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THE LOOSE SUBORDINATION OF DIFFERENTIAL PROCESSES TO BROWNIAN MOTION[1]

By Barthel W. Huff

Arizona State University

1. Introduction. Our terminology is, in general, that of [6].

A *differential process* $\{X(t)/t \varepsilon [0, \infty)\}$ is a stochastic process with stationary, independent increments that is continuous in law and satisfies the initial condition $P[X(0) = 0] = 1$. We shall assume that our processes are separable and have sample paths that are almost surely right-continuous. A *random time* $\{Y(T)\}$ is a nonnegative differential process with sample paths that are almost surely nondecreasing.

Every differential process is an infinitely divisible process. That is the characteristic functions are of the form

$$f_{X(t)}(u) = \exp\{t\psi_X(u)\}, \tag{1.1}$$

where $\psi_X(u) = i\gamma_X u - \sigma_X^2 u^2/2 + \int_{-\infty}^{\infty} (e^{iux} - 1 - iux/(1 + x^2))\, dM_X(x) \cdot \gamma_X$, σ_X^2, and M_X are the Lévy parameters uniquely associated with the infinitely divisible random variable $X(1)$. The Lévy spectral function M_X is nondecreasing on $(-\infty, 0)$ and on $(0, \infty)$, is asymptotically zero $(M_X(-\infty) = 0 = M_X(+\infty))$, and satisfies the integrability condition

$$\int_{-1}^{-0} + \int_{+0}^{+1} x^2\, dM_X(x) < \infty.$$

The Lévy spectral function for a random time vanishes on the negative half-axis and satisfies the stronger integrability condition

$$\int_{+0}^{+1} x\, dM_Y(x) < \infty.$$

Consequently, the characteristic functions for a random time can be written in the form

$$f_{Y(T)}(u) = \exp\{T(i\gamma_Y u + \int_0^{\infty} (e^{iux} - 1)\, dM_Y(x))\}, \tag{1.2}$$

where $\gamma_Y \geqq 0$ is the trend term of the random time.

A standard Brownian motion $\{W(t)\}$ is a separable differential process with sample paths that are almost surely continuous and such that $\mathfrak{L}(W(t)) = \mathfrak{N}(0, t)$. Any random time $\{Y(T)\}$ independent of the standard Brownian motion is *loosely subordinate* in the sense that there exists a random time (*loose sub-*

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ordinator) $\{S(T)\}$ such that $W(S(T)) = Y(T)$ almost surely. In particular, $S(T) = \tau(Y(T)) = \inf\{t/W(t) = Y(T)\}$ would be such a loose subordinator. The random variable $\tau(Y(T))$ is the first time of hitting the random variable $Y(T)$. We would like to extend this concept of loose subordination to more general differential processes than the random times.

The following result due to A. V. Skorokhod (page 163 of [5]) seemed very suggestive. Suppose $X_1, \cdots, X_n$ are independent random variables (satisfying certain conditions) that are independent of $\{W(t)\}$. Then there exist independent nonnegative random variables $\tau_1, \cdots, \tau_n$ such that

$$W(\tau_1), W(\tau_1 + \tau_2) - W(\tau_1), \cdots, W(\textstyle\sum_{k=1}^{n} \tau_k) - W(\sum_{k=1}^{n-1} \tau_k)$$

are independent and

$$\mathcal{L}(W(\textstyle\sum_{k=1}^{j} \tau_k) - W(\sum_{k=1}^{j-1} \tau_k)) = \mathcal{L}(X_j), \qquad j = 1, \cdots, n.$$

One might feel that if the X_k are increments of a differential process $\{X(T)\}$, the τ_k behave in the manner that would be expected of the corresponding increments of a loose subordinator. If our intuition was correct in this instance, we could use the Daniel-Kolmogorov Theorem (as in Section 7.4 of [6], where the existence of a Brownian motion is proved) to generate a random time equivalent to the loose subordinator.

