The weights

$$(11.6) \quad \rho(j_1, \cdots, j_k) = \{\textstyle\prod_{h=1}^{k} (a_h + b_h)\}^{-1} \prod_{h=1}^{k} \{a_h \epsilon_2(1, j_h) + b_h \epsilon_1(2, j_h)\}$$

provide the required result. Next we show that there exists a vector $g \, \varepsilon \, R^m$, or equivalently a strategy for Player I, such that the other conditions of Lemma 4.2 are satisfied. The coordinates

$$(11.7) \quad g(i_1, \cdots, i_k) = \{\textstyle\prod_{h=1}^{k} (a_h + b_h)\}^{-1} \prod_{h=1}^{k} \{a_h \epsilon_2(i_h, 1) + b_h \epsilon_1(i_h, 0)\}$$

provide the required result. Hence the proof is complete.

The referee remarked that we can obtain more insight into Theorem 11.1 by establishing $W_3 \subset W_1$, using the simple remark that the value of the game with $2^k \times 2^k$ matrix (11.2) is equal to the sum of the values of the k games with 2×2 matrices (11.1). This elucidates our formula for $\bar{w}^*$. For suppose $\delta = \delta_1 \times \cdots \times \delta_k \, \varepsilon \, W_3$, then each δ_h has M.R. for the corresponding component problem (Theorem 5.1). By applying (11.5) in case $\theta = \theta'$ we obtain that the maximum risk of δ is not larger than $\bar{w}^*$. Hence $\delta \, \varepsilon \, W_1$ (see (3.1) in the proof of Theorem 3.1). Moreover $W_3 \subset W_2$ was established as a simple consequence of (11.5). Unfortunately (11.5) suggests that there may exist product problems such that $W_3 \neq W_2$ or in other words such that there may exist *unbiased product procedures which are not products of unbiased procedures.* Lehmann ([19] page 16) gave sufficient conditions for $W_2 = W_3$ or equivalently such that the unbiasedness of a product implies the unbiasedness of the components. His way of proving the sufficiency of these conditions applies to many situations from practice and particularly to the Examples 11.1 and 11.2. Nevertheless $W_3 \neq W_2$ will be possible in some situations and Theorem 11.1 provides the extra information that $W_2 \subset W_1$ still holds. Theorem 11.1 suggests that it is reasonable for unrestricted Type I product problems to confine the attention to the class W_2 or the class W_3. Remark 11.1 will show that this is a true restriction in general.

THEOREM 11.2. (*Lehmann* [19] *page* 18) *For the unrestricted Type I product* $\Pi = \Pi_1 \times \cdots \times \Pi_k$ *let* $\omega(h) = [\Omega_1(h)] \cap [\Omega_2(h)]$ *and* $\delta^* = \delta_1^* \times \cdots \times \delta_k^*$ *with* $\delta_h^* = \delta(\varphi_0^{(h)}, \varphi_1^{(h)})$ *be such that with* $\alpha_h = a_h/(a_h + b_h)$ (i) $\varphi_1^{(h)}$ *is* U.M.P. *similar* *size-α_h for testing $\theta \varepsilon \omega(h)$ against $\theta \varepsilon \Omega_2(h) - \omega(h)$ and* (ii) $\varphi_0^{(h)}$ *is* U.M.P. *similar size-$(1 - \alpha_h)$ for testing $\theta \varepsilon \omega(h)$ against $\theta \varepsilon \Omega_1(h) - \omega(h)$. Then δ^* has* U.M.R. (W_3) *for Problem* Π. *If moreover* (iii) *for each $\theta \varepsilon \omega(h)$ there exist indices $i_1, \cdots, i_k$ such that $\theta \varepsilon [\Omega(i_1, \cdots, i_k)]$ both when $i_h = 1$ and when $i_h = 2$, then δ^* has also* U.M.R. (W_2) *for Problem* Π.

PROOF. First show that the conditions (i) and (ii) imply that $\varphi_1^{(h)}$ is U.M.P. unbiased size-α_h for testing the hypothesis $\theta \varepsilon \Omega_1(h)$ against $\theta \varepsilon \Omega_2(h)$ and that $\varphi_0^{(h)}$ is U.M.P. unbiased size-$(1 - \alpha_h)$ for testing $\theta \varepsilon \Omega_2(h)$ against $\theta \varepsilon \Omega_1(h)$. Hence $\delta^* \varepsilon W_3$ and if $\delta = \delta_1 \times \cdots \times \delta_k \varepsilon W_3$ then it is easily seen that on account of the unbiasedness of δ_h for Problem Π_h we have $R(\theta, \delta_h) \geqq R(\theta, \delta_h^*)$ for all $\theta \varepsilon \Omega$. Then (11.5) with $\theta = \theta'$ provides that δ^* has U.M.R. (W_3).

