ASYMPTOTIC NORMALITY OF LINEAR COMBINATIONS OF FUNCTIONS OF ORDER STATISTICS

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1. Introduction. Let $X_1, \ldots, X_N$ be i.i.d. uniform $(0, 1)$ r.v.’s defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $F_N$ denote the empirical df, and let $X_{N_1} \leq \cdots \leq X_{N_N}$ denote the ordered $X_1, \ldots, X_N$. We wish to consider statistics of the form

$$T_N = N^{-1} \sum_{i=1}^{N} c_{Ni} g(X_{Ni})$$

where $g$ is a specified function and $\{c_{Ni}: 1 \leq i \leq N, N \geq 1\}$ is a set of specified constants.

Remark. We may suppose the $X_i$’s to have an arbitrary continuous df $F$ provided we replace $g$ by $g(F^{-1})$.

We define inverses of df’s to be left continuous; thus

$$F_N^{-1}(t) = \inf \{x: F_N(x) \geq t\};$$

and we write $g \circ F_N^{-1}$ for the composed function $g(F_N^{-1})$. Note that $T_N = \int g \circ F_N^{-1} d\nu_N$ when the signed measure $\nu_N$ puts mass $c_{Ni}/N$ at $i/N$ for $i = 1, \cdots, N$ and puts 0 mass elsewhere. Let $\nu$ denote a signed measure on $(0, 1)$. (The signed measures $\nu_N$ will not be used, but $\nu$ is in some sense their limit.) For technical reasons to be displayed below, we bound ourselves away from 0 and 1 by an amount $\beta_N$ where $\beta_N \to 0$ as $N \to \infty$ at a rate to be specified later. Let

$$\mu_N = \int_{\beta_N}^{1-\beta_N} g \, d\nu$$

where $\int_a^b d\nu = \int_{(a, b)} d\nu$. Let $J_N$ be defined on $(0, 1]$ by

$$J_N(t) = c_{Ni} \quad \text{for} \quad (i-1)/N < t \leq i/N, \quad 1 \leq i \leq N.$$ 

Let $I(t) = t$ be the identity function on $[0, 1]$ and let $\int \cdot \, dI$ denote integrals with respect to Lebesgue measure. Let

$$T_N^* = N^4 (T_N - \mu_N).$$

Define stochastic processes $\{L_N(t): 0 < t < 1\}$ for $N \geq 1$ by

$$L_N(t) = N^4 [g \circ F_N^{-1}(t) - g(t)].$$

Then

$$T_N^* = S_N^* + \gamma_N$$

where

$$S_N^* = \int_{\beta_N}^{1-\beta_N} L_N \, d\nu$$

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and

$$\gamma_N = N^{-1} \left[ \sum_{i=1}^{N} c_N g(Y_{ni}) - N \int_{\beta_N}^{1} g \circ F_N^{-1} \, dv \right].$$

We wish to obtain asymptotic normality of statistics of the form $N^1(T_N - \mu)$ where

$$\mu = \int_{0}^{1} g \, dv.$$ 

However, our technique is tailor-made for $S_N^*$. Thus, after preliminaries in Sections 2 and 3, we obtain asymptotic normality of $S_N^*$ in Section 4 by first showing that the $L_N$-processes converge to a natural limiting $L_\infty$-process. Lemma 4.3 may be of interest in its own right. In Section 5 asymptotic normality of $T_N^*$ is obtained by giving conditions under which $\gamma_N \to_p 0$; from this we go quickly to $N^1(T_N - \mu)$. The theorem of Section 5 does not apply to the Winsorized mean; in Section 6 results are extended to include statistics of this type. Section 7 extends the results to get joint asymptotic normality of a vector of statistics of the type considered. Section 8 extends all results to the case that $N_*$ is a random sample size satisfying $N_* / v \to_p 1$ as $v \to \infty$.

Related work is contained in [1], [3], [4], [5], [6], [7] and [10]. See Section 9 for a short discussion of these.

2. The basic identity. Let

$$\rho(f_1, f_2) = \sup_{0 < t < 1} |f_1(t) - f_2(t)|$$

for arbitrary functions $f_1$ and $f_2$.