Unfortunately, neither the Skorokhod variables nor the first hitting times satisfy the necessary consistency requirements. In particular, if $\{S(T)\}$ is to be a nonnegative process with stationary, independent increments it should be true that

$$P[S(2T) = 0] = P[S(T) = 0, S(2T) - S(T) = 0] = P[S(T) = 0]^2.$$

But suppose $\{X(T)\}$ is a symmetric Poisson process; i.e., $X(T)$ has distribution $\mathcal{P}_1(T) - \mathcal{P}_2(T)$, where $\mathcal{P}_1(T)$ and $\mathcal{P}_2(T)$ are independent Poisson random variables with parameter T. Then if $\tau(X(T))$ is either the Skorokhod variable or the first hitting time corresponding to the singleton $X(T)$

$$\begin{aligned} P[\tau(X(T)) = 0] &= P[X(T) = 0] \\ &= e^{-2T}(1 + \textstyle\sum_{n=1}^{\infty} T^{2n}/(n!)^2). \end{aligned}$$

We see that

$$P[\tau(X(T)) = 0]^2 < e^{-4T} \cdot e^{2T^2}$$

and

$$P[\tau(X(2T)) = 0] > e^{-4T}(1 + 4T^2 + 4T^4).$$

Thus for $T = 1$, $P[\tau(X(T)) = 0]^2 < P[\tau(X(2T)) = 0]$.

The problem seems to be that the Skorokhod variables and first hitting times do not in general reflect the behavior of the $\{X(T)\}$ sample paths. We shall develop a loose subordination which is valid for exactly those differential processes whose sample paths are almost surely of bounded variation over [0, 1] (over every bounded interval). We shall also obtain the Lévy spectral function of the loose subordinator in terms of the spectral function of the $\{X(T)\}$ process and the trend term of the total variation of the $\{X(T)\}$ process. We conclude by considering the case when $\{X(T)\}$ is symmetric stable with characteristic exponent $\alpha \varepsilon (0, 1)$ and showing that loose subordination does not correspond to subordination in the sense of Bochner.

2. First hitting times and loose subordination. If X is a random variable independent of $\{W(t)\}$, we define the *first hitting time* via

$$\tau(X) = \inf\{t/W(t) = X\}.$$

Theorem 1 and its corollaries are known but we include them for completeness.

THEOREM 1. *For* $\lambda > 0$, *the first hitting time has distribution*

$$\text{(A)} \quad F_{\tau(X)}(\lambda) = \int_{-\infty}^{\infty} (2/(2\pi\lambda)^{\frac{1}{2}}) \int_{|x|}^{\infty} \exp\{-y^2/2\lambda\}\, dy\, dF_X(x) = F_{\tau(|X|)}(\lambda).$$

PROOF. Set $X_m = \sum_{k=0}^{\infty} k/2^m \cdot I[k/2^m \leqq X < (k+1)/2^m] + \sum_{k=0}^{-\infty} k/2^m \cdot I[(k-1)/2^m < X \leqq k/2^m]$. Then $X_m^+ \uparrow X^+$ and $-X_m^- \downarrow -X^-$. The continuity of the Brownian sample paths then implies that $\tau(X_m) \to \tau(X)$ and $\tau(|X_m|) \to \tau(|X|)$ almost surely. Thus $F_{\tau(X_m)} \to_c F_{\tau(X)}$ and $F_{\tau(|X_m|)} \to_c F_{\tau(|X|)}$. Applying the reflection principle of Désiré André (Theorem 2 of Section 8.3 in [6]), we see that for $x \geqq 0$,

$$P[\tau(x) \leqq \lambda] = P[\sup_{[0,\lambda]} W(t) \geqq x] = (2/(2\pi\lambda)^{\frac{1}{2}}) \cdot \int_x^{\infty} \exp\{-y^2/2\lambda\}\, dy.$$