In order to show the second part, suppose $\bar{\delta} = \bar{\delta}_1 \times \cdots \times \bar{\delta}_k \varepsilon \mathrm{W}_2$ then we have to show $R(\theta, \bar{\delta}) \geqq R(\theta, \delta^*)$ for all $\theta \varepsilon \Omega$ but it is sufficient when we show $R(\theta, \bar{\delta}_h) \geqq R(\theta, \delta_h^*)(\theta \varepsilon \Omega)$ for the component Problem Π_h. For that purpose we show that the unbiasedness of $\bar{\delta}$ implies for $\bar{\delta}_h = \delta(\bar{\varphi}_0^{(h)}, \bar{\varphi}_1^{(h)})$ that $\bar{\varphi}_1^{(h)}$ is similar size-α_h and $\bar{\varphi}_0^{(h)}$ is similar size-$(1 - \alpha_h)$ for testing the hypothesis $\theta \varepsilon \omega(h)$. For then (i) and (ii) easily provide the required result for Problem Π_h. Hence let $\bar{\delta}$ be unbiased for Problem Π and $\theta \varepsilon \omega(h)$. Condition (iii) and the continuity of $E_\theta(\varphi)$ as a function of θ (see definition of Type I m.d.p.) imply that

$$\sum_{j_1=0}^{1} \cdots \sum_{j_k=0}^{1} \sum_{h=1}^{k} \{a_h \epsilon_1(i_h, j_h) + b_h \epsilon_2(i_h, j_h)\} E_\theta\{\bar{\varphi}_{j_1 \cdots j_k}(X)\}$$

has the same value for $i_h = 1$ and for $i_h = 2$ when $\bar{\delta}$ is defined by the 2^k test functions $\bar{\varphi}_{j_1 \cdots j_k}$. Then (11.3) shows that this implies $b_h E_\theta\{\bar{\varphi}_1^{(h)}(X)\} = a_h E_\theta\{\bar{\varphi}_0^{(h)}(X)\}$. By using $\bar{\varphi}_0^{(h)} + \bar{\varphi}_1^{(h)} = 1$, the similarity follows.

Lehmann's Theorem 2 in [19] constitutes a generalization of Theorem 11.3 to restricted products. If Theorem 11.3 provides that δ^* has U.M.R. (W_2) where W_2 is the class of all unbiased *product* procedures then Lemma 11.1 shows that δ^* has also U.M.R. (W_2') where W_2' denotes the larger class of all unbiased procedures for Problem Π. In Remark 11.1 we shall see that δ^* does not have U.M.R. (W_1) in general: there may exist much more M.R. procedures than unbiased procedures for Problem Π.

EXAMPLE 11.1. (Combining k one-sided tests of significance; see Example 12.1 for a more reasonable formulation in the $k = 2$ case.) Let $X_1, \cdots, X_k$ have independent normal $N(\mu_i, 1)$ distributions $(i = 1, \cdots, k)$. Ω is the R^k of all points $\theta = (\mu_1, \cdots, \mu_k)$ (the case (see [34]) where Ω is the positive orthant of R^k consisting of all θ with $\mu_i \geqq 0$ for all i, can be treated along the same lines and provides the same optimum procedure δ^*). Π_h is defined by the partition $\Omega = \Omega_1(h) \cup \Omega_2(h)$ where $\Omega_1(h)$ is defined by $\mu_h \leqq 0$ and $\Omega_2(h)$ by $\mu_h > 0$. We verify the conditions of Theorem 11.2. The subset $\omega(h)$ consists of all θ with $\mu_h = 0$. The test $\varphi_1^{(h)}$ with $\varphi_1^{(h)}(x_1, \cdots, x_k) = 1$ (or 0) for all sample points with $x_h \geqq$ (or $<$) u_α is the U.M.P. similar size-α_h test for testing $\theta \varepsilon \omega(h)$ against $\theta \varepsilon \Omega_2(h) - \omega(h)$ while $\varphi_0^{(h)} = 1 - \varphi_1^{(h)}$ satisfies condition (ii) of Theorem 11.2. Also condition (iii) is easily verified. Hence the natural procedure $\delta^* = \delta_1^* \times \cdots \times \delta_k^*$ with $\delta_h^* = \delta(\varphi_0^{(h)}, \varphi_1^{(h)})$ which combines the results of the k

component one-sided size-α_h tests has U.M.R. (W_r) for the compound Problem $\Pi(r = 2, 3)$.