**Lemma 2.1.** We have

$$\frac{\rho(L_T, -A_N U_N (F_N^{-1}) + \delta_N)}{N^1} = 0$$

where

$$U_N = N^1(F_N - I),$$

$$A_N = (g \circ F_N^{-1} - g) / (F_N^{-1} - I)$$

and

$$\delta_N = A_N N^1(F_N \circ F_N^{-1} - I).$$

(In (2.4) $A_N$ is defined by left continuity at any otherwise undefined points.)

**Proof.** Now at any fixed $t$

$$L_N = \left[ (g \circ F_N^{-1} - g) / (F_N^{-1} - I) \right] N^1 \left[ - (F_N \circ F_N^{-1} - F_N^{-1}) + (F_N \circ F_N^{-1} - I) \right] = A_N \left[ U_N (F_N^{-1}) + N^1(F_N \circ F_N^{-1} - I) \right];$$

where in dividing by $F_N^{-1} - I$ for fixed $t$ we are dividing by a function which is a.s. never 0. Since the quantity in (2.2) is determined by its values for rational $t$, the result follows. □
3. Some definitions. Let \( \{U_0(t): 0 \leq t \leq 1\} \) denote a separable tied-down Wiener process; that is a Gaussian process with \( E[U_0(t)] = 0 \) and \( E[U_0(s)U_0(t)] = s(1-t) \) for all \( 0 \leq s \leq t \leq 1 \). Let \( g'(t) = (d/dt)g(t) \) whenever this derivative exists. When defined let

\[
L_0(t) = -g'(t)U_0(t), \quad 0 < t < 1.
\]

It is well-known that the \( U_N \)-processes (2.3) converge weakly to the \( U_0 \)-process. However, in Pyke and Shorack (1968a) the \( U_0 \)- and \( U_N \)-processes are replaced by \( \bar{U}_{0-} \) and \( \bar{U}_N \)-processes on a new probability space \( (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}) \); these new processes have the same finite dimensional distributions as do the old ones, but in addition satisfy the strong condition

\[(3.2) \quad \rho(\bar{U}_N, \bar{U}_0) \to_{a.s.} 0 \text{ as } N \to \infty.\]

Also \( \bar{P}_N = N^3\bar{U}_N + I \) is a.s. a df having exactly \( N \) jump discontinuities each of magnitude \( N^{-1} \). It is these new processes on this new probability space with which we work below; however, we drop the symbol \( \sim \) from the notation.

**Remark.** Results obtained below which involve convergence stronger than convergence in law may apply only to the specially constructed process. Thus the conclusion \( S_N^* \to \rho N(0, \sigma^2) \) in Corollary 4.1 yields only that \( S_N^* \) converges in law to a \( N(0, \sigma^2) \) rv when viewed on the original probability space.

Let \( L_N^*, U_N^*(F_N^{-1}), A_N^*, \delta_N^* \) equal \( L_N, U_N(F_N^{-1}), A_N, \delta_N \) respectively for \( \beta_N \leq t \leq 1- \beta_N \) and let them equal 0 otherwise. Let

\[
\rho_\phi(f_1, f_2) = \sup_{0 < t < 1} |f_1(t) - f_2(t)|/\phi(t)
\]

for arbitrary functions \( f_1, f_2 \) and any fixed positive function \( \phi \).

**Definition 3.1.** Let \( Q \) denote the class of all non-negative continuous functions \( q \) defined on \([0, 1]\) which are bounded below by functions \( \bar{q} \) non-decreasing (non-increasing) on \([0, \frac{1}{2}] \) (on \([\frac{1}{2}, 1]\)) with \( \int_0^1 |\bar{q}(t)|^{-1} dt < \infty \).

Two examples are worth noting: \( q(t) = [t(1-t)]^{-\frac{1}{2}} \) for any \( \delta > 0 \) and \( q(t) = -t^{\delta} \log t \).

Let \( q_0 \) denote a fixed function in the class \( Q \) for which \( q_0(t)/t \) increases as \( t \searrow 0 \). We now suppose that \( \beta_N \) satisfies \( \beta_N \geq 1/N, N\beta_N \) is an integer and

\[(3.3) \quad q_0(\beta_N)q_0(1/N)/\beta_N \to 0 \text{ as } N \to \infty.\]

**Remark.** One useful choice is \( q_0(t) = -t^\delta \log t \) and \( \beta_N = \lceil N^\delta \rceil/N \), where \( \lceil \cdot \rceil \) denotes the greatest integer function and where \( \delta > 0 \).