The symmetry of the Brownian motion yields the same value for $P[\tau(-x) \leqq \lambda]$. Thus, since X is independent of $\{W(t)\}$, $F_{\tau(X_m)}(\lambda) = \sum_{k=-\infty}^{\infty} P[\tau(k/2^m) \leqq \lambda] \cdot P[X_m = k/2^m] = F_{\tau(|X_m|)}(\lambda)$. We conclude by noting that

$$\begin{aligned} F_{\tau(X_m)}(\lambda) &= \int_{-\infty}^{\infty} P[\tau(x) \leqq \lambda]\, dF_{X_m}(x) \\ &= \int_{-\infty}^{\infty} (2/(2\pi\lambda)^{\frac{1}{2}}) \int_{|x|}^{\infty} \exp\{-y^2/2\lambda\}\, dy\, dF_{X_m}(x) \end{aligned}$$

and applying the Helly-Bray Theorem. □

Applying the Helly-Bray Theorem to (A), we obtain

COROLLARY 1A. *Suppose* $\langle X_n \rangle$ *and* X *are random variables independent of* $\{W(t)\}$ *and* $X_n \to_{\mathcal{L}} X$. *Then* $\tau(X_n) \to_{\mathcal{L}} \tau(X)$.

COROLLARY 1B. *For* $\lambda > 0$

$$\text{(B)} \qquad \frac{dF_{\tau(X)}(\lambda)}{d\lambda} = \int_{-\infty}^{\infty} (|x|/(2\pi\lambda^3)^{\frac{1}{2}}) \exp\{-x^2/2\lambda\}\, dF_X(x).$$

PROOF. The Lebesgue Dominated Convergence Theorem allows us to take derivatives inside integrals in the following argument. From (A) we see that

$$\frac{dF_{\tau(X)}(\lambda)}{d\lambda} = \int_{-\infty}^{\infty} \frac{d}{d\lambda}\left[(2/(2\pi\lambda)^{\frac{1}{2}}) \int_{|x|}^{\infty} \exp\{-y^2/2\lambda\}\, dy\right] dF_X(x)$$
$$= \int_{\infty}^{\infty} \left[(-1/(2\pi\lambda^3)^{\frac{1}{2}}) \int_{|x|}^{\infty} \exp\{-y^2/2\lambda\}\, dy \right.$$
$$\left. + (1/(2\pi\lambda^5)^{\frac{1}{2}}) \int_{|x|}^{\infty} y^2 \exp\{-y^2/2\lambda\}\, dy\right] dF_X(x).$$

But integrating by parts, we obtain

$$(1/(2\pi\lambda^5)^{\frac{1}{2}}) \textstyle\int_{|x|}^{\infty} y^2 \exp\{-y^2/2\lambda\}\, dy$$
$$= (1/(2\pi\lambda^3)^{\frac{1}{2}})(-y \exp\{-y^2/2\lambda\}|_{|x|}^{\infty} + \textstyle\int_{|x|}^{\infty} \exp\{-y^2/2\lambda\} dy$$
$$= (1/(2\pi\lambda^3)^{\frac{1}{2}})(|x| \exp\{-x^2/2\lambda\} + \textstyle\int_{|x|}^{\infty} \exp\{-y^2/2\lambda\}\, dy). \ \square$$

This does not imply that $F_{\tau(X)}$ is absolutely continuous since it will take a jump of size $P[X = 0]$ at the origin.

We now note that if $\{X(t)\}$ is a random time, then $\{\tau(X(t))\}$ has no trend term. This follows from the Markov property and the observations that (see pages 25–27 of [3])

$$f_{\tau(x)}(u) = \exp\{(2\pi)^{-\frac{1}{2}} |x| \textstyle\int_0^{\infty} (e^{iuy} - 1) y^{-\frac{3}{2}}\, dy\} \tag{2.1}$$

has no trend term and that the first times of hitting a jump process will form a jump process.