REMARK 11.1. The U.M.R. (W_2) procedure δ^* does not have U.M.R. (W_1) in case $k = 2$, $a_1 = a_2 = a$, $b_1 = b_2 = b$. If $\theta_0 = (0, 0)$, $\delta^* = \delta(\varphi_{00}, \cdots, \varphi_{11})$ and $p_{ij} = E_{\theta_0}(\varphi_{ij})$ then $p_{00} = b^2/(a + b)^2$, $p_{10} = p_{01} = ab/(a + b)^2$ and $p_{11} = a^2/(a + b)^2$. Let W_1^* denote the class of all M.R. product procedures for which these equalities hold. Then obviously $\delta^* \varepsilon W_1^* \subset W_1$ where the inclusion is strict, for the unique minimax and maximin point of the convex set S does not have a unique convex representation: apart from the coefficients p_{ij} corresponding with those determined by (11.6) one may also choose $p_{00} = b/(a + b)$, $p_{10} = p_{01} = 0$, $p_{11} = a/(a + b)$ for example. Hence there exist many M.R. product procedures not belonging to W_1^* and in order to show that δ^* does not have U.M.R. (W_1) it is sufficient when we show the much stronger result that δ^* does not have U.M.R. (W_1^*). For that purpose we have to construct a procedure $\delta' \varepsilon W_1^*$ such that $R(\theta, \delta') < R(\theta, \delta^*)$ holds for some $\theta \varepsilon \Omega$. A simple example is obtained as follows. Let Q denote the set of all (x_1, x_2) with $x_1 < u_\alpha$ where $\alpha = ab/(a + b)^2$ and let I_Q denote the indicator function of Q. Define $\varphi'_{00} = b^2 I_Q/c$; $\varphi'_{01} = abI_Q/c$; $\varphi'_{10} = 1 - I_Q$; $\varphi'_{11} = a^2 I_Q/c$ where $c = a^2 + ab + b^2$. Then it is shown easily that $\delta(\varphi'_{00}, \cdots, \varphi'_{11})$ is a procedure satisfying the equalities mentioned at the beginning of this remark. Moreover the corresponding risk is not larger than $\bar{w}^* = 2ab/(a + b)$ for all $\theta \varepsilon \Omega$ (provided that $a \leqq b$; we have to consider respectively all 4 cases $\theta \varepsilon \Omega(i_1, i_2)$). Let δ' denote the product procedure corresponding with $\delta(\varphi'_{00}, \cdots, \varphi'_{11})$ (first apply (11.3) and next (11.4)). Lemma 11.1 shows that $\delta' \varepsilon W_1^*$. Moreover $R(\theta, \delta') < R(\theta, \delta^*)$ will hold for $\theta = (\mu_1, 0)$ with μ_1 sufficiently large. The remark shows that apart from the criterion U.M.R. (W_2) also other criterions might be of interest, like the criterion *minimax regret* (W_1).

REMARK 11.2. The U.M.R. (W_2) procedure δ^* provides in a natural way a test $1 - \varphi_{00}$ for testing $H: \theta \varepsilon \Omega_{11}$ against $K: \theta \varepsilon \Omega - \Omega_{11}$ in the case of Remark 11.1. This test rejects if and only if max (X_i) is sufficiently large. Tests of this form may be rather reasonable (though not unbiased) if $\Omega = R^2$ but they are not very attractive if Ω is the positive orthant of R^2 (see [28] and [34]). Darroch and Silvey [7] constructed similar product problems where the test of the form $1 - \varphi_{00}$ has very poor power properties for testing H against K. Obviously this results from the formulation of our product problems where no special attention is paid to testing H against K. In many practical situations it seems to be very natural to be concerned about the testing problem (H, K). This shows that for such situations the formulation is oversimplified if we consider them as unrestricted products. In Section 12 we try to remedy this.

EXAMPLE 11.2. (An m.d.p. for a $(k + 1)$-sample trend situation; see Example 12.2 for a more reasonable formulation in the $k = 2$ case). Let $X_{i1}, \cdots, X_{in_i}$ be an independent sample from the normal $N(\mu_i, \sigma^2)$-distribution $(i = 0, \cdots, k)$. Theoretical a priori considerations show that one may assume $\mu_0 \leqq \cdots \leqq \mu_k$. Hence Ω consists of all points $\theta = (\mu_0, \cdots, \mu_k, \sigma^2)$ in R^{k+2} satisfying