**Lemma 4.1.** Let \( q \in Q \). Then \( \rho_q(U_N^*(F_N^{-1}), U_0) \to \rho 0 \) as \( N \to \infty \).

**Proof.** This is essentially Theorem 2.2 of [8]; though slight modifications in the proofs of Lemmas 2.3, 2.4, 2.5 and Theorem 2.2 of [8] must be made to make the present proof completely rigorous. A proof based on 13.17 and 13.8 of Breiman (1968) is also possible. []

Let \( |\nu| \) denote the total variation measure for \( \nu \).
Lemma 4.2. If \(|v| (\{|t: g'(t) fails to exist\}) = 0\), then a.s. \(P\) we have

\[ A_N \rightarrow g' \text{ a.s. } |v| \text{ as } N \rightarrow \infty. \]

Proof. Now \(A_N = (g \circ F_N^{-1} - g)/(F_N^{-1} - I)\) where

\[ \rho(F_N^{-1}, I) \leq \rho(F_N^{-1}, F_N \circ F_N^{-1}) + \rho(F_N \circ F_N^{-1}, I) \leq \rho(F_N, I) + 1/N \rightarrow 0 \text{ a.s. } P. \]

Thus \(P[\omega: A_N(t) \rightarrow g'(t) \text{ for all } t \text{ such that } g'(t) \text{ exists}] = 1. \)

For \(q \in Q\) let

\[ \rho_q^*(f_1, f_2) = \sup_{\|f_1 - f_2\| \leq 1} \|f_1(t) - f_2(t)\|/q(t). \]

We also write \(Z_N = o_p(b_N)\) if \(Z_N/b_N \rightarrow_p 0 \) as \(N \rightarrow \infty\) and \(Z_N = O_p(b_N)\) if for all \(\varepsilon > 0\) there exists \(M_\varepsilon\) and \(N_\varepsilon\) such that for each \(n > N_\varepsilon\) \(|Z_n/b_n|\) exceeds \(M_\varepsilon\) with probability less than \(\varepsilon\). We use the fact that \(o_p(1)O_p(1) = o_p(1)\).

Lemma 4.3. Given \(\delta, \varepsilon > 0\) and \(0 < \lambda < 1\) there exists an integer \(N_\varepsilon\) and sets \(S_{N, \varepsilon}\) having \(P(S_{N, \varepsilon}) > 1 - \varepsilon\) such that

\[ t - |F_N^{-1}(t) - t| \geq \lambda t \text{ for all } \beta_N \leq t \leq \frac{1}{2}, \]

provided \(N > N_\varepsilon\) and \(\omega \in S_{N, \varepsilon}\).

Proof. Now \((N^\delta q_0(1/N))^{-1} \rightarrow 0\) as \(N \rightarrow \infty\). Thus

\[ \begin{align*}
\rho_q^*(F_N^{-1}, I) &\leq N^{-1} [p_q^*(U_N(F_N^{-1}), U_0) + p_q^*(U_0, 0)] + [Nq_0(1/N)]^{-1} \\
&= o_p(q_0(1/N)).
\end{align*} \]

Thus for \(N\) exceeding some \(N_\varepsilon\) and \(\beta_N \leq t \leq \frac{1}{2}\)

\[ |F_N^{-1}(t) - t| \leq q_0(t)q_0(1/N) \]

on some set \(S_{N, \varepsilon}\) whose probability exceeds \(1 - \varepsilon\).