Now let $\{X(T)\}$ be a differential process, independent of $\{W(t)\}$, and fix $T \geqq 0$. Then for any positive integer n, there exists a nonnegative integer k_n such that $k_n/2^n \leqq T < (k_n + 1)/2^n$. Setting $\tau_n^0 \equiv 0$ and applying the strong Markov property (see Lemma 2, page 166 in [5] and [2]), we obtain independent, identically distributed random variables

$$\tau_n^i(T) = \inf\{t/W(t + \textstyle\sum_{k=1}^{i-1} \tau_n^k(T)) = X(i/2^n)\}, \qquad i = 1, \cdots, k_n$$

such that $W(\sum_{i=1}^{k_n} \tau_n^i(T)) = X(k_n/2^n)$ almost surely. It is obvious from the construction that $\sum_{i=1}^{k_n} \tau_n^i(T) = \tau_n(T) \uparrow$. Indeed,

$$\tau_n(T) = \inf\{t/W(t_i) = X(i/2^n), W(t) = X(k_n/2^n); 0 \leqq t_1 \leqq \cdots \leqq t_{k_n-1} \leqq t\}.$$

If $S(T) = \lim_{n\to\infty} \tau_n(T)$ is almost surely finite for some $T > 0$, we have achieved a loose subordination of the $\{X(T)\}$ process. Since random time processes have sample paths that are almost surely nondecreasing, $S(T) = \tau(X(T))$ almost surely when $\{X(T)\}$ is itself a random time.

THEOREM 2. *Suppose there exists $T_0 > 0$ such that $S(T_0)$ is almost surely finite. Then $W(S(T)) = X(T)$ almost surely and $\{S(T)\}$ is a random time process.*

PROOF. Assume $T_0 = 1$. Then $T \leqq 1$ implies $\tau_n(T) \leqq \tau_n(1)$ and thus $S(T)$ is almost surely finite. If k is some positive integer $f_{\tau_n(k)}(u) = (f_{\tau_n(1)}(u))^k$. Thus $\tau_n(k)$ converges in law and $S(k)$ is almost surely finite.

$W(\tau_n(T)) \to W(S(T))$ almost surely via the continuity of the Brownian sample paths. At the same time

$$X(k_n/2^n) = \sum_{j=1}^{n} [X(k_j/2^j) - X(k_{j-1}/2^{j-1})] \to_{\mathcal{L}} X(T).$$

Applying Theorem 1 of Section 5.2 in [6], we see that $X(k_n/2^n) \to X(T)$ almost surely. Thus $W(S(T)) = X(T)$ with probability one.

Now suppose that $0 < T_1 < T_2$. Then $\tau_n(T_2) - \tau_n(T_1)$ and $\tau_n(T_1)$ are independent by construction. Since the $\tau_n^{\,i}$ are independent and identically distributed, we see that $\tau_n(T_2) - \tau_n(T_1)$ has the same distribution as one of $\tau_n(T_2 - T_1)$, $\tau_n(T_2 - T_1) - \tau_n^{\,1}$, or $\tau_n(T_2 - T_1) + \tau_1^*$, where r_1^* is another independent copy of $\tau_n^{\,1}$. Theorem 1 implies that $\tau_n^{\,1} \to_{\mathcal{L}} 0$ and the obvious limit argument completes the proof that $\{S(T)\}$ has stationary, independent increments. □

THEOREM 3. *Let $B(T)$ be the total variation of the sample path $X(t)$ over $[0, T]$. Then $\mathcal{L}(S(T)) = \mathcal{L}(\tau(B(T)))$. That is, the loose subordinator has the same distribution as the first time of hitting the total variation. Thus $S(T)$ is almost surely finite if and only if $B(T)$ is almost surely finite.*

PROOF. For convenience we assume $T = 1$. Then

$$B_n(1) = \sum_{i=1}^{2^n} |X(i/2^n) - X((i-1)/2^n)| \uparrow B(1),$$

implying that $\tau(B_n(1)) \to \tau(B(1))$ almost surely. The random variables $|X(1/2^n)|, \cdots, |X(1) - X((2^n - 1)/2^n)|$ are independent and independent of $\{W(t)\}$. The strong Markov property allows us to represent $\tau(B_n(1))$ as a sum of independent, identically distributed random variables

$$\tau(B_n(1)) = \sum_{i=1}^{2^n} {}^*\tau_n^{\,i}, \qquad \text{where} \quad {}^*\tau_n^{\,0} \equiv 0 \quad \text{and}$$

$$ {}^*\tau_n^{\,i} = \inf\{t/W(t + \textstyle\sum_{k=1}^{i-1} {}^*\tau_n^{\,k}) = \sum_{k=1}^{i} |X(k/2^n) - X((k-1)/2^n)|\}.$$

