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THE LAW OF THE ITERATED LOGARITHM FOR MIXING STOCHASTIC PROCESSES[1]

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1. Introduction. Let $\langle \xi_n, n = 1, 2, \cdots \rangle$ be a sequence of random variables centered at expectations with finite variances. Suppose that

$$(1) \qquad s_N^2 = E(\textstyle\sum_{n \leqq N} \xi_n)^2 \to \infty \qquad (N \to \infty)$$

$$(2) \qquad s_{N+1}/s_N \to 1 \qquad (N \to \infty)$$

and that

$$(3) \qquad s_{MN}^2 = E(\textstyle\sum_{n=M+1}^{N} \xi_n)^2 = (s_N^2 - s_M^2)(1 + o(1)) \quad (\text{as } s_N^2 - s_M^2 \to \infty).$$

Let M_{ab} be the σ-algebra generated by the events $\{\xi_n < \alpha\}$, $a \leqq n \leqq b$. We say that the Borel-Cantelli Lemma holds for the process $\langle \xi_n \rangle$ if $\sum P(A_k) = \infty$ implies that $P(A_k \text{ i.o.}) = 1$ where $A_k \, \varepsilon \, M_{n_{k-1}n_k-1}$ $(1 \leqq n_0 < n_1 < \cdots)$.

The standard proof of the law of the iterated logarithm yields the following

THEOREM 0. *Let $\langle \xi_n \rangle$ be any stochastic process satisfying* (1)–(3) *for which the Borel-Cantelli Lemma holds. Suppose that uniformly in M and x*

$$(4) \qquad P(s_{MN}^{-1} \textstyle\sum_{n=M+1}^{N} \xi_n < x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\, dt + O((\log s_{MN})^{-1-\eta}), \ \eta > 0,$$

and that for some constants $C > 0$, $0 < \rho_N = O((\log \log s_N)^{\frac{1}{2}})$ and ϵ sufficiently large

$$(5) \qquad P(\max_{1 \leqq n \leqq N} \textstyle\sum_{k \leqq n} \xi_k > \epsilon) \leqq CP(\sum_{k \leqq N} \xi_k > \epsilon - \rho_N s_N).$$

Moreover, suppose that (5) *holds with ξ_n replaced by $-\xi_n$. Then*

$$(6) \qquad P(\limsup_{N \to \infty} (2 s_N^2 \log \log s_N^2)^{-\frac{1}{2}} \textstyle\sum_{n \leqq N} \xi_n = 1) = 1.$$

In short the law of the iterated logarithm holds for any process for which the Borel-Cantelli Lemma, the central limit theorem with a reasonably good remainder and a certain maximal inequality are valid. The proof of Theorem 0 can be found in Loève [4, pages 260–263] (see also [1], [5]) where instead of the exponential bounds we use the fact that for $\tau > 0$

$$P(\textstyle\sum_{n=M+1}^{N} \xi_n > \tau s_{MN}) = (2\pi)^{-\frac{1}{2}} \tau^{-1} \exp(-\frac{1}{2}\tau^2)(1 + \theta\tau^{-2}) + O((\log s_{MN})^{-1-\eta})$$

with $0 < \theta < 1$. This follows from (4) and the well-known [1, page 175] estimate

$$\int_x^\infty e^{-\frac{1}{2}t^2}\, dt = x^{-1} \exp(-\tfrac{1}{2}x^2)(1 + \theta x^{-2}).$$

Moreover, we choose n_k to be the largest integer n with $s_n \leqq c^k$, where $c > 1$ is the constant occurring in [4, page 261], (s_n is not assumed to be monotone).

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The purpose of this paper is to prove the law of the iterated logarithm for sequences of random variables satisfying two kinds of mixing conditions. Roughly speaking, we shall assume that as time passes events concerning the "future" of the stochastic process become almost independent of the events in the "past". More precisely, we shall assume that either one of the following two conditions holds:

(I) For any events $A \varepsilon M_{1t}$ and $B \varepsilon M_{t+n,\infty}$ we have

$$|P(AB) - P(A)P(B)| \leqq \psi(n)P(A)P(B)$$

with $\psi(n) \downarrow 0$.