Thus for \(N > N_\varepsilon\), \(\omega \in S_{N, \varepsilon}\) and \(\beta_N \leq t \leq \frac{1}{2}\) we may assume by (3.3) that

\[ t - |F_N^{-1}(t) - t| \geq t[1 - q_0(t)q_0(1/N)/t] \]

\[ \geq t[1 - q_0(\beta_N)q_0(1/N)/\beta_N] \]

\[ \geq \lambda t. \]

For a given function \(h\) and \(0 < \lambda < 1\) define

\[ h_\lambda(t) = h(\lambda t) \text{ for } 0 < t \leq \frac{1}{2}, \]

\[ = h(1 - \lambda(1 - t)) \text{ for } \frac{1}{2} < t \leq 1. \]

For functions \(f\) on \((0, 1)\) define

\[ \|f\|_\nu = \int_0^1 |f| \, d|\nu|. \]

Theorem 4.1. If

(C1) \(g\) is absolutely continuous on \((\epsilon, 1 - \epsilon)\) for all \(\epsilon > 0\) and \(|g'| \leq h\) a.e. \(|\nu|\) where the function \(h\) is increasing (decreasing) on \([\frac{1}{2}, 1)\) (on \((0, \frac{1}{2})\))
and

\[ (C2) \int_{0}^{\theta} h_{\theta} q d|\nu| < \infty \text{ for some } q \in Q \text{ and some } 0 < \lambda < 1, \text{ then} \]

\[ \|L_{N}^* - L_{0}\|_{\nu} \rightarrow_{p} 0 \text{ as } N \rightarrow \infty. \]

**Proof.** Now

\[ \|L_{N}^* - L_{0}\|_{\nu} \leq \rho_{q}(U_{N}^*(F_{N}^{-1}), U_{0}) \int_{0}^{\theta} |A_{N}^*| q d|\nu| + \rho_{q}(U_{0}, 0) \int_{0}^{\theta} |A_{N}^* - g'| q d|\nu| + N^{1/2} \rho_{q}^{*}(F_{N} \circ F_{N}^{-1}, I) \int_{0}^{\theta} |A_{N}^*| q d|\nu| \]

\[ = \int_{0}^{\theta} |g'| q d|\nu| o_{p}(1) + \int_{0}^{\theta} |A_{N}^* - g'| q d|\nu| O_{p}(1) \]

\[ = o_{p}(1) + \int_{0}^{\theta} |A_{N}^* - g'| q d|\nu| O_{p}(1), \]

using Lemma 4.1 and Lemma 2.2 of [8] in the first equality and \( \int_{0}^{\theta} |g'| q d|\nu| < \infty \) by hypotheses in the second. (Everything so far in the present proof is also true if the symbol \( * \) restricted the range of a quantity to \( 1/N \leq t \leq 1 - 1/N \). What follows is not.) Since an absolutely continuous function is the integral of its derivative, (C1) gives \( |A_{N}(t)| \leq \max [h(t), h(F_{N}(t))] \). Thus by Lemma 4.3 for \( N > N_{*}, \omega \in S_{N}, \) and \( \beta_{N} \leq t \leq \frac{1}{2} \) we have \( |A_{N}(t)| \leq h(x_{t}) = h_{\lambda}(t) \); with a symmetric result for \( \frac{1}{2} \leq t \leq 1 - \beta_{N} \). Thus we may suppose that for \( N > N_{*} \), \( \omega \in S_{N}, \) and \( \beta_{N} \leq t \leq \frac{1}{2} - \beta_{N} \) (\( |A_{N}(t)| \leq h_{\lambda}(t) \)) Now a.s. \( P \) we have \( A_{N} \rightarrow g' \) a.e. \( |\nu| \) by Lemma 4.2. Thus

\[ I_{S_{N}}, \int_{0}^{\theta} |A_{N}^* - g'| q d|\nu| \rightarrow_{a.s.} 0, \]

by applying the dominated convergence theorem, with dominating function \( 2h_{\theta} q \) by (C2), separately to each \( \omega \) point. Thus

\[ \int_{0}^{\theta} |A_{N}^* - g'| q d|\nu| \rightarrow_{p} 0. \]

**Remark.** If for some \( M \) and \( \theta > 0 \) the function \( M[t(1 - t)]^{\theta} \) can be used for \( h \) in Theorem 4.1, then \( h_{\theta} \leq M h \) for some \( M > 0 \). This is very useful because often \( q \) and a bound on the Radon-Nikodym derivative of \( |\nu| \) with respect to Lebesgue measure can also be taken to be of this same functional form. Often, then, the integral in (C2) can be bounded by a constant times \( \int_{0}^{\theta} |t(1 - t)|^{1/2} d|\nu| \) for some \( \delta > 0 \).