$\mu_0 \leqq \cdots \leqq \mu_k$, $\sigma^2 > 0$. Let Π_h denote the two-decision problem for comparing μ_{h-1} and μ_h. The corresponding partition $\Omega = \Omega_1(h) \cup \Omega_2(h)$ is determined by $\Omega_1(h)$ and $\Omega_2(h)$ which are respectively the subsets of Ω defined by $\mu_{h-1} = \mu_h$ and $\mu_{h-1} < \mu_h$. The decision $d_0(h)$ states that there exists no sufficient evidence for "$\mu_{h-1} < \mu_h$" whereas $d_1(h)$ states that such evidence exists. The decisions for the compound problem $\Pi = \Pi_1 \times \cdots \times \Pi_k$ can be visualized by drawing lines above the nonoverlapping subsets of consecutive sample means for which no pair $(x_{i\cdot}x_{i+1\cdot})$ showed a significant difference. In the $k = 2$ case the 4 decisions are $d_{00} = \overline{x_{1\cdot}x_{2\cdot}x_{3\cdot}}$; $d_{10} = x_{1\cdot}\overline{x_{2\cdot}x_{3\cdot}}$; $d_{01} = \overline{x_{1\cdot}x_{2\cdot}}x_{3\cdot}$ and $d_{11} = x_{1\cdot}x_{2\cdot}x_{3\cdot}$. In Example 12.2 we shall argue that for many practical situations one will wish to extend the decision space by adding the decision $\overline{x_{1\cdot}\overline{x_{2\cdot}}x_{3\cdot}}$ which expresses that there exists sufficient evidence for rejecting $\theta \varepsilon \Omega_{11}$ or equivalently for rejecting $\mu_1 = \mu_3$ but that it remains undecided where the shift occurs. If we restrict the attention to the product problem Π and the case in which σ^2 *is known* (we assume $\sigma^2 = 1$) then the conditions of Theorem 11.2 are satisfied when $\varphi_1^{(h)}$ is the test rejecting $H: \theta \varepsilon \omega(h) = \Omega_1(h)$, if and only if $x_{h\cdot} - x_{h-1\cdot} \geqq u_{\alpha_h}(n_h^{-1} + n_{h-1}^{-1})^{\frac{1}{2}}$ and the U.M.R. (W_2) procedure δ^* for Problem Π is obtained by combining the results of the k component tests in the natural way. This result suggests that in case σ^2 *is unknown* $\varphi_1^{(h)}$ will be the Student-t test, rejecting if and only if

$$x_{h\cdot} - x_{h-1\cdot} \geqq t_{n-k;\alpha_h}(n_h^{-1} + n_{h-1}^{-1})^{\frac{1}{2}}s; \qquad s^2 = \textstyle\sum_i \sum_j (x_{ij} - x_{i\cdot})^2/(n-k).$$

Though the corresponding compound procedure will be a very attractive one, especially when $n = \sum n_i$ is large, we believe that $\varphi_1^{(h)}$ does not satisfy the conditions of Theorem 11.2 on account of the restrictions imposed upon Ω.

12. A Type I problem with $m = n = 4$ and two examples. In Remark 11.2 and Example 11.2 we argued that the formulation of Section 11 might constitute an oversimplification of the actual situation. In this section we try to remedy this for the special $k = 2$ case by introducing a fifth decision (see Example 11.2) which states that there exists sufficient evidence for "$\theta \not\varepsilon \Omega_{11}$" but that it is not clear from the observations whether the result is significant for Problem Π_1 or for Problem Π_2. As the double-index notation of Section 11 loses its sense we denote d_{00}, d_{10}, d_{01} and d_{11} respectively by d_0, d_2, d_3 and d_4 while Ω_{11}, Ω_{21}, Ω_{12}, Ω_{22} is changed to Ω_1, Ω_2, Ω_3 and Ω_4. The fifth decision will be denoted by d_1, thus expressing that its implications are situated somewhere between those of d_0 and the decisions d_2, d_3. We shall give arguments for the following loss function

(12.1)

	d_0 $(= d_{00})$	d_1	d_2 $(= d_{10})$	d_3 $(= d_{01})$	d_4 $(= d_{11})$
Ω_1 $(= \Omega_{11})$	0	b	$2b$	$2b$	$3b$
Ω_2 $(= \Omega_{21})$	$2a$	a	0	$a + b$	b
Ω_3 $(= \Omega_{12})$	$2a$	a	$a + b$	0	b
Ω_4 $(= \Omega_{22})$	$3a$	$2a$	a	a	0.

For that purpose we remark that there are three testing problems playing their part: Π_1, Π_2 and the problem where "$\theta \varepsilon \Omega_{11}$" has to be tested against

"$\theta \varepsilon \Omega - \Omega_{11}$." Thus $w_{12} = 2b$ expresses that for $\theta \varepsilon \Omega_{11}$ one makes two errors of the first kind when one decides upon d_2. We remark that (12.1) is not in agreement with (11.2) in case $k = 2$; $a_1 = a_2 = a$; $b_1 = b_2 = b$, for (11.2) implies, for example, that $w_{12} = w(\theta \varepsilon \Omega_{11}, d = d_{10}) = b$. This gives a new illustration of Remark 10.1: our notions of the errors of the first and second kind are not always compatible.