(II) $$\sup_t \sup_{B\varepsilon M_{t+n,\infty}} |P(B \mid M_{1t}) - P(B)| \leqq \varphi(n) \downarrow 0$$

with probability 1.

Moreover, we shall assume throughout this paper that the random variables x_n are centered at expectations with $\sup_n E(x_n^4) \leqq 1$ and that $s_N \to \infty$. From Theorem 0 we shall derive the following two theorems.

THEOREM 1. *Suppose that the process* $\langle x_n \rangle$ *satisfies* (I) *with* $\sum \psi^{\frac{1}{2}}(n) < \infty$ *and that uniformly in* $M = 0, 1, 2, \cdots$

$$\sum_{n=M+1}^{M+H} E\,|x_n| \ll (E(\textstyle\sum_{n=M+1}^{M+H} x_n)^2)^{\frac{3}{4}} \qquad H \to \infty \tag{7}$$

$$\sum_{n=M+1}^{M+H} E(x_n^4) \ll (E(\textstyle\sum_{n=M+1}^{M+H} x_n)^2)^{3} \qquad H \to \infty. \tag{8}$$

Then the law of the iterated logarithm (6) *holds.*

Observe that if $\sup \|x_n\|_\infty \leqq 1$ then (7) implies (8), which thus can be omitted.

THEOREM 2. *Suppose that the process* $\langle x_n \rangle$ *satisfies* (II) *with* $\sum \varphi^{1/5}(n) < \infty$ *and that uniformly in* $M = 0, 1, 2, \cdots$ *with* $\rho < 208/111$

$$\sum_{n=M+1}^{M+H} \|x_n\|_4 \ll (E(\textstyle\sum_{n=M+1}^{M+H} x_n)^2)^{\rho} \qquad H \to \infty_t \tag{9}$$

Then (6) *holds.*

There are several variations of the hypothesis possible. For example in Theorem 2 one can replace $\frac{1}{5}$ by $\frac{1}{6}$ and ρ by 2 and the conclusion still remains valid.

For strict sense stationary processes the law of the iterated logarithm has been proved independently by Iosifescu [3] and the author [6] using different hypotheses and different methods. We shall discuss the stationary case in Section 4 (Theorem 4).

2. Some lemmas.

LEMMA 1 [8]. *Suppose that condition* (I) *is satisfied, that the random variables* ξ *and* η *are measurable over* M_{1t} *and* $M_{t+n,\infty}$ *respectively and that both are integrable. Then*

$$|E(\xi\eta) - E(\xi)E(\eta)| \leqq \psi(n)\, E\,|\xi|\, E\,|\eta|.$$

LEMMA 2 [2]. *Suppose that* (II) *holds and that* ξ *and* η *are measurable over* M_{1t} *and* $M_{t+n,\infty}$ *respectively. If* $E\,|\xi|^p < \infty$ *and* $E\,|\eta|^q < \infty$ *with* $p, q > 1$ *and* $p^{-1} + q^{-1} = 1$ *then*

$$|E(\xi\eta) - E(\xi)E(\eta)| \leqq 2\varphi^{p^{-1}}(n)\|\xi\|_p\, \|\eta\|_q.$$

Moreover, if ξ and η are essentially bounded then

$$|E(\xi\eta) - E(\xi)E(\eta)| \leqq 4\varphi(n)\|\xi\|_\infty \|\eta\|_\infty .$$

Let us observe that (I) implies (II) since (II) is equivalent to (II*). For any events $A \varepsilon M_{1t}$ and $B \varepsilon M_{t+n,\infty}$ we have

$$|P(AB) - P(A)P(B)| \leqq \varphi(n)P(A).$$

Let $\langle\eta_n\rangle$ be any stochastic process with $E(\eta_n) = 0$ and $E(\eta_n^4) \leqq 1$.