**Proposition 4.1.** If \( \|L_{N}^* - L_{0}\|_{\nu} \rightarrow_{p} 0 \) as \( N \rightarrow \infty \) and if \( \int_{0}^{\theta} |g'| q d|\nu| < \infty \) for some \( q \in Q \), then

\[ S_{N}^* \rightarrow_{p} \int_{0}^{\theta} L_{0} d\nu \text{ as } N \rightarrow \infty. \]

The limiting rv is \( N(0, \sigma^2) \) where

\[ \sigma^2 = 2 \int_{0}^{\theta} \int_{0}^{\theta} g'(u) g'(v) u(1 - v) d\nu(u) d\nu(v). \]

If, moreover, \( \gamma_{N} \rightarrow_{p} 0 \) as \( N \rightarrow \infty \), then \( T_{N}^* \rightarrow_{p} \int_{0}^{\theta} L_{0} d\nu. \)

**Proof.** Now \( |S_{N}^* - \int_{0}^{\theta} L_{0} d\nu| \leq \|L_{N}^* - L_{0}\|_{\nu} \rightarrow_{p} 0, \) by (1.6). The limiting rv is normal since \( \sigma^2 = E[(\int_{0}^{\theta} L_{0} d\nu)^2] \) is finite by the hypotheses. The final claim follows from (1.5). \( \square \)
The remainder of this section constitutes a digression.

**Theorem 4.1'.** If

(C1') $g$ is absolutely continuous on $(\epsilon, 1 - \epsilon)$ for all $\epsilon > 0$ and $g'$ exists a.e. $|\nu|$, and if

(C2') (a) $\int_0^t |g'|d|\nu| < \infty$ for some $q \in Q$

(b) for some $0 < \lambda < 1$ there exists $0 < M < \infty$ such that

\[ |g'(u)/g'(t)| < M \]

whenever $\lambda t < u < t/\lambda \leq 1/2\lambda$ or whenever $\lambda(1 - t) < 1 - u < (1 - t)/\lambda \leq 1/2\lambda$, then

\[ ||L_{N^*} - L_0||_{\nu} \rightarrow 0 \text{ as } N \rightarrow \infty. \]

**Proof.** From the proof of Theorem 4.1 it suffices to show that $I_{S_{N^*}}|A_{N^*} - g'|$ times $q$ is a.s. bounded by a $|\nu|$-integrable function. But by (C1')

\[ A_N(t) - g'(t) = \int_{t_{N^{-1}(t)}}^{t} g'(u) du/[F_N^{-1}(t) - t] - g'(t) \]

so that on the set $S_{N^*}$ of the proof of Theorem 4.1 we have

\[ |A_{N^*}(t) - g'(t)| \leq |g'(t)| |(\int_{t_{N^{-1}(t)}}^{t} |g'(u)/g'(t)| du/[F_N^{-1}(t) - t]| + 1) \leq |g'(t)| (M + 1); \]

where the last inequality uses (C2') (b). Finally (C2') (a) shows that $|g'|q$ is the desired $|\nu|$-integrable function. \[ \square \]

**Theorem 4.1''.** If

(C1'') $g$ satisfies a first order Lipschitz condition on $(\epsilon, 1 - \epsilon)$ for all $\epsilon > 0$ and $g'$ exists a.e. $|\nu|$ and

(C2'') (a) $\int_0^t |g'|d|\nu| < \infty$ for some $q \in Q$,

(b) there exists $\delta > 0$ such that for some $0 < \lambda < 1$ there exists $0 < M < \infty$ such that $|g'(u)/g'(t)| < M$ whenever $\lambda t < u < t/\lambda \leq \delta/\lambda$ or whenever $\lambda(1 - t) < 1 - u < (1 - t)/\lambda \leq \delta/\lambda$. Or there exists $\delta_0 > 0$ such that $c_{N^*} = 0$ for $i \leq N\delta_0$ and $i \geq N(1 - \delta_0)$ and $N$ exceeding some $N_0$,

(c) $|\nu|(\delta_1, 1 - \delta_1) < \infty$ for some $0 < \delta_1 < \delta$,

then

\[ ||L_{N^*} - L_0||_{\nu} \rightarrow 0 \text{ as } N \rightarrow \infty. \]

