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A NOTE ON EXCHANGEABLE PROCESSES WITH STATES OF FINITE RANK

By S. W. Dharmadhikari

Indian Statistical Institute

1. Introduction. Let $\{Y_n, n \geqq 1\}$ be a stationary process with state-space $J = \{1, \cdots, D\}$. States of J will be denoted by ∂ and finite sequences of states of J will be denoted by s or t. If $s = (\partial_1, \cdots, \partial_n)$ then we write $p(s) = P[(Y_1, \cdots, Y_n) = s]$. The *rank* $n(\partial)$ of ∂ is defined to be the largest integer n such that we can find $2n$ sequences $s_1, \cdots, s_n, t_1, \cdots, t_n$ such that the $n \times n$ matrix $\|p(s_i \partial t_j)\|$ is non-singular. Gilbert [4] conjectured that if $n(\partial) < \infty$ for all $\partial \varepsilon J$, then $\{Y_n\}$ is a function of a finite stationary Markov chain. This conjecture was disproved by Heller [6] and by Fox and Rubin [3]. The purpose of this note is to prove that, in the special case when $\{Y_n\}$ is exchangeable, Gilbert's conjecture is correct.

2. Main results. Assume that $\{Y_n\}$ is exchangeable. Let Q denote the space of all probability distributions on J. If U denotes the unit interval [0, 1], then Q is just the subset of U^D corresponding to those vectors $q = (q(1), \cdots, q(D))$ satisfying $q(\partial) \geqq 0$ for all ∂ and $\sum_\partial q(\partial) = 1$. By de Finetti's theorem there is a unique probability measure μ on the σ-algebra of Borel subsets of Q such that if $s = (\partial_1, \cdots, \partial_n)$, then

$$p(\partial) = \int_Q q(\partial_1) \cdots q(\partial_n)\, d\mu(q). \tag{1}$$

Define the projection π_∂ by $\pi_\partial(q) = q(\partial)$, $q \varepsilon Q$. The probability measure μ and the function π_∂ induce a measure μ_∂ on the Borel σ-algebra in U.

Lemma. *If $n(\partial) < \infty$ then μ_∂ has finite support.*

Proof. Let ∂^n denote the sequence having n ∂'s in succession. From (1) and from Theorem C on page 163 of Halmos [5], it follows that

$$p(\partial^n) = \int_Q [q(\partial)]^n\, d\mu(q) = \int_U u^n\, d\mu_\partial(u), \qquad n = 1, 2, \cdots.$$

Thus $p(\partial^n)$ is just the nth moment of μ_∂. Now section 12.6 of Cramér [1] shows that, if we take $s_i = \partial^{i+1}$ and $t_j = \partial^j$, then the $(n \times n)$ matrix $\|p(s_i \partial t_j)\| = \|p(\partial^{i+j+2})\|$ is non-singular for every n whenever the support of μ_∂ is infinite. Thus $n(\partial) = \infty$ whenever μ_∂ has infinite support. This proves the lemma.

Suppose that $n(\partial) < \infty$ for all $\partial \varepsilon J$. Then the lemma shows that each μ_∂ has finite support. Since J is finite, μ itself has finite support. The remark at the end of [2] now shows that $\{Y_n\}$ is a function of a finite stationary Markov chain. We thus have the

Theorem. *A finite-state exchangeable process having all its states of finite rank is a function of a finite stationary Markov chain.*

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