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ON A CHARACTERIZATION OF THE WIENER PROCESS BY CONSTANT REGRESSION

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1. Introduction. Recently Cacoullos [1] has proved the following theorem which characterizes Normal Distribution by constant regression of one linear statistic on another linear statistic.

THEOREM 1.1. *Let X_i, $1 \leqq i \leqq n$ be a random sample from a univariate population with non-degenerate distribution function $F(x)$, and assume that $F(x)$ has moments of every order. Consider the linear statistics*

$$U = a_1 X_1 + \cdots + a_n X_n; \qquad V = b_1 X_1 + \cdots + b_n X_n$$

where a_i, $1 \leqq i \leqq n$; b_i, $1 \leqq i \leqq n$ have the property that $\sum_{i=1}^n a_i b_i = 0$ implies that $\sum_{i=1}^n a_i b_i^k \neq 0$ for all $k > 1$. Then U has constant regression on V, i.e.,

$$E[U \mid V] = E[U] \quad \text{a.e.}$$

if and only if (1) *the population distribution F is normal, and* (2) $\sum_{i=1}^n a_i b_i = 0$.

In this paper we derive a similar result for a characterization of the Wiener process. We would like to mention that a theorem similar to Theorem 1.1 has also been obtained by Rao [4].

2. Preliminaries. Let $T = [A, B]$. We shall consider stochastic processes $\{X(t), t \in T\}$ which have finite moments of all orders. In particular, $\{X(t), t \in T\}$ will be a stochastic process of the second order. Let $a(\cdot)$ be a function which is continuous on $[A, B]$, and suppose that the mean function $m(t) = E[X(t)]$ and the covariance function $r(s, t) = E[X(t)X(s)] - E[X(t)]E[X(s)]$ are of bounded variation in $[A, B]$. It can be shown that the integral

$$\int_A^B a(t)\, dX(t) \tag{2.1}$$

exists as the limit in the mean (lim) of the corresponding Riemann-Stieltjes sums.

A stochastic process $\{X(t), t \in T\}$ is said to be a homogeneous process with independent increments if the distribution of the increments $X(t+h) - X(t)$ depends only on h but is independent of t, and if the increments over non-overlapping intervals are stochastically independent. The process is said to be continuous if $X(t)$ converges in probability to $X(s)$ as t tends to s for every $s \in T$. Let $\{X(t), t \in T\}$ be a continuous homogeneous process with independent increments. Let $\varphi(u; h)$ denote the characteristic function of $X(t+h) - X(t)$. It is well known that $\varphi(u; h)$ is infinitely divisible and $\varphi(u; h) = [\varphi(u; 1)]^h$. See Lukacs [2].

A homogeneous process $\{X(t), t \in T\}$ with independent increments is called a Wiener process if the increments $X(t+h) - X(t)$ are normally distributed with variance proportional to h.

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3. Characterization.

THEOREM 3.1. *Let $\{X(t), t \in T\}$ be a continuous homogeneous process with independent increments and suppose that the increments have non-degenerate distributions. Further suppose that the process has moments of all orders and its mean function as well as its covariance function are of bounded variation in $T = [A, B]$. Let $a(\cdot)$, $b(\cdot)$ be continuous functions defined on $[A, B]$ with the property that*

$$\int_A^B a(t)b(t)\,dt = 0 \tag{3.1}$$

implies that

$$\int_A^B a(t)[b(t)]^k\,dt \neq 0 \tag{3.2}$$

for all $k > 1$. Let

$$U = \int_A^B a(t)\,dX(t), \tag{3.3}$$

$$V = \int_A^B b(t)\,dX(t). \tag{3.4}$$

Then U has constant regression on V, i.e.,

$$E[U \mid V] = E[U] \quad \text{a.e.} \tag{3.5}$$

if and only if

(i) *$\{X(t), t \in T\}$ is a Wiener process with a linear mean function,*
(ii) $\int_A^B a(t)b(t)\,dt = 0$.

The proof of the theorem is given in Section 5. We shall state and prove some lemmas in the next section which will be used in the proof of the theorem.