It is an interesting and urgent task (Theorem 3.1) to determine all minimax strategies for Player II in the game with matrix (12.1). Karlin [15] (Section 2.4) described a very beautiful general method for characterizing the extreme point optimal strategies in zero-sum matrix games. This method can be applied to (12.1) in order to find the set Y^0 of all minimax strategies $(p_0, \cdots, p_4)$. By considering all square submatrices of the payoff matrix (12.1) it turns out that Y^0 is the convex hull of the following strategies where $\alpha = a/(a + b)$

$$(12.2)\quad (1 - \alpha, 0, 0, 0, \alpha), \quad (1 - 3\alpha/2, 0, \alpha, \tfrac{1}{2}\alpha, 0), \quad (1 - 3\alpha/2, 0, \tfrac{1}{2}\alpha, \alpha, 0)$$
$$(1 - 2\alpha, \alpha, \alpha, 0, 0), \quad (1 - 2\alpha, \alpha, 0, \alpha, 0), \quad (1 - 3\alpha, 3\alpha, 0, 0, 0).$$

In the following lemma we shall give a similar characterization by using special features of (12.1).

LEMMA 12.1. *$(p_0, \cdots, p_4)$ is a minimax strategy if and only if there exist constants ϵ, δ and ρ such that*

$$(12.3)\quad p_0 = b/(a + b) - \epsilon; \quad p_1 = 2\epsilon - \delta; \quad p_2 = \delta - \tfrac{1}{2}\epsilon - \rho;$$
$$p_3 = \delta - \tfrac{1}{2}\epsilon + \rho; \quad p_4 = a/(a + b) - \delta$$

with

$$(12.4)\quad 0 \leqq \epsilon \leqq b/(a + b); 0 \leqq \delta \leqq a/(a + b); \quad \epsilon \leqq 2\delta;$$
$$\delta \leqq 2\epsilon; \quad |\rho| \leqq \min(\tfrac{1}{2}\epsilon, \delta - \tfrac{1}{2}\epsilon).$$

If (12.3) and (12.4) hold and $w = \sum_{j=0}^{4} p_j w_{\cdot j}$, then w is a minimax point of the convex set S and its coordinates are determined by

$$(12.5)\quad w = (\bar{w}^*, \bar{w}^* - (\tfrac{1}{2}\epsilon - \rho)(a + b), \bar{w}^* - (\tfrac{1}{2}\epsilon + \rho)(a + b), \bar{w}^*)$$

where $\bar{w}^* = 3ab/(a + b)$. The point $w^* = (\bar{w}^*, \cdots, \bar{w}^*)$ is the unique maximin point of S and has a unique convex representation determined by $\epsilon = \delta = \rho = 0$.

PROOF. The vectors $w_{\cdot 0}, \cdots, w_{\cdot 4}$ lie in the 3-dimensional hyperplane V defined by $ax_1 + bx_4 = 3ab$ in R^4. Consequently we have $\max(x_i) \geqq \bar{w}^*$ and $\min(x_i) \leqq \bar{w}^*$ for all $x = (x_1, \cdots, x_4) \varepsilon S \subset V$. But equality holds for $w^* \varepsilon S$. Hence w^* is both a minimax and a maximin point of S.

Next we characterize the class Y^0 of all minimax strategies for Player II. Obviously the conditions (12.4) are sufficient on account of (12.5). We must show that they are necessary. The first paragraph of this proof shows that x cannot be a minimax point of S unless $x_1 = x_4 = \bar{w}^*$. Hence if $(p_0, \cdots, p_4)$ is a minimax strategy for Player II, then

$$p_j \geqq 0 (j = 0, \cdots, 4); \qquad \sum p_j = 1 \tag{12.6}$$

$$bp_1 + 2bp_2 + 2bp_3 + 3bp_4 = \bar{w}^* \tag{12.7}$$

$$2ap_0 + ap_1 + (a + b)p_3 + bp_4 \leqq \bar{w}^* \tag{12.8}$$

$$2ap_0 + ap_1 + (a + b)p_2 + bp_4 \leqq \bar{w}^* \tag{12.9}$$

$$3ap_0 + 2ap_1 + ap_2 + ap_3 = \bar{w}^* \tag{12.10}$$

The equalities (12.7) and (12.10) show that p_0 and p_4 can be determined by (12.3) where ϵ and δ have to satisfy the first two inequalities of (12.4). Next (using (12.6)), substitute $p_2 + p_3 = 1 - p_0 - p_1 - p_4 = \epsilon + \delta - p_1$ into (12.7). We obtain $p_1 = 2\epsilon - \delta$ and δ must satisfy $\delta \leqq 2\epsilon$ (see (12.4) and (12.6)). Moreover $p_2 + p_3 = 2\delta - \epsilon$ and ϵ must satisfy $\epsilon \leqq 2\delta$. Next we can introduce ρ with $|\rho| \leqq \delta - \frac{1}{2}\epsilon$ and write p_2 and p_3 according to (12.3). The inequalities (12.8), (12.9) are satisfied if and only if $|\rho| \leqq \frac{1}{2}\epsilon$.