From Theorem 1, ${}^*\tau_n^{\,1} = \tau(|X(1/2^n)|)$ and $\tau_n^{\,1} = \tau(X(1/2^n))$ have the same distribution. Thus $\tau(B_n(1))$ and $\tau_n(1)$ have the same distribution. Convergence in the extended-real sense and convergence in law imply that the limit is almost surely finite. Thus $S(1)$ is almost surely finite if and only if $\tau(B(1))$ is almost surely finite. Brownian sample path properties complete the proof. □

THEOREM 4. *Let $\{X(t)\}$ be a differential process whose sample paths are almost surely of bounded variation over* [0, 1] *and such that*

$$f_{B(t)}(u) = \exp\{t(iu\gamma + \textstyle\int_0^\infty (e^{iux} - 1)\, dM_B(x)\}, \qquad \gamma \geqq 0.$$

Then

$$f_{S(T)}(u) = \exp\{T \textstyle\int_0^\infty (e^{iux} - 1)\, dM_S(x)\},$$

where the last Lévy spectral function is given at points of continuity by

$$M_S(x) = -(2\gamma(2\pi x)^{-\frac{1}{2}} + \textstyle\int_x^\infty \int_{-\infty}^\infty |t|\,(2\pi y^3)^{-\frac{1}{2}}(\exp\{-t^2/2y\}\, dM_X(t)\, dy).$$

PROOF. We first assume that $\gamma = 0$ and remark that $M_B(x) = M_X(x) -$

$M_X(-x)$ except for at most countably many points. Applying Theorem 3 of Section 6.5 in [6] and Theorem 3 of this paper, we obtain

$$M_S(x) = -\int_x^\infty dM_S(y) = -\lim_{n\to\infty} 2^n \int_x^\infty dF_{S(1/2^n)}(y)$$
$$= -\lim_{n\to\infty} 2^n \int_x^\infty dF_{T(B(1/2^n))}(y).$$

Corollary 1B implies that the last expression equals

$$-\lim_{n\to\infty} 2^n \int_x^\infty \int_0^\infty |t| (2\pi y^3)^{-\frac{1}{2}} \exp\{-t^2/2y\}\, dF_{B(1/2^n)}(t)\, dy.$$

The Helly-Bray Theorem and the integrability condition $\int_0^1 t\, dM_B(t) < \infty$ imply that this converges to

$$-\int_x^\infty \int_0^\infty |t| (2\pi y^3)^{-\frac{1}{2}} \exp\{-t^2/2y\}\, dM_B(t)\, dy$$
$$= -\int_x^\infty \int_{-\infty}^\infty |t| (2\pi y^3)^{-\frac{1}{2}} \exp\{-t^2/2y\}\, dM_X(t)\, dy.$$

If $\gamma \neq 0$, the Markov property and observation (2.1) complete our proof. □

Interchanging the order of integration and setting $y = t^2x/\lambda^2$, we obtain the alternate form

$$M_S(x) = -2(2\pi x)^{-\frac{1}{2}}(\gamma + \int_{-\infty}^\infty \int_0^{|t|} \exp\{-\lambda^2/2x\}\, d\lambda\, dM_X(t)).$$