4. Some lemmas.

LEMMA 4.1. *A random variable Y, with finite expectation, has constant regression on a random variable Z*, i.e., $E[Y \mid Z] = E[Y]$ a.e. *if and only if*

$$E[Y e^{itZ}] = E[Y]E[e^{itZ}]. \tag{4.1}$$

Proof of this lemma can be found in Lukacs and Laha [3].

LEMMA 4.2. *Let $\{X(t), t \in T\}$ be a continuous homogeneous stochastic process with independent increments on $T = [A, B]$. Further, suppose that the process is a second order process and its mean function and covariance function are of bounded variation in $[A, B]$. Let*

$$Y = \int_A^B g(t)\,dX(t); \qquad Z = \int_A^B h(t)\,dX(t)$$

for continuous functions $g(\cdot)$ and $h(\cdot)$ on $[A, B]$. Denote by $\varphi(u; h)$ and $\theta(u, v)$ the characteristic functions of $X(t+h) - X(t)$ and (Y, Z) respectively. Then $\theta(u, v)$ is different from zero for all u and v and

$$\log \theta(u, v) = \int_A^B \psi[ug(t) + vh(t)]\,dt$$

where $\psi(u) = \log \varphi(u; 1)$.

Proof of this lemma can be found in Lukacs and Laha [3].

LEMMA 4.3. *Let $\{X(t), t \in T\}$ be a continuous homogeneous stochastic process with independent increments on $T = [A, B]$. Further suppose that the process is a second order process and its mean function and covariance function are of bounded variation on $[A, B]$. Let $g(\cdot)$ and $h(\cdot)$ be continuous functions on $[A, B]$. Denote*

$$Y = \int_A^B g(t)\,dX(t); \qquad Z = \int_A^B h(t)\,dX(t).$$

Then for any real number v,

$$E[Y e^{ivZ}] = -i(\int_A^B g(t)\psi'[vh(t)]\,dt) \cdot \exp(\int_A^B \psi[vh(t)]\,dt) \tag{4.2}$$

where $\psi(u) = \log \varphi(u; 1)$ is the logarithm of the characteristic function of $X(t+1)-X(t)$.

PROOF. Let $\theta(u, v)$ denote the characteristic function of the bivariate random variable (Y, Z). By Lemma 4.2, $\log \theta(u, v)$ is well-defined and

$$\log \theta(u, v) = \int_A^B \psi[ug(t)+vh(t)]\,dt,$$

i.e.,

$$E[\exp(iuY+ivZ)] = \exp(\int_A^B \psi[ug(t)+vh(t)]\,dt).$$

Differentiating on both sides with respect to u, we get that for all u and v,

$$\begin{aligned} &E[iY\exp(iuY+ivZ)] \\ &= \exp(\int_A^B \psi[ug(t)+vh(t)]\,dt) \cdot \int_A^B \psi'[ug(t)+vh(t)]g(t)\,dt \end{aligned} \tag{4.3}$$

where $\psi'(u)$ denotes the derivative of $\psi(\cdot)$ at u. This differentiation is valid since the random vector (Y, Z) has moments of all orders. Take $u = 0$ in (4.3). Then it follows that

$$E[iY e^{ivZ}] = \exp(\int_A^B \psi[vh(t)]\,dt) \cdot \int_A^B \psi'[vh(t)]g(t)\,dt$$

and hence

$$E[Y e^{ivZ}] = -i\exp(\int_A^B \psi[vh(t)]\,dt) \cdot \int_A^B \psi'[vh(t)]g(t)\,dt$$

which completes the proof of the lemma.

5. Proof of Theorem 3.1.

"*Only if*" *part*. Suppose $\{X(t), t \in T\}$ is a Wiener process with mean $m(t) = \lambda t$ and covariance function $r(s, t) = \sigma^2 \min(s, t)$ where $-\infty < \lambda < \infty$, $\sigma^2 > 0$. Let $a(\cdot)$ and $b(\cdot)$ be continuous functions on $[A, B]$ and define U and V as in the theorem. Further suppose that $a(\cdot)$ and $b(\cdot)$ are such that

$$\int_A^B a(t)b(t)\,dt = 0. \tag{5.1}$$

Since $\psi(\cdot)$ is the logarithm of the characteristic function of $X(t+1)-X(t)$, it is well-known that $\psi(t) = i\lambda t - \frac{1}{2}\sigma^2 t^2$. In order to show that $E[U \mid V] = E[U]$ a.e. it is enough to prove that