In order to show the last part of Lemma 12.1, let $w = \sum p_j w_{\cdot j}$ be a maximin point of S. The first paragraph of this proof shows that w cannot be a maximin point unless both the first and the last coordinate are equal to $\bar{w}^*$. Hence the equalites (12.6), (12.7) and (12.10) must hold and the discussion above holds apart from the last sentence. (12.8) and (12.9) must hold with reversed inequality signs. This is only possible if (see (12.5)) $\frac{1}{2}\epsilon - \rho \leqq 0$ and $\frac{1}{2}\epsilon + \rho \leqq 0$. Combining these inequalities we obtain $\epsilon \leqq 0$, but $\epsilon \geqq 0$. Hence $\epsilon = 0$ and $\delta = \rho = 0$ follows immediately.

In order to see that (12.2) and Lemma 12.1 determine the same set Y^0 of minimax strategies for Player II, we remark that both sets of strategies are in 1:1 correspondence with the convex hull of the points (0, 0, 0), $(\frac{1}{2}\alpha, \alpha, -\alpha/4)$, $(\frac{1}{2}\alpha, \alpha, \alpha/4)$, $(\alpha, \alpha, -\frac{1}{2}\alpha)$, $(\alpha, \alpha, \frac{1}{2}\alpha)$ and $(2\alpha, \alpha, 0)$ in the R^3 of points (ϵ, δ, ρ) and where of course $\alpha = a/(a + b)$.

Lemma 12.1 and Theorem 3.1 show that there does not exist a unique M.R. procedure if the problem is of Type I: there exist for example many constant procedures with M.R.

COROLLARY 12.1. *If W denotes the class of all unbiased procedures and M that of all* M.R. *procedures, then $W \subset M$.*

This corollary is a consequence of Lemma 12.1 and Theorem 4.1. The conditions of Lemma 4.2 are not satisfied, for w^* is not the unique minimax point of S (see Remark 4.2). We remark that Lehmann's unbiasedness condition is not attractive here (notwithstanding Corollary 12.1) because (i) *w^* is the least attractive minimax point of S*; each other minimax point has some coordinates smaller than w^* and will consequently have a more attractive risk function (at least in the neighborhood of Ω_0') (ii) the uniqueness of the convex representation $w^* = b/(a + b)w_{\cdot 0} + a/(a + b)w_{\cdot 4}$ (Lemma 12.1) shows that for each unbiased procedure $E_{\theta_0}\{\varphi_j(X)\} = 0$ for all $\theta_0 \in \Omega_0'$ and $j = 1, 2, 3$: *the decisions d_1, d_2 and d_3 will never occur if we apply an unbiased procedure.*

LEMMA 12.2. *If* $\Omega_1 = \Omega_0'$, $\Omega_i = [\Omega_i] \cap [\Omega_4] (i = 2, 3)$ *and for some* $\theta_0 \varepsilon \Omega_1$ *the condition* $E_{\theta_0}(\varphi) = 0$ *implies* $\varphi = 0$ a.e. (μ) *for each test function* φ, *then* $\delta(\varphi_0, \cdots, \varphi_4)$ *is unbiased if and only if* (i) $\varphi_1 = \varphi_2 = \varphi_3 = 0$ a.e. (μ), (ii) $E_\theta\{\varphi_0(X)\} = 1 - \alpha$ *and* $E_\theta\{\varphi_4(X)\} = \alpha$ *for all* $\theta \varepsilon \Omega - \Omega_4$ *and* (iii) $E_\theta\{\varphi_4(X)\} \geqq \alpha$ *for all* $\theta \varepsilon \Omega_4$, *where* $\alpha = a/(a + b)$.

PROOF. The necessity of (i) was shown above. $\theta \varepsilon \Omega_2$ is a limit point both of Ω_2 and Ω_4. Unbiasedness implies continuity of the risk function. Thus (12.1) proves the necessity of (ii). (iii) follows from Lemma 4.1. The sufficiency of (i), (ii) and (iii) follows by verifying the conditions of Lemma 4.1.