Let us consider an example. Suppose $\{X(t)\}$ is a symmetric stable process with $f_{X(t)}(u) = \exp\{-t|u|^\alpha\}$, where $\alpha \,\varepsilon\, (0, 1)$. It is well known that these are precisely the symmetric stable processes with sample paths of bounded variation. Moreover, $\{B(t)\}$ would have no trend term. The Lévy spectral function of the stable process is given (see page 330 of [4] and integrate by parts) by

$$M_X(t) = k/|t|^\alpha, \qquad t < 0;$$
$$= -k/t^\alpha, \qquad t > 0,$$

where $k = 1/2\Gamma(1-\alpha)\cos(\pi\alpha/2)$. Using the alternate form in Theorem 4, we obtain

$$M_S(x) = -4k\alpha(2\pi x)^{-\frac{1}{2}} \int_0^\infty \int_0^t \exp\{-\lambda^2/2x\} t^{-(\alpha+1)}\, d\lambda\, dt$$
$$= -4k\alpha(2\pi x)^{-\frac{1}{2}} \int_0^\infty \int_\lambda^\infty (1/t^{\alpha+1}) \exp\{-\lambda^2/2x\}\, dt\, d\lambda$$
$$= -4k\alpha(2\pi x)^{-\frac{1}{2}} \int_0^\infty (1/\lambda^\alpha) \exp\{-\lambda^2/2x\}\, d\lambda.$$

If we set $\lambda^2/2x = y$, the last expression becomes

$$-2k(\pi 2^\alpha)^{-\frac{1}{2}} x^{-\alpha/2} \int_0^\infty y^{-(\alpha+1)/2} e^{-y}\, dy = \frac{-2k\Gamma(\frac{1}{2}(1-\alpha))}{(\pi 2^\alpha)^{\frac{1}{2}}} \cdot \frac{1}{x^{\alpha/2}}.$$

Thus the loose subordinator is one-sided stable with characteristic exponent $\alpha/2$.

If a random time $\{Y(T)\}$ is independent of the standard Brownian motion $\{W(t)\}$, then the superposition $\{X(T) = W(Y(T))\}$ is again a differential process. $\{X(T)\}$ is subordinate to the Brownian motion in the sense of Bochner and $\{Y(T)\}$ is the corresponding subordinator. Since the symmetric stable process with characteristic exponent α has a subordinator which is one-sided stable with characteristic exponent $\alpha/2$, we might suspect that our loose subordination has simply recaptured the Bochner subordination.

However, it is well known that the characteristic function of the superposition is given by the Laplace-Stieltjes transformation

$$f_{W(Y(T))}(u) = \int_{-\infty}^{\infty} e^{-u^2y/2} dFY(T)^{(y)}.$$

Thus if the Bochner subordinator corresponding to the symmetric stable process with characteristic exponent α has Lévy spectral function $M_Y(x) = -c/x^{\alpha/2}$, we may calculate that

$$\begin{aligned} e^{-|u|^\alpha} &= \int_0^\infty \exp\,(-u^2y/2)\, dF_{Y(1)}(y) \\ &= \exp\,\{c \int_0^\infty [\exp\,(-u^2x/2) - 1]\, \alpha/2x^{-(\alpha/2+1)}\, dx \\ &= \exp\,\{-c \int_0^\infty u^2 \exp\,(-u^2x/2)(2x^{\alpha/2})^{-1}\, dx\}. \end{aligned}$$

If we set $u^2x/2 = t$, the last expression becomes

$$\exp\left\{-c \int_0^\infty \frac{e^{-t}}{(2t/u^2)^{\alpha/2}}\, dt\right\} = \exp\left\{\frac{-c\,|u|^\alpha\Gamma(1-\alpha/2)}{2^{\alpha/2}}\right\}.$$

Thus $c = 2^{\alpha/2}/\Gamma(1-\alpha/2)$ and $M_Y(x) = -2^{\alpha/2}x^{-\alpha/2}/\Gamma(1-\alpha/2)$. Comparing the Bochner and loose subordinators, we obtain

$$\frac{M_Y(x)}{M_S(x)} = \frac{\pi^{\frac{1}{2}}2^\alpha\Gamma(1-\alpha)\cos\,(\pi\alpha/2)}{\Gamma(1-\alpha/2)\Gamma(\frac{1}{2}(1-\alpha))} = \cos\,(\pi\alpha/2)$$

via Legendre's relation (page 24 of [1]). We conclude that the two subordinators differ in scale and that loose subordination is apparently a new concept.

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