Lemma 3. *Suppose that $\langle\eta_n\rangle$ satisfies* (II) *with $\sum \varphi^{\frac{1}{3}}(n) < \infty$. Then with a fixed constant $O(1)$*

$$E(\textstyle\sum_{n=M+1}^N \eta_n)^2 = s_N^2 - s_M^2 + O(1).$$

Proof. We have from Lemma 2

$$\begin{aligned} s_N^2 &= E(\textstyle\sum_{n=1}^N \eta_n)^2 = E(\sum_{n=1}^M \eta_n + \sum_{n=M+1}^N \eta_n)^2 \\ &= s_M^2 + E(\textstyle\sum_{n=M+1}^N \eta_n)^2 + 2\sum_{m=1}^M \sum_{n=M+1}^N E(\eta_n\eta_m) \\ &= s_M^2 + E(\textstyle\sum_{n=M+1}^N \eta_n)^2 + O(\sum_{m=1}^M \sum_{n=M+1}^N \varphi^{3/4}(n-m)\cdot\|\eta_n\|_4 \|\eta_m\|_{4/3} \\ &= s_M^2 + E(\textstyle\sum_{n=M+1}^N \eta_n)^2 + O(\sum_{m=-\infty}^M \sum_{n=M+1}^\infty (n-m)^{-9/4}) \\ &= s_M^2 + E(\textstyle\sum_{n=M+1}^N \eta_n)^2 + O(1) \end{aligned}$$

since the monotonicity of $\varphi(n)$ and $\sum \varphi^{\frac{1}{3}}(n) < \infty$ implies $n^3\varphi(n) \to 0$.

Lemma 4. *Suppose that $\langle\eta_n\rangle$ satisfies* (II) *with $\varphi(1) < \frac{1}{2}$. Then for any N, ϵ with the constant $O(1)$ from Lemma* 3 *we have*

$$P(\max_{1\leqq n\leqq N} \textstyle\sum_{k\leqq n} \eta_k > \epsilon) \leqq (\frac{1}{2} - \varphi(1))^{-1} P(\sum_{k\leqq N} \eta_k > \epsilon - 2^{\frac{1}{2}} s_N + O(1)).$$

For a proof see Loève [4, page 248]. We apply (II*) at the place where the independence of the events A_k and B_k is used and Lemma 3 for the estimate of the median.

Lemma 5. *Suppose that the process $\langle\eta_n\rangle$ satisfies either the hypotheses of Theorem* 1 *or Theorem* 2. *Then* (4) *holds uniformly in M.*

A theorem of this type has been proved in an earlier paper [9]. The proof of Lemma 5, apart from a few minor modifications, is the same. I shall indicate the changes necessary under the hypotheses of Theorem 2. In [9, Lemma 2] we choose $\alpha > \frac{1}{3}$ such that $\rho < (1 + \alpha)52/111\alpha < 208/111$ and $k = [s_N^\beta]$ with $\frac{1}{3} < \beta < \alpha$. We obtain in place of [9, (6)-(11)]—note that we do not have a double sequence anymore—

$$E(y_j^2) = s_N^\alpha + O(1) \qquad E(y_{l+1}^2) < s_N^\alpha$$

uniformly in $1 \leqq j \leqq l$. Moreover, there is a constant $c_1 > 0$ such that for all $1 \leqq j \leqq l$ we have $h_j \geqq c_1 s_N^\alpha$. Also $l = s_N^{2-\alpha}(1 + O(s_N^{\beta-\alpha}))$, $E(Y_N^2) = s_N^2(1 + O(s_N^{\beta-\alpha}))$, $E(Z_N^2) \ll s_N^{2-\alpha+\beta}$. Lemmas 3 and 4 [9] remain unchanged whereas (14) transforms to

$$P(Y_N/s_N < x) - \phi(x) \ll g_N^3 + s_N^{-\alpha} + s_N^{\alpha-2} + s_N^{2-\alpha-5\beta} \log s_N$$

and (15) becomes if we choose $A_1 = s_N^{8\alpha/9}$, $A_2 = s_N^{12\alpha/13}$

$$g_N^3 \ll lE\,|y_j^3|/s_N^3 \ll s_N^{-3+2-\alpha+111\alpha\rho/52}$$

since by the hypotheses of Theorem 2, $P(h_j) \ll s_N^{\alpha\rho}$ and $\varphi(n) \ll n^{-5}$. Moreover, by Chebyshev's inequality