The conditions in Lemma 12.2 are very strong and one might conjecture that the trivial constant procedure with $\varphi_0 = \alpha$ and $\varphi_4 = 1 - \alpha$ for all $x \varepsilon \mathfrak{X}$, is the unique unbiased procedure. Remark 12.1 shows that this is not true in general.

EXAMPLE 12.1 (An m.d.p. for combining two one-sided tests of significance). Let X_1, X_2 have independent $N(\mu_i, 1)$ distributions with $\mu_i \geqq 0$ $(i = 1, 2)$. $\Omega = \Omega_1 \cup \cdots \cup \Omega_4$ where in the R^2 of points $\theta = (\mu_1, \mu_2)$ we have $\Omega_1 = \{(0, 0)\}$, Ω_i is the positive μ_{i-1}-axis $(i = 2, 3)$ and Ω_4 is the open positive orthant. Decision d_0 expresses that the evidence for $(\mu_1, \mu_2) \neq (0, 0)$ is not sufficient, d_1 expresses that $(\mu_1, \mu_2) \neq (0, 0)$ but that no sufficient evidence exists for deciding whether μ_1, μ_2 or both are positive, d_i states that $\mu_{i-1} > 0$ but whether the other mean is also positive remains undecided $(i = 2, 3)$ and d_4 states that both means are positive. The preceding discussions can be applied to this example. Lemma 12.2 shows that each unbiased procedure δ satisfies $R(\theta, \delta) = 3ab/(a + b) = \bar{w}^*$ for all $\theta \varepsilon \Omega - \Omega_4$ whereas of course $R(\theta, \delta) \leqq \bar{w}^*$ for all $\theta \varepsilon \Omega_4$ but in such a way that $R(\theta, \delta) \to \bar{w}^*$ if $\theta \varepsilon \Omega_4$ tends to a point on one of the two nonnegative axes (for the unbiasedness of δ implies the continuity of its risk function). We obtain new arguments for rejecting the unbiasedness restriction in this section, by constructing *a special* M.R. *procedure* $\delta' = \delta(\varphi_0', \cdots, \varphi_4')$ *whose risk function is not continuous but much more attractive than that of any unbiased procedure.* For that purpose suppose

$$\alpha' = 2 - \{4 - 3a/(a + b)\}^{\frac{1}{2}} \tag{12.11}$$

and define $\varphi_0'(x_1, x_2) = 1$ when $x_1, x_2 < u_{\alpha'}$; $\varphi_1'(x_1, x_2) = 0$ for all (x_1, x_2); $\varphi_i'(x_1, x_2) = 1$ when $x_{i-1} \geqq u_{\alpha'}$, $x_{4-i} < u_{\alpha'} (i = 2, 3)$ and $\varphi_4'(x_1, x_2) = 1$ when $x_1, x_2 \geqq u_{\alpha'}$. With respect to the corresponding risk function $R(\theta, \delta')$ we obviously have $R(\theta, \delta') = \bar{w}^*$ if $\theta \varepsilon \Omega_1$,

$$R(\theta, \delta') = \alpha' b + (2 - \alpha')a\Phi(u_{\alpha'} - \kappa) \tag{12.12}$$

if $\theta = (\kappa, 0) \varepsilon \Omega_2$ or $\theta = (0, \kappa) \varepsilon \Omega_3 (\kappa > 0)$ and

$$R(\theta, \delta') = a\{1 + \Phi(u_{\alpha'} - \kappa)\}\{1 + \Phi(u_{\alpha'} - \nu)\} - a \tag{12.13}$$

if $\theta = (\kappa, \nu) \varepsilon \Omega_4 (\kappa, \nu > 0)$. One easily verifies that $R(\theta, \delta') \leqq \bar{w}^*$ for all $\theta \varepsilon \Omega$ and consequently δ' has minimax risk. Moreover it follows from (12.12) and (12.13) that δ' has an appropriate risk function with for example (12.13) $\to 0$ as $\kappa, \nu \to \infty$; (12.12) $\to \alpha' b < \bar{w}^*$ as $\kappa \to \infty$; (12.13) $\to a(1 - \alpha') < \bar{w}^*$ as $(\kappa, \nu) \varepsilon \Omega_4$ with

$\kappa \rightarrow \infty$ and $\nu \rightarrow 0$. The discontinuity of the risk function follows from the last two limits. We remark that it is quite natural that procedures have discontinuous risk functions for m.d.p.'s of Type I. The unbiasedness condition implies continuity of the risk function *everywhere*. In our special case each unbiased procedure δ was shown to satisfy $R(\theta, \delta) = \bar{w}^*$ for all $\theta \varepsilon \Omega - \Omega_4$ with as a result that δ' is obviously a much better procedure. Of course δ' is not necessarily the best M.R. procedure. We studied δ' because of the simplicity of its risk function. In our opinion the most beautiful appropriate criterion for the problem of this section is the *minimax regret* (M) *criterion* where M is the class of all M.R. procedures. Unfortunately the construction of minimax regret (M) procedures goes beyond our abilities.