**Proof.** As in the previous proofs, we seek to bound $|A_{N^*} - g'|$ on $S_{N^*}$ by a $|\nu|$-integrable function. (C1'') and (C2'') (c) yield the bound on $[\delta, 1 - \delta]$ with probability exceeding 1 - $2\epsilon$ for large $N$ and (C1'') and (C2'') (b) yield the bound on $(0, \delta)$ and $(1 - \delta, 1)$ by the argument of Theorem 4.1'. \[ \square \]

**Remark.** This form of the theorem corresponds closely to Theorem 3 of Chernoff, et al. (1967) which gives asymptotic normality under conditions labeled $A^*$, $B^{**}$ and $E$. Our (C1'') is essentially slightly weaker than their $A^*$, our (C2'') (a) (c) is essentially slightly stronger than their $B^{**}$ and our (C2'') (b) is essentially their $E$. The statistics to which the conclusions apply are different.
Remark. The proof of Theorem 4.1 shows that \( \|L_N^* - L_0\|_\nu \to_p 0 \) as \( N \to \infty \) provided we have \( g' \) exists a.e. \( \nu \), (C2') (a) and \( \int_0^\beta |A_N^* - g| d\nu \to_p 0 \) as \( N \to \infty \).

5. Limiting distribution of \( T_N^* \). Throughout this section we assume \( \nu = \nu_J \) for some function \( J \); that is \( \nu_J \) is given by \( \nu_J((a, b)) = \int_a^b J \, dI \) for all \( (a, b) \subset (0, 1) \). Note that \( |\nu_J((a, b))| = \int_a^b |J| \, dI \). Now by (1.7) \( \gamma_N = \gamma_{N1} + \gamma_{N2} + \gamma_{N3} \) where

\[
\gamma_{N1} = N^{-1} \sum_{i=1}^{N\beta_N} c_{Ni} g(X_{Ni}), \quad \gamma_{N2} = N^{-1} \sum_{i=1}^{N(1-\beta_N)+1} c_{Ni} g(X_{Ni}),
\]

\[
\gamma_{N3} = N^{-1} \sum_{i=N(1-\beta_N)+1}^{i/N} |c_{Ni} - N \int_{(i-1)/N}^{i/N} J \, dI| g(X_{Ni}).
\]

Theorem 5.1. Suppose that \( \|L_N^* - L_0\|_\nu \to_p 0 \) as \( N \to \infty \) (see Section 4), \( \int_0^1 |g'|^q \, d\nu < \infty \) for some \( q \in Q \) and

(C3) \( N^{-1} \int_{(1-\beta_N)^N}^{1-\beta_N} |J_N - J| k_\lambda \, dI \to 0 \) as \( N \to \infty \) for some \( 0 < \lambda < 1 \), where \( |g| \leq k \), \( k \) is increasing on \( [\frac{1}{2}, 1] \) and \( k \) is symmetric about \( \frac{1}{2} \);

(C4) (a) For some \( K \) and \( \theta > 0 \) \( |g(u)| \leq K[u(1 - u)]^{-\theta} \) \( 0 < u < 1 \) and

\( b) \quad N^{-1} \sum_{i=0}^{N\beta_N} |c_{Ni}| \frac{1}{i(N)^{-\theta}} \to 0 \)
and \( N^{-1} \sum_{i=N(1-\beta_N)+1}^N |c_{Ni} - (1 - i/N)^{-\theta} \to 0 \) as \( N \to \infty \);

and

(C5) \( N^{-1} \int_{(1-\beta_N)^N}^{1-\beta_N} g \, d\nu \to 0 \) as \( N \to \infty \).

Then \( N^2 (T_N - \mu) \) is asymptotically \( N(0, \sigma^2) \) with \( \mu = \int_0^1 g \, d\nu \) and \( \sigma^2 \) given by

(4.4).

Remark. (C4) (b) may be replaced by

(C4) (b') there exists \( K, \Delta \) and a sequence \( \beta_N \) (see Section 3) such that \( |c_{Ni}| \leq K[(i/N)(1 - i/N)]^\Delta \) for \( i \leq N\beta_N \) and \( i > N(1 - \beta_N) \) and \( N\beta_N^{1+\Delta-\theta} \to 0 \) as \( N \to \infty \).

Remark. (C4) (a) implies \( \int_0^1 |g|^m \, dI \leq K^m \int_0^1 [u(1 - u)]^{-m\theta} \, du \) and hence implies the existence of absolute \( m \)th moments of \( g(X) \) provided \( m < \theta^{-1} \).

(C4) (b) and (b') shows that if the constants \( c_{Ni} \) for extreme values of \( i \) are of a small enough order, then certain moments of \( g(X) \) need not exist.

Proof. Now by definition of \( J_N \) in Section 1

\[
\gamma_{N2} = \frac{1}{N^{1/2}} \int_{(1-\beta_N)^N}^{1-\beta_N} (J_N - J) \circ F_N^{-1} \, dI.
\]

Thus, as in the proof of Theorem 4.1, for \( N > N_\epsilon \) and \( \omega \in S_{N, \epsilon} \) with \( P(S_{N, \epsilon}) > 1 - \epsilon \) we have

\[
I_{S_{N, \epsilon}}(\gamma_{N2}) \leq N^{1/2} \int_{(1-\beta_N)^N}^{1-\beta_N} (J_N - J) \circ k_\lambda \, dI;
\]

so that \( \gamma_{N2} \to_p 0 \) by (C3). \( \gamma_{N1} = \gamma_{N3} \to_p 0 \) under (C4) as in the proof of Theorem 4 of Stigler (1967); which is a short, self-contained theorem. Thus by Proposition 4.1 we have \( T_N^* \to_p \int_0^1 L_0 \, d\nu \) as \( N \to \infty \). Under (C5) the constant \( \mu \) may replace \( \mu_N \).
6. Point mass in the limiting measure. In this section we apply Theorem 4.1 to statistics of the form

\[ T_N' = T_N + \tau_N \]

with \( T_N \) of the form (1.1) treated in Theorem 5.1 and

\[ \tau_N = \sum_{i=1}^{s} d_{Ni} g(X_{N,(x_{p_i})+1}) \]

where the constants \( d_{Ni} \) satisfy \( (d_{Ni} - d_i) = o(N^{-1}) \) with \( 0 < d_i < \infty \) for \( 1 \leq i \leq s \), where \( 0 < p_1 < p_2 < \cdots < p_s < 1 \) and where \([ \ ]\) denotes the greatest integer function. We suppose that \( g'(p_i) \) exists for \( 1 \leq i \leq s \).

**Theorem 6.1.** Under the hypotheses of Theorem 5.1 and the assumptions of this section \( N^4(T_N' - \mu - \mu_s) \) is asymptotically distributed as a \( N(0, \sigma^2) \) rv where

\[ \mu_s = \sum_{i=1}^{s} d_i g(p_i) \]

and

\[ \sigma^2 = \text{Var} \left[ \int_0^1 L_0 J \, dI + \sum_{i=1}^{s} d_i L_0(p_i) \right] \]

\[ = 2 \int_0^1 \int_0^1 g(u)g'(v)u(1-v)J(u)J(v) \, du \, dv \]

\[ + 2 \sum_{i=1}^{s} d_i \int_0^1 g(u)g'(p_i)[\min(u, p_i) - up_i]J(u) \, du \]

\[ + \sum_{i=1}^{s} [d_i g'(p_i)]^2 p_i(1 - p_i) + 2 \sum_{i<j} d_i d_j g'(p_i)g'(p_j) p_i(1 - p_i). \]

**Proof.** Now

\[ N^4(\tau_N - \mu_s) = N^4 \sum_{i=1}^{s} (d_{Ni} - d_i) g(X_{N,(x_{p_i})+1}) \]

\[ + N^4 \sum_{i=1}^{s} d_i L_0(p_i) \]

Now

\[ N^4[g(X_{N,(x_{p_i})+1}) - g(p_i)] = N^4[g \circ F_N^{-1}((Np_i) + 1)/N) - g(p_i)] \]

\[ = N^4[g \circ F_N^{-1}(p_i) - g(p_i)] \]

\[ \rightarrow_p L_0(p_i), \]

by Lemma 6.1 below and (1.4). Thus

\[ N^4(\tau_N - \mu_s) \rightarrow_p \sum_{i=1}^{s} d_i L_0(p_i). \]

Thus

\[ N^4(T_N' - \mu - \mu_s) \rightarrow_p \int_0^1 L_0 J \, dI + \sum_{i=1}^{s} d_i L_0(p_i); \]

where the limiting rv is \( N(0, \sigma^2) \) with \( \sigma^2 \) as given. \( \square \)

**Lemma 6.1.** If \( g'(t_0) \) exists for some \( t_0 \) in \( (0, 1) \), then

\[ L_N^*(t_0) \rightarrow_p L_0(t_0) \] as \( N \rightarrow \infty. \)

**Proof.** Now at \( t_0 \) we have

\[ |L_N^* - L_0| \leq |A_N^*||U_N^*(F_N^{-1}) - U_0| + |A_N^*|g'(t_0)|U_0| + N^{-1}|A_N^*| \]

\[ = |A_N^* - g'(t_0)|U_0(1) + o_p(1) = o_p(1) \]
where the first equality uses Lemma 4.1 and the second equality uses Lemma 4.2. 

7. **Joint asymptotic normality.** Suppose the conclusion of Theorem 6.1 applies to each to the statistics \( T_{nk} \), \( k = 1, \cdots, K \) of the form (6.1). Then the vector

\[
(N^t(T'_{x1} - \mu_1 - \mu_1), \cdots, N'_{x1k} - \mu_k - \mu_k)
\]

is asymptotically multivariate normal \( N(0, \Sigma) \) where the \( j, k \)th element of \( \Sigma \) is

\[
\sigma_{jk} = \text{Cov} \left( \int_0^1 L_0 J_j \, dI + \sum_{i \ge 1} \beta_i \, d_i^{(j)} L_0(p_{i}^{(j)}), \int_0^1 L_0 J_k \, dI + \sum_{i \ge 1} \beta_i \, d_i^{(k)} L_0(p_{i}^{(k)}) \right);
\]

the meaning of the above notation is clear. This is true since the convergence in probability of a vector is implied by the convergence in probability of each of its coordinates.

8. **Random sample size.** All results of Sections 1–7 carry over if \( N \) is replaced by \( N_v \) and \( N \to \infty \) is replaced by \( v \to \infty \) where the stochastic process \( \{N_v: v > 0\} \) satisfies \( N_v/v \to_p 1 \). (All of the previous sections may be recopied except for Lemma 4.1. To prove Lemma 4.1 in the case of random sample size we cannot modify Theorem 2.2 of [8]; rather we must make slight modifications in the random sample size version of it which is given in [9].

9. **Comparison with other results.** The results of this paper are a straightforward application of the results and techniques of [8] in which the value of replacing weakly convergent processes by a.s. convergent equivalent processes is demonstrated. However, bounding the coefficient \( A_N \) (see the proof of Theorem 4.1) introduced a problem not encountered in [8]; and Lemma 4.3 provided the key to a solution. But in order to prove Lemma 4.3 we found it necessary to introduce \( \beta_N \). This in turn required conditions in Theorem 5.1 which would make \( \gamma_{x1} + \gamma_{x3} \) negligible.

In the expository paper [7] Pyke also uses the technique of [8] to obtain some results for this problem; however a different basic identity was exploited.

Bickel [1] also uses an approach in which the statistic is viewed as a functional over a stochastic process. His technique uses weak convergence of the stochastic processes, rather than the stronger versions in [8]; and he uses the metric corresponding to \( q = 1 \). He proves asymptotic normality when \( \sum_{i \le N} c_{ni} \) converges to a function \( J(t) \) of bounded variation, \( F \) has a continuous density positive on its support, and either no weight is put on observations below the \( p \)th and above the \( q \)th percentile, or the more extreme observations are not weighted more than in the sample mean.

The techniques used by other authors are quite different from those in the papers cited above. Charnoff et al [4] use a device of Rényi to express \( T_N \) as a linear combination of exponential rv’s plus a remainder term. Stigler [10] uses a projection technique due to Hájek. The results of these two papers and the present one seem to be of approximately equal strength. (A remark in Section 4 compares our results with [4].) Moore’s [6] result is an elegant proof of a special case. Mark Brown [3], by a neat application of a theorem of Sethuraman on the convergence of stochastic integrals, also obtains a special case.
REFERENCES