$$E[U e^{isV}] = E[U]E[e^{isV}] \tag{5.2}$$

by Lemma 4.1. By Lemmas 4.3 and 4.2 and the condition (5.1), it follows that

$$\begin{aligned} E[U e^{isV}] &= -i(\textstyle\int_A^B a(t)\psi'[sb(t)]\,dt)\exp(\int_A^B \psi[sb(t)]\,dt) \\ &= -i(\textstyle\int_A^B a(t)\psi'[sb(t)]\,dt)E[e^{isV}] \\ &= -i(\textstyle\int_A^B a(t)\{i\lambda-\sigma^2 sb(t)\}\,dt)E[e^{isV}] \quad (5.3) \\ &= -i(\textstyle\int_A^B i\lambda a(t)\,dt)E[e^{isV}] \\ &= \{\textstyle\int_A^B \lambda a(t)\,dt\}E[e^{isV}]. \end{aligned}$$

It can be shown easily that $E[U] = \lambda\int_A^B a(t)\,dt$ which proves that

$$E[U e^{isV}] = E[U]E[e^{isV}]$$

in view of (5.3). This completes the proof of the "only if" part.

"If" part. Let U and V be as defined in the theorem. Let $\psi(\cdot)$ denote the logarithm of the characteristic function of $X(t+1)-X(t)$. Further suppose that U has constant regression on V, i.e.,

$$E[U \mid V] = E[U] \quad \text{a.e.} \tag{5.4}$$

This implies that

$$E[U e^{isV}] = E[U]E[e^{isV}]$$

by Lemma 4.1. Hence by Lemmas 4.2 and 4.3, it follows that

$$\begin{aligned} -i(\textstyle\int_A^B \psi'[sb(t)]a(t)\,dt)&\exp(\textstyle\int_A^B \psi[sb(t)]\,dt) \\ &= E[U e^{isV}] \quad (5.5) \\ &= E[U]E[e^{isV}] \\ &= E[U]\exp(\textstyle\int_A^B \psi[sb(t)]\,dt). \end{aligned}$$

The above equality gives the relation

$$\int_A^B \psi'[sb(t)]a(t)\,dt = iE[U] \tag{5.6}$$

for any real number s. Since the process has moments of all orders by assumption, $\psi(\cdot)$ has derivatives of all orders and the differentiations with respect to s under integral sign in (5.6) are valid. Differentiating once with respect to s, we get that

$$\int_A^B \psi''[sb(t)]a(t)b(t)\,dt = 0. \tag{5.7}$$

Let $s = 0$. Then we have

$$\psi''(0)\int_A^B a(t)b(t)\,dt = 0.$$

$\psi''(0)$ is different from zero, since by assumption the increments of the process have non-degenerate distributions. Hence it follows from the above equality, that

$$\int_A^B a(t)b(t)\,dt = 0. \tag{5.8}$$

Differentiating k times with respect to s under the integral sign in (5.7), we obtain that

$$\int_A^B \psi^{(k)}[sb(t)]a(t)[b(t)]^{k-1}\,dt = 0 \tag{5.9}$$

for $k \geqq 3$ where $\psi^{(k)}(s)$ denotes the kth derivative of $\psi(\cdot)$ at s. Take $s = 0$ in (5.9). Then we have

$$\psi^{(k)}(0)\int_A^B a(t)[b(t)]^{k-1}\,dt = 0 \tag{5.10}$$

for $k \geqq 3$. Since the functions $a(\cdot)$, $b(\cdot)$ have the property that $\int_A^B a(t)b(t)\,dt = 0$ implies that $\int_A^B a(t)[b(t)]^k \neq 0$ for $k > 1$, (5.9) and (5.10) together imply that

$$\psi^{(k)}(0) = 0 \qquad \text{for} \qquad k \geqq 3 \tag{5.11}$$

which shows that $\psi(t) = i\lambda t - \frac{1}{2}\sigma^2 t^2$ where $-\infty < \lambda < \infty$, $\sigma^2 > 0$ for some λ and σ^2. Hence the process $\{X(t),\, t \in T\}$ is a Wiener process. This completes the proof of the "If" part.

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