REMARK 12.1. (An example of A. I. van de Vooren of a nonconstant unbiased procedure). The function

$$\varphi_4(x_1, x_2) = \alpha + \epsilon\{1 - e \cos (2^{\frac{1}{2}}x_1)\}\{1 - e \cos (2^{\frac{1}{2}}x_2)\}$$

is a test function provided that ϵ is sufficiently small. Moreover

$$E_\theta\{\varphi_4(X_1, X_2)\} = \alpha + \epsilon\{1 - \cos (2^{\frac{1}{2}}\kappa)\}\{1 - \cos (2^{\frac{1}{2}}\nu)\}$$

in $\theta = (\kappa, \nu) \varepsilon \Omega$. Lemma 12.2 shows that $\delta(1 - \varphi_4, 0, 0, 0, \varphi_4)$ is an example of a nonconstant unbiased procedure. The proof is based on the following equality

$$\int_{-\infty}^{\infty} \exp \{-(x - y)^2\} \cos 2x \, dx = e^{-1}\pi^{\frac{1}{2}} \cos 2y.$$

The problem to construct a nonrandomized (indicator-) function φ_4 satisfying the conditions in Lemma 12.2 (or to show the nonexistence of such a function) has not yet been solved.[3]

EXAMPLE 12.2 (A multiple decision three-sample trend problem). Consider $k = 2$ in Example 11.2; $\Omega = \Omega_1 \cup \cdots \cup \Omega_4$ where Ω_1 is the subset of Ω defined by $\mu_0 = \mu_1 = \mu_2$; Ω_2 is defined by $\mu_0 < \mu_1 = \mu_2$; Ω_3 by $\mu_0 = \mu_1 < \mu_2$ and Ω_4 by $\mu_0 < \mu_1 < \mu_2$. Decision d_0 $(= \overline{x_{0\cdot}x_{1\cdot}x_{2\cdot}})$ states that there does not exist sufficient evidence for $\theta \not\varepsilon \Omega_1$; d_1 $(= \overline{\overline{x_{0\cdot}x_{1\cdot}}x_{2\cdot}})$ states that the hypothesis of homogeneity $\mu_0 = \mu_1 = \mu_2$ has to be rejected but it remains undecided whether $\mu_0 < \mu_1$ or $\mu_1 < \mu_2$ or both; d_2 $(= x_{0\cdot}\overline{x_{1\cdot}x_{2\cdot}})$ states that there is sufficient evidence for $\mu_0 < \mu_1$ but not for $\mu_1 < \mu_2$; d_3 $(= \overline{x_{0\cdot}x_{1\cdot}}x_{2\cdot})$ is formulated when $x_{2\cdot}$ is significantly larger than $x_{1\cdot}$ whereas the evidence for $\mu_0 < \mu_1$ is not sufficient and d_4 $(= x_{0\cdot}x_{1\cdot}x_{2\cdot})$ corresponds with the statement "$\theta \varepsilon \Omega_4$" or "$\mu_0 < \mu_1 < \mu_2$."

This (and other) problem(s) may be regarded from various points of view: (i) as a problem of *multiple comparisons* where at the same time $\mu_0 = \mu_1$ has to be tested against $\mu_0 < \mu_1$; $\mu_1 = \mu_2$ against $\mu_1 < \mu_2$ and $\mu_0 = \mu_2$ against $\mu_0 < \mu_2$; (ii) as a problem where two decision problems are considered *consecutively*, first the problem to test $\mu_0 = \mu_1 = \mu_2$ against an upward trend (see [27]) and next, if the hypothesis of homogeneity is rejected, the problem to test at the same time (see Section 11) $\mu_0 = \mu_1$ against $\mu_0 < \mu_1$ and $\mu_1 = \mu_2$ against $\mu_1 < \mu_2$; (iii) as a

[3] In the meantime, Mrs. W. Stefansky, Department of Statistics, Berkeley, constructed such indicator functions φ_4 in the case α is rational.

restricted product (in a more general sense than in [19] because there are more decisions than subsets of Ω) of the three problems under (ii). In all these cases (12.1) seems to be an attractive loss function. The fact that the problem can be regarded from different points of view entails that there does not exist a "natural" procedure: a procedure which is natural from one point of view is not natural from another one. Accordingly the problem to find an "optimum" procedure (the minimax regret (M) procedure) seems to be forbidding.

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