$$P(|Z_N|/s_N \geqq s_N^{\frac{1}{2}(\beta-\alpha)}) \ll s_N^{\beta-\alpha}.$$

This proves Lemma 5 for $M = 0$ with an error term $O(s_N^{\beta-\alpha})$. We observe that the constant implied by O depends only on $\langle\varphi(n)\rangle$ and the constant implied by $\ll$ in (9). Hence defining a new process $\eta_n^{(M)} = \eta_{M+n}$ we see that (4) holds uniformly for all M thus completing the proof of Lemma 5 under the hypotheses of Theorem 2. The proof under the hypotheses of Theorem 1 is similar.

LEMMA 6 [7]. *Let $\langle E_n, n \geqq 1\rangle$ be a sequence of events and denote by $A(N, \omega)$ the number of integers $n \leqq N$ with $\omega \in E_n$. Put*

$$\phi(N) = \textstyle\sum_{n\leqq N} P(E_n).$$

Suppose there is a convergent series $\sum \varphi(n)$ with $\varphi(n) \geqq 0$ such that for all positive integers n, t

$$P(E_tE_{n+t}) \leqq P(E_t)P(E_{n+t}) + \varphi(n)P(E_t).$$

Then with probability 1

$$A(N, \omega) - \phi(N) \ll \phi^{\frac{1}{2}+\epsilon}(N) \qquad \epsilon > 0.$$

Note that Lemma 6 was proven under the assumption that

$$P(E_tE_{n+t}) \leqq P(E_t)P(E_{n+t}) + \varphi(n)P(E_{n+t}).$$

However, this does not affect the estimate [7, (3)]. For an improved version of Lemma 6 see Theorem 3 below.

Observe that if $\varphi(1) < \frac{1}{2}$ or $\psi(1) < \frac{1}{2}$ respectively the proof of Theorems 1 and 2 is complete.

3. Proof of Theorems 1 and 2. Suppose that either the hypotheses of Theorem 1 or 2 are satisfied. Let j be with $\psi(j) \leqq \frac{1}{4}$ or $\varphi(j) \leqq \frac{1}{4}$ respectively. Given $1 > \delta > 0$ we define two new processes $\langle y_n\rangle$ and $\langle z_n\rangle$ as follows:

$$\begin{array}{ll} y_1 = x_1 + \cdots + x_{h_1} & z_1 = x_{h_1+1} + \cdots + x_{h_1+j} \\ \vdots & \vdots \\ y_n = x_{\rho_n+1} + \cdots + x_{\rho_n+h_n} & z_n = x_{\rho_n+h_{n+1}} + \cdots + x_{\rho_{n+1}} \\ \vdots & \vdots \end{array}$$

where $\rho_n = \sum_{\nu<n} (h_\nu + j)$. We choose the h_n inductively as the largest integer h such that

$$E\left(\textstyle\sum_{i=\rho_n+1}^{\rho_n+h} x_i\right)^2 \leqq (j/\delta)^2.$$

LEMMA 7. *We have uniformly in* $n = 1, 2, \cdots$

$$(10) \qquad E(y_n^2) = (j/\delta)^2 + O(1), \qquad E(z_n^2) \leqq j^2, \qquad h_n \geqq c_1(j/\delta)^2$$

with a constant $O(1)$ *depending only on* $\sum \psi^{\frac{1}{2}}(n)$ *or* $\sum \varphi^{\frac{1}{2}}(n)$ *respectively. Moreover, let* l *be the index such that* x_N *occurs in either in* y_l *or* z_l. *Then*

$$(11) \qquad s_N^2 = (j/\delta)^2 l(1 + \delta^2 O(1)) = E(\textstyle\sum_{n \leqq l} y_n)^2(1 + \delta^2 O(1)),$$
$$E(\textstyle\sum_{n \leqq l} z_n)^2 \ll \delta^2 s_N^2.$$

The proof is similar to [9, Lemma 2] or [8, Lemma 4].

We observe that both $\langle y_n \rangle$ and $\langle z_n \rangle$ are mixing in the same sense as is $\langle x_n \rangle$ and from Lemma 7 that $\psi_y(1) < \frac{1}{4}$, $\psi_z(1) < \frac{1}{4}$ or $\varphi_y(1) < \frac{1}{4}$, $\varphi_z(1) < \frac{1}{4}$ respectively. Hence from Lemmas 3, 4, 5 and 6 we conclude that the hypotheses of Theorem 0 are satisfied and thus—that $\sup_n E(y_n^4) \ll 1$ follows in the same way as (15) below—

$$P\ (\limsup_{l \to \infty} [2E(\textstyle\sum_{n \leqq l} y_n)^2 \log\log E(\sum_{n \leqq l} y_n)^2]^{-\frac{1}{2}} \sum_{n \leqq l} y_n = 1) = 1$$

and similarly for $\sum z_n$ and $|\sum z_n|$ by symmetry. Therefore with probability 1

$$\limsup_{N \to \infty} (2s_N^2 \log\log s_N^2)^{-\frac{1}{2}} \textstyle\sum_{i \leqq N} x_i \geqq \limsup_{n \to \infty} (2s_{\rho_n}^2 \log\log s_{\rho_n}^2)^{-\frac{1}{2}} \sum_{i \leqq \rho_n} x_i$$
$$\geqq \limsup\ ((\textstyle\sum y_n)) - \limsup\ ((|\sum z_n|))$$
$$\geqq 1 + \delta \cdot O(1)$$

by (11) which proves the lower estimate of the law of the iterated logarithm. Similarly we have with probability 1

$$(12) \qquad \limsup_{n \to \infty} (2s_{\rho_n}^2 \log\log s_{\rho_n}^2)^{-\frac{1}{2}} \textstyle\sum_{i \leqq \rho_n} x_i \leqq 1 + \delta \cdot O(1).$$

Now let N be arbitrary and let n be the largest integer with $\rho_n \leqq N$. Write $R = \rho_{n+1} - \rho_n$, $R^* = [R/j]$ and

$$(13) \qquad \eta_{rs} = x_{\rho_n + s + rj} \qquad (0 \leqq r \leqq R^*, 1 \leqq s \leqq j).$$

For fixed s the process $\langle \eta_{rs}, 0 \leqq r \leqq R^* \rangle$ is mixing in the same sense as $\langle x_n \rangle$ with $\psi_\eta(1) \leqq \frac{1}{4}$ or $\varphi_\eta(1) \leqq \frac{1}{4}$ respectively. For fixed s we have from Lemma 7 and (9)—now under the hypotheses of Theorem 2—

$$(14) \qquad \begin{aligned} E(\textstyle\sum_{r \leqq R^*} \eta_{rs})^2 &\ll \textstyle\sum_{s \leqq j} \sum_{r \leqq R^*} E(\eta_{rs}^2) + \sum_{r < r_1 \leqq R^*} |E(\eta_{rs}\eta_{r_1 s})| \\ &\ll (\textstyle\sum_{r,s} \|\eta_{rs}\|_4)^2 + \sum_{r < r_1 \leqq R^*, s \leqq j} \varphi^{3/4}(r_1 - r) \|\eta_{rs}\|_4 \\ &\ll (E(\textstyle\sum_{h=\rho_n}^{\rho_n+1} x_h)^2)^{2\rho} \ll (j/\delta)^8. \end{aligned}$$

Hence we obtain from [8, Lemma 10] with $P \ll (j/\delta)^4$ and $A_1 = A_2 = P$

$$(15) \qquad E(\textstyle\sum_{r \leqq R^*} \eta_{rs})^4 \ll (j/\delta)^{12}.$$

Thus from Lemma 4 for fixed s

$$P(\max_{1 \leqq r^* \leqq R^*} \sum_{r \leqq R^*} \eta_{rs} > s_{\rho_n})$$
$$\leqq 4P(\sum_{r \leqq R^*} \eta_{rs} > s_{\rho_n} - (j/\delta)^8) \ll (s_{\rho_n} - (j/\delta)^8)^{-4} E(\sum_{r \leqq R^*} \eta_{rs})^4$$
$$\ll (s_{\rho_n} - (j/\delta)^8)^{-4} (j/\delta)^{12} \ll s_{\rho_n}^{-4}$$

and since $s_n^{2\rho} \approx (j/\delta)^2 n$ by (10) and (11) we obtain

$$(16) \qquad P(\max_{1 \leqq r \leqq R} \sum_{t \leqq r} x_{\rho_n + t} > s_{\rho_n}) \ll j s_{\rho_n}^{-4} \ll n^{-2}.$$

The result follows now from (12), (16) and the fact $\sum n^{-2} < \infty$.

4. Some applications. In this section we shall consider two special cases. Let $\langle E_n, n = 1, 2, \cdots \rangle$ be a sequence of events with indicators χ_n. Let M_{ab} be the σ-algebra generated by the $E_n (a \leqq n \leqq b)$ and $A(N, \omega) = \sum_{n \leqq N} \chi_n(\omega)$ the number of integers $n \leqq N$ such that $\omega \varepsilon E_n$. (By ω we denote the elements of the sample space.) With this notation we have

THEOREM 3. *Suppose that the sequence $\langle E_n \rangle$ satisfies* (I) *with* $\sum \psi^{\frac{1}{3}}(n) < \infty$ *and that* $P(E_n) \to 0$. *Moreover, let*

$$\phi(N) = \sum_{n \leqq N} P(E_n) \to \infty \qquad (N \to \infty).$$

Then almost surely

$$\limsup_{N \to \infty} (2\phi(N) \log \log \phi(N))^{-\frac{1}{2}} [A(N, \omega) - \phi(N)] = 1.$$

PROOF. In [8, Theorem 6] we showed that

$$s_N^2 = \phi(N)(1 + o(1))$$
$$\phi(M, N) = \sum_{n=M+1}^{M+N} P(E_n) \ll E(\sum_{n=M+1}^{M+N} \chi_n - \phi(M, N))^2 \qquad (N \to \infty)$$

uniformly in M and that $E|\chi_n - P(E_n)| \leqq 2P(E_n)$.

If we set $x_n = \chi_n - P(E_n)$, Theorem 1 applies.

The second application deals with stationary processes considered already in [3] and [6].

THEOREM 4. *Suppose that the weak sense stationary process* $\langle x_n, n = 1, 2, \cdots \rangle$ *has the random variables* x_n *centered at expectations with* $\sup_n (E(x_n^4)) < \infty$. *Further assume that condition* (II) *holds with* $\sum \varphi^{1/5}(n) < \infty$. *Then*

$$\sigma^2 = \lim N^{-1} E(\sum_{n \leqq N} x_n)^2 \qquad (N \to \infty)$$

exists. Moreover if $\sigma \neq 0$ *then*

$$P(\limsup_{N \to \infty} (2\sigma^2 N \log \log N)^{-\frac{1}{2}} \sum_{n \leqq N} x_n = 1) = 1.$$

The proof of the theorem is an immediate application of Theorem 2. As is well known (see e.g. [8, Theorem 9] we have $s_N^2 = N\sigma^2 + O(1)$ and using Lemma 3 we have for $\sigma \neq 0$

$$\sum_{n=M+1}^{M+N} \|x_n\|_4 \leqq N \ll N\sigma^2 \ll E(\sum_{n=M+1}^{M+N} x_n)^2.$$

It is not difficult to see that it is enough to assume $\sum \varphi^{\frac{1}{2}}(n) < \infty$ and the conclusion of Theorem 4 still remains valid. The same is true if we assume $\sum \varphi(n) < \infty$ and $\sup \|x_n\|_\infty \leqq 1$. This improves a theorem of Iosifescu [3]. He additionally assumed $\varphi(1) < 1$, which in many applications seems to be rather hard to verify, see, e.g., [10]. Theorem 4 has also been proved independently under somewhat different hypotheses by the author [6]. For a comparison of the hypotheses see [8, Chapter 2].

Note added in proof. For strict sense stationary processes Reznik [*Teoria Verojat. Primen.* **13,** 642–656 (1968)] has Theorem 4 in a slightly sharper form. As a matter of fact, the remainder of the remark to [8, Theorem 9] applies for the present situation, too